EXAMPLES OF CIRCLE ACTIONS ON SYMPLECTIC SPACES

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Many interesting results in the study of symplectic torus actions can be proved by purely cohomological methods. All one needs is a closed orientable topological 2n-manifold $M$ (or, more generally, a reasonably pleasant topological space whose rational cohomology satisfies Poincaré duality with formal dimension $2n$), which is cohomologically symplectic ($\omega$-symplectic) in the sense that there is a class $\omega \in H^2(M; \mathbb{Q})$ such that $\omega^n \neq 0$. Sometimes one requires that $M$ satisfies the Lefschetz condition that multiplication by $\omega^{n-1}$ is an isomorphism $H^1(M; \mathbb{Q}) \to H^{2n-1}(M; \mathbb{Q})$. And an action of a torus $T$ on $M$ is said to be cohomologically Hamiltonian ($\omega$-Hamiltonian) if $\omega \in \text{Im}[i^* : H^*(M_T; \mathbb{Q}) \to H^*(M; \mathbb{Q})]$, where $M_T$ is the Borel construction; and $i : M \to M_T$ is the inclusion of the fibre in the fibre bundle $M_T \to BT$.

Some examples of some results which can be proved easily by cohomological methods are the following.

(a) If the group $G = T^k$, the $k$-dimensional torus, acts symplectically on a closed symplectic manifold $M$, and if the action is Hamiltonian, then the fixed point set $M^G \neq \emptyset$. (In the cohomological context, there is no reason to expect the existence of a moment map.)

(b) If $G = T^k$ acts symplectically on a closed symplectic manifold $M$, if $M$ satisfies the Lefschetz condition and if $M^G \neq \emptyset$, then the action is Hamiltonian.

(c) If a compact connected Lie group $G$ acts on a closed symplectic manifold $M$ with only finite isotropy subgroups, then $G$ is a torus.

(d) If $G = T^k$ acts on a closed symplectic manifold $M$ with only finite isotropy subgroups (i.e., almost–freely), and if $M$ satisfies the Lefschetz condition, then $H^*(M; \mathbb{Q}) \cong H^*(G; \mathbb{Q}) \otimes H^*(M/G; \mathbb{Q})$.

(For the results above, see, for example, [Al], [AP], [B], [F] and [LO].)
On the other hand, the following results require some more geometrical reasoning: they do not hold in the purely cohomological context.

(1) If $G = S^1$ acts symplectically on a closed symplectic manifold $M$, then every component of $M^G$ is symplectic. ([F])

(2) If $G = S^1$ acts symplectically on a closed symplectic manifold $M$, and if the action is Hamiltonian, then
\[ \dim_{\mathbb{Q}} H^*(M; \mathbb{Q}) = \dim_{\mathbb{Q}} H^*(M^G, \mathbb{Q}). \]  

(3) If $G = S^1$ acts symplectically on a closed symplectic 4-manifold $M$, and if $M^G \neq \emptyset$, then the action is Hamiltonian. ([McD])

In [A2] we gave cohomological examples in which no component of $M^G$ is $c$-symplectic. In this note we give cohomological examples which do not satisfy the conclusions of (2) and (3).

**Example 1.** Let $G = S^1$ act freely on $S^3 \times S^3$. Consider a tube $S^1 \times D^5$ around an orbit. The group is acting by left translations on the first factor. Remove the tube, and replace it with $D^2 \times S^4$ with $G$ acting by standard rotations on the first factor. Call the resulting $G$-manifold $N$. So $G$ is acting semi-freely on $N$ with $N^G = S^4$. Furthermore, a typical Mayer–Vietoris sequence argument shows that $H^*(N; \mathbb{Z})$ is free with Betti numbers 1, 0, 1, 2, 1, 0, 1.

Now let $G$ act semi-freely on $\mathbb{C}P^3$ with fixed point set $P + \mathbb{C}P^2$, where $P$ is an isolated point. Form the equivariant connected sum $M = \mathbb{C}P^3 \# N$ by removing small open discs centered on fixed points in $\mathbb{C}P^2$ and $S^4$. Thus $G$ acts semi-freely on $M$ with $M^G = P + \mathbb{C}P^2$. Clearly $M$ is $c$-symplectic and satisfies the Lefschetz condition, and the action is $c$-Hamiltonian. However
\[ \dim_{\mathbb{Q}} H^*(M^G; \mathbb{Q}) = 4 < \dim_{\mathbb{Q}} H^*(M; \mathbb{Q}) = 8. \]

Thus this example satisfies the conditions of Frankel’s theorem (2) above as far as the cohomology is concerned, but it does not satisfy the conclusion.

Before giving Example 2 we shall prove two lemmas. The first lemma shows that, in a large number of examples similar to Example 2, there are always $c$-symplectic classes which are not $c$-Hamiltonian. The second lemma shows that in Example 2, in particular, no $c$-symplectic class is $c$-Hamiltonian.

