HOMOTOPY AND GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 45
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

EXAMPLES OF CIRCLE ACTIONS ON SYMPLECTIC SPACES

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Many interesting results in the study of symplectic torus actions can be proved by purely cohomological methods. All one needs is a closed orientable topological $2n$-manifold M (or, more generally, a reasonably pleasant topological space whose rational cohomology satisfies Poincaré duality with formal dimension $2n$), which is cohomologically symplectic (c-symplectic) in the sense that there is a class $w \in H^2(M;\mathbb{Q})$ such that $w^n \neq 0$. Sometimes one requires that M satisifes the Lefschetz condition that multiplication by w^{n-1} is an isomorphism $H^1(M;\mathbb{Q}) \to H^{2n-1}(M;\mathbb{Q})$. And an action of a torus T on M is said to be cohomologically Hamiltonian (c-Hamiltonian) if $w \in \mathrm{Im}[i^* : H^*(M_T;\mathbb{Q}) \to H^*(M;\mathbb{Q})]$, where M_T is the Borel construction; and $i : M \to M_T$ is the inclusion of the fibre in the fibre bundle $M_T \to BT$.

Some examples of some results which can be proved easily by cohomological methods are the following.

(a) If the group $G = T^k$, the k-dimensional torus, acts symplectically on a closed symplectic manifold M, and if the action is Hamiltonian, then the fixed point set $M^G \neq \emptyset$. (In the cohomological context, there is no reason to expect the existence of a moment map.)

(b) If $G = T^k$ acts symplectically on a closed symplectic manifold M, if M satisfies the Lefschetz condition and if $M^G \neq \emptyset$, then the action is Hamiltonian.

(c) If a compact connected Lie group G acts on a closed symplectic manifold M with only finite isotropy subgroups, then G is a torus.

(d) If $G = T^k$ acts on a closed symplectic manifold M with only finite isotropy subgroups (*i.e.*, almost–freely), and if M satisfies the Lefschetz condition, then $H^*(M;\mathbb{Q}) \cong H^*(G;\mathbb{Q}) \otimes H^*(M/G;\mathbb{Q})$.

(For the results above, see, for example, [Al], [AP], [B], [F] and [LO].)

1991 *Mathematics Subject Classification*: 57S99, 55P99, 58F99.
The paper is in final form and no version of it will be published elsewhere.

On the other hand, the following results require some more geometrical reasoning: they do not hold in the purely cohomological context.

(1) If $G = S^1$ acts symplectically on a closed symplectic manifold M, then every component of M^G is symplectic. ([F])

(2) If $G = S^1$ acts symplectically on a closed symplectic manifold M, and if the action is Hamiltonian, then

$$\dim_{\mathbb{Q}} H^*(M;\mathbb{Q}) = \dim_{\mathbb{Q}} H^*(M^G,\mathbb{Q}). \qquad ([\mathrm{F}])$$

(3) If $G = S^1$ acts symplectically on a closed symplectic 4-manifold M, and if $M^G \neq \emptyset$, then the action is Hamiltonian. ([McD])

In [A2] we gave cohomological examples in which no component of M^G is c-symplectic. In this note we give cohomological examples which do not satisfy the conclusions of (2) and (3).

Example 1. Let $G = S^1$ act freely on $S^3 \times S^3$. Consider a tube $S^1 \times D^5$ around an orbit. The group is acting by left translations on the first factor. Remove the tube, and replace it with $D^2 \times S^4$ with G acting by standard rotations on the first factor. Call the resulting G-manifold N. So G is acting semi-freely on N with $N^G = S^4$. Furthermore, a typical Mayer–Vietoris sequence argument shows that $H^*(N;\mathbb{Z})$ is free with Betti numbers 1, 0, 1, 2, 1, 0, 1.

Now let G act semi-freely on $\mathbb{C}P^3$ with fixed point set $P + \mathbb{C}P^2$, where P is an isolated point. Form the equivariant connected sum $M = \mathbb{C}P^3 \# N$ by removing small open discs centered on fixed points in $\mathbb{C}P^2$ and S^4. Thus G acts semi-freely on M with $M^G = P + \mathbb{C}P^2$. Clearly M is c-symplectic and satisfies the Lefschetz condition, and the action is c-Hamiltonian. However

$$\dim_{\mathbb{Q}} H^*(M^G,\mathbb{Q}) = 4 < \dim_{\mathbb{Q}} H^*(M;\mathbb{Q}) = 8.$$

Thus this example satisfies the conditions of Frankel's theorem (2) above as far as the cohomology is concerned, but it does not satisfy the conclusion.

Before giving Example 2 we shall prove two lemmas. The first lemma shows that, in a large number of examples similar to Example 2, there are always c-symplectic classes which are not c-Hamiltonian. The second lemma shows that in Example 2, in particular, no c-symplectic class is c-Hamiltonian.

Lemma 1. *Let M be a closed topological 4-manifold. Suppose that $G = S^1$ acts on M such that $M^G \neq \emptyset$ and*

$$\dim_{\mathbb{Q}} H^*(M^G;\mathbb{Q}) < \dim_{\mathbb{Q}} H^*(M;\mathbb{Q}).$$

Then there is a class $y \in H^2(M;\mathbb{Q})$ such that $y^2 \neq 0$ and $y \notin \operatorname{Im}\big(i^* : H^*(M_G;\mathbb{Q}) \to H^*(M;\mathbb{Q})\big)$.

Proof. Let $y_1, \ldots, y_k$ be a basis for $H^2(M;\mathbb{Q})$ such that $y_i^2 \neq 0$ for $1 \leq i \leq k$ and $y_i y_j = 0$ for $i \neq j$. (We shall prove and not assume, however, that $H^2(M;\mathbb{Q}) \neq 0$.)

Consider the E_2 term of the Serre spectral sequence for $M_G \to BG$ in rational cohomology. Since $\dim_{\mathbb{Q}} H^*(M^G;\mathbb{Q}) < \dim_{\mathbb{Q}} H^*(M;\mathbb{Q})$, the spectral sequence does not collapse. (See, e.g., [AP], Theorem (3.10.4).) Hence $H^3(M;\mathbb{Q}) \neq 0$. Let $v \in H^3(M;\mathbb{Q})$ and let $d_2(v) = \sum_{j=1}^{k} \lambda_j y_j t$, where $t \in H^2(BG;\mathbb{Q})$ is a generator. (Here $y_j t$ is an abbreviation for $t \otimes y_j \in H^2(BG;\mathbb{Q}) \otimes H^2(M;\mathbb{Q}) = E_2^{2,2}$.)

For degree reasons, $d_2(y_i v) = 0$. And $d_2(y_i v) = \lambda_i y_i^2 t$ if $d_2(y_i) = 0$. So $d_2(v) = 0$ if $d_2(y_i) = 0$ for all i. But, since $M^G \neq \emptyset$ and $E_2 \neq E_\infty$, $d_2(v) \neq 0$ for some $v \in H^3(M;\mathbb{Q})$. Thus $d_2(y_i) \neq 0$ for some i.

LEMMA 2. *Let M be a closed c-symplectic topological $2n$-manifold, and let $G = S^1$ act on M in an effective c-Hamiltonian way. Then M^G has at least two components.*

Proof. Suppose that M^G is connected. Let $y \in H^2(M;\mathbb{Q})$ be a c-Hamiltonian class: *i.e.*, $y^n \neq 0$, and there is $\overline{y} \in H^2(M_G;\mathbb{Q})$ such that $i^*(\overline{y}) = y$, where i, as before, is the inclusion of the fibre $M \to M_G$.

Let $\varphi : M^G \to M$ be the inclusion, and consider

$$\varphi^* : H^*(M_G;\mathbb{Q}) \to H^*\big((M^G)_G;\mathbb{Q}\big) \cong H^*(BG;\mathbb{Q}) \otimes H^*(M^G;\mathbb{Q}).$$

By subtracting a rational multiple of the generator $t \in H^2(BG;\mathbb{Q})$ from $\overline{y}$, if necessary, we can assume that $\varphi^*(\overline{y}) \in H^0(BG;\mathbb{Q}) \otimes H^2(M^G;\mathbb{Q})$. Thus $\varphi^*(\overline{y}^{m+1}) = 0$ where $\dim M^G = 2m$.

Now by the Localization Theorem, $\overline{y}^{m+1}$ is torsional in $H^*(M_G;\mathbb{Q})$ viewed as a $\mathbb{Q}[t]$-module. But $\overline{y}^n$ is not torsional, since $i^*(\overline{y}^n) = y^n \neq 0$. (There can be no torsion on the top row of the Serre spectral sequence.) Hence $m \geq n$, which contradicts the effectiveness of the action.

REMARKS. Lemma 2 can be generalized as follows. Let M be a closed c-symplectic topological $2n$-manifold, and let $G = T^k$, the k-dimensional torus, act on M in an effective, uniform (see below), c-Hamiltonian way. Then M^G has at least $k+1$ components.

See [AP], Definition (3.6.17), for the definition of a uniform action. Note that an actual Hamiltonian action is uniform by [AP], Corollary (3.6.19) and Frankel's Theorem (2) above, which is also valid for torus actions (as follows from the circle case).

Lemma 2 is another example of a well-known geometric theorem which has a purely cohomological proof. See, e.g., [Au], Chapter III, Corollary 4.2.3 and its proof, for the geometric version, which follows from the Atiyah–Guillemin–Sternberg Convexity Theorem.

Now we conclude with Example 2 which shows that McDuff's Theorem (3) above does not have a purely cohomology proof.

EXAMPLE 2. The example begins with two copies of $\mathbb{C}P^2$ with different orientations, and with $G = S^1$ acting on each copy semi-freely fixing $P + S^2$ where P is an isolated point. Now let N be the equivariant connected sum formed by removing small open discs centered on the isolated fixed points. So $N = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$; and G acts semi-freely on N with $N^G = S^2 + S^2$. Next remove two small open discs centered on fixed points, one in each component of N^G. Let M be the result of equivariantly attaching $S^3 \times I^1$. Clearly this can be done so that M is orientable; and G is acting semi-freely on M with $M^G = S^2$.

Again, a Mayer–Vietoris sequence argument shows that $H^*(M;\mathbb{Z})$ is free with Betti numbers 1, 1, 2, 1 and 1. Since $H^2(M;\mathbb{Q}) \neq 0$, M is c-symplectic. Since M^G is connected, the action is not c-Hamiltonian with respect to any c-symplectic class by Lemma 2.

Acknowledgements. I would like to thank John Oprea for raising the questions which led to this paper; and I would like to thank Volker Puppe for some helpful discussions, especially concerning Example 1.

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