**Lemma 1.** Let $M$ be a closed topological 4-manifold. Suppose that $G = S^1$ acts on $M$ such that $M^G \neq \emptyset$ and
\[ \dim_{\mathbb{Q}} H^*(M^G; \mathbb{Q}) < \dim_{\mathbb{Q}} H^*(M; \mathbb{Q}). \]

Then there is a class $y \in H^2(M; \mathbb{Q})$ such that $y^2 \neq 0$ and $y \notin \text{Im} (i^*: H^*(M^G; \mathbb{Q}) \to H^*(M; \mathbb{Q})).$

**Proof.** Let $y_1, \ldots, y_k$ be a basis for $H^2(M; \mathbb{Q})$ such that $y_i^2 \neq 0$ for $1 \leq i \leq k$ and $y_iy_j = 0$ for $i \neq j$. (We shall prove and not assume, however, that $H^2(M; \mathbb{Q}) \neq 0$.)

Consider the $E_2$ term of the Serre spectral sequence for $M_G \to BG$ in rational cohomology. Since $\dim_{\mathbb{Q}} H^*(M^G; \mathbb{Q}) < \dim_{\mathbb{Q}} H^*(M; \mathbb{Q})$, the spectral sequence does not collapse. (See, e.g., [AP], Theorem (3.10.4).) Hence $H^2(M; \mathbb{Q}) \neq 0$. Let $v \in H^1(M; \mathbb{Q})$ and
Thus $d_2(y, v) = 0$ and $d_2(y_i, v) = \lambda_i y_i t$ if $d_2(y_i) = 0$. So $d_2(v) = 0$ if $d_2(y_i) = 0$ for all $i$. But, since $M^G \neq \emptyset$ and $E_2 \neq E_\infty$, $d_2(v) \neq 0$ for some $v \in H^3(M; \mathbb{Q})$. Thus $d_2(y_i) \neq 0$ for some $i$.

**Lemma 2.** Let $M$ be a closed $c$-symplectic topological $2n$-manifold, and let $G = S^1$ act on $M$ in an effective $c$-Hamiltonian way. Then $M^G$ has at least two components.

**Proof.** Suppose that $M^G$ is connected. Let $y \in H^2(M; \mathbb{Q})$ be a $c$-Hamiltonian class; i.e., $y^n \neq 0$, and there is $\eta \in H^2(M_G; \mathbb{Q})$ such that $i^*(\eta) = y$, where $i$, as before, is the inclusion of the fibre $M \rightarrow M_G$.

Let $\varphi : M^G \rightarrow M$ be the inclusion, and consider

$$
\varphi^* : H^*(M_G; \mathbb{Q}) \rightarrow H^*(M^G; \mathbb{Q}) \cong H^*(BG; \mathbb{Q}) \otimes H^*(M^G; \mathbb{Q}).
$$

By subtracting a rational multiple of the generator $t \in H^2(BG; \mathbb{Q})$ from $\eta$, if necessary, we can assume that $\varphi^*(\eta) \in H^0(BG; \mathbb{Q}) \otimes H^*(M^G; \mathbb{Q})$. Thus $\varphi^*(\eta^{m+1}) = 0$ where $\dim M^G = 2m$.

Now by the Localization Theorem, $\eta^{m+1}$ is torsional in $H^*(M_G; \mathbb{Q})$ viewed as a $\mathbb{Q}[t]$-module. But $\eta$ is not torsional, since $i^*(\eta^t) = y^n \neq 0$. (There can be no torsion on the top row of the Serre spectral sequence.) Hence $m \geq n$, which contradicts the effectiveness of the action.

**Remarks.** Lemma 2 can be generalized as follows. Let $M$ be a closed $c$-symplectic topological $2n$-manifold, and let $G = T^k$, the $k$-dimensional torus, act on $M$ in an effective, uniform (see below), $c$-Hamiltonian way. Then $M^G$ has at least $k + 1$ components.

See [AP], Definition (3.6.17), for the definition of a uniform action. Note that an actual Hamiltonian action is uniform by [AP], Corollary (3.6.19) and Frankel’s Theorem (2) above, which is also valid for torus actions (as follows from the circle case).

Lemma 2 is another example of a well-known geometric theorem which has a purely cohomological proof. See, e.g., [Au], Chapter III, Corollary 4.2.3 and its proof, for the geometric version, which follows from the Atiyah-Guillemin-Sternberg Convexity Theorem.

Now we conclude with Example 2 which shows that McDuff’s Theorem (3) above does not have a purely cohomology proof.

**Example 2.** The example begins with two copies of $\mathbb{C}P^2$ with different orientations, and with $G = S^1$ acting on each copy semi-freely fixing $P + S^2$ where $P$ is an isolated point. Now let $N$ be the equivariant connected sum formed by removing small open discs centered on the isolated fixed points. So $N = \mathbb{C}P^2 \# \mathbb{C}P^2$; and $G$ acts semi-freely on $N$ with $N^G = S^2 + S^2$. Next remove two small open discs centered on fixed points, one in each component of $N^G$. Let $M$ be the result of equivariantly attaching $S^3 \times I^1$. Clearly this can be done so that $M$ is orientable; and $G$ is acting semi-freely on $M$ with $M^G = S^2$.

Again, a Mayer–Vietoris sequence argument shows that $H^*(M; \mathbb{Z})$ is free with Betti numbers $1, 1, 2, 1$ and $1$. Since $H^2(M; \mathbb{Q}) \neq 0$, $M$ is $c$-symplectic. Since $M^G$ is connected, the action is not $c$-Hamiltonian with respect to any $c$-symplectic class by Lemma 2.
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References


