

CATEGORY WEIGHT: NEW IDEAS CONCERNING LUSTERNIK–SCHNIRELMANN CATEGORY

YULI B. RUDYAK

*Mathematisches Institut, Universität Heidelberg
Im Neuenheimer Feld 288, D-69120 Heidelberg 1, Germany
E-mail: july@mathi.uni-heidelberg.de*

Introduction. The concept of category weight was introduced by Fadell–Husseini [FH] and developed by Rudyak and Strom. Here we give a survey, some further development and applications of category weight.

The Lusternik–Schnirelmann category of a topological space X , $\text{cat } X$, is defined as the minimal number k such that X admits a numerable covering $\{A_1, \dots, A_{k+1}\}$ where each A_i is contractible in X . Lusternik and Schnirelmann [LS] introduced the invariant $\text{cat } X$ for manifolds. They proved that, for every connected smooth ($=C^\infty$) closed manifold M ,

$$1 + \text{cat } M \leq \text{Crit } M := \min\{\text{crit } f \mid f \in C^\infty(M, \mathbb{R})\}$$

where $\text{crit } f$ is the number of critical points of a smooth real-valued function f on M . Afterwards Fox [Fox] suggested considering $\text{cat } X$ as an invariant of a *space* X . The basic information concerning the Lusternik–Schnirelmann category can be found in [Fox], [Sv], [J].

This (homotopy) invariant is quite far from other invariants like homotopy and homology groups, so it is difficult to compute $\text{cat } X$. For example, in [G], Ganea asked whether $\text{cat}(X \times S^n) = \text{cat } X + 1$, $n > 0$, X connected. The affirmative claim is usually referred to as the Ganea conjecture. Recently it was disproved by Iwase [I], but it is still unclear whether it is true for manifolds. Here is another rather naive question. Let $f : M \rightarrow N$ be a map of degree 1 of closed manifolds. Is it true that $\text{cat } M \geq \text{cat } N$?

One of the favorite and famous ways to estimate the Lusternik–Schnirelmann category is a so-called cup-length estimation (Froloff–Elsholz [FE], Eilenberg). Namely, if $u_1 \cdots u_n \neq 0$ for some $u_i \in \tilde{H}^*(X)$ then $\text{cat } X \geq n$. The idea of the proof is quite simple: if $X = A_1 \cup \cdots \cup A_n$ where each A_i is contractible in X then $u_i|_{A_i} = 0$, and so $u_1 \cdots u_n|(A_1 \cup \cdots \cup A_n) = 0$. However, the cup-length estimation is not perfect. For example, if $L = L_p^{2n+1}$ is the lens space with p an odd prime, then $\text{cat } L = 2n+1$ (Krasnosel'ski,

1991 *Mathematics Subject Classification*: Primary 55M30; Secondary 55N20, 55S30, 58F05.
The paper is in final form and no version of it will be published elsewhere.

1955), while the cup-length estimation gives us $\text{cat } L \geq n+1$ only. Fadell–Husseini refined the cup-length estimation by suggesting that elements u_i be equipped with “weights”. Speaking informally, we say that the category weight of u ($\text{cwg}t u$) is $\geq k$ if

$$u|(A_{i_1} \cup \cdots \cup A_{i_k}) = 0$$

for A_i as above. Clearly, $\text{cat } X \geq \text{cwg}t u$ if $u \neq 0$. Furthermore, $\text{cwg}t(u_1 \cdots u_n) \geq \sum \text{cwg}t u_i$, and, since $\text{cwg}t u_i \geq 1$, we conclude that

$$\text{cat } X \geq \text{cwg}t(u_1 \cdots u_n) \geq \sum \text{cwg}t u_i \geq n$$

provided $u_1 \cdots u_n \neq 0$. Certainly, this improves the cup-length estimation. For example, this establishes a short calculation of $\text{cat } L_p^{2n+1}$, see [FH]. So, it seems reasonable to find (indecomposable) elements of high category weight and, more generally, to be able to compute category weight. Unfortunately, category weight is not a homotopy invariant, i.e., $\text{cwg}t h^*u$ is not necessarily equal to $\text{cwg}t u$ for a homotopy equivalence h . This makes category weight difficult for calculations.

Because of this, it makes sense to introduce a homotopy invariant version of category weight as the author did in a talk at the AMS Summer Research Institute, Seattle, July 1996 (see [R2]). This invariant is called *strict* category weight and denoted by swgt . A similar concept was also introduced by Strom [S2]. There are many ways to define $\text{swgt } u, u \in H^*(X)$. One of them is:

$$\text{swgt } u = \min\{\text{cwg}t h^*u\} \text{ where } h \text{ runs over all homotopy equivalences } Y \rightarrow X.$$

Clearly, $\text{cwg}t u \geq \text{swgt } u$, so we can use swgt to estimate $\text{cwg}t$. Furthermore, swgt has better multiplicative properties than $\text{cwg}t$, see §3. It turns out that swgt is quite manageable: for example, it is possible both to find many elements of high swgt (see §4) and to apply swgt to certain geometric problems (see §§6,7). Also, notice that strict category weight yields a decreasing filtration $\{F_n(X)\}$ of $H^*(X)$, $F_n(X) := \{u \in H^*(X) | \text{swgt } u \geq n\}$.

Here is another description of swgt : we have $X \simeq B\Omega X$, the classifying space for the loop space of X . Let $\{B_n\Omega X\}$ be the Milnor filtration of $B\Omega X$, and let $i_n : B_n\Omega X \subset B\Omega X \simeq X$. (Notice that i_n is homotopy equivalent to a map (1.5) described in §1.) Then

$$\text{swgt } u = \sup\{k | i_k^*(u) = 0\}.$$

In other words, the above filtration $\{F_n(X)\}$ is just the filtration

$$\text{Ker}\{i_n^* : H^*(X) \rightarrow H^*(B_n\Omega X)\}.$$

This also shows how to apply the Eilenberg–Moore spectral sequence to the study of the Lusternik–Schnirelmann category, cf. [To].

It is clear that category weight can be defined in more general situations. For example, we can consider an arbitrary (extraordinary) cohomology theory instead of $H^*(-)$. In this paper we consider even a little bit more general functors than cohomology.

Throughout this paper, we reserve the term “map” for continuous functions and the term “inessential map” for homotopy trivial maps (i.e., for maps which are homotopic to constant maps).

We use the sign \simeq for homotopy of maps, and we write $f \simeq^\bullet g$ when there is a pointed homotopy between pointed maps f and g .

The reduced cone of a pointed map f is denoted by Cf .

“Connected” always means “path connected”.

Given a pointed topological space X , the (reduced) cohomology group $H^k(X; \pi)$ is defined as $[X, K(\pi, n)]$ (pointed homotopy classes) where the Eilenberg–Mac Lane space $K(\pi, n)$ is assumed to be (homotopy equivalent to) a CW -space. Two exceptions to this agreement (in 3.10 and 4.2) are mentioned explicitly.

Given $u \in H^k(X)$, the notation $u|_{\pi_k(X)} = 0$ means that $\langle u, a \rangle = 0$ for every $a \in \text{Im}\{h : \pi_k(X) \rightarrow H_k(X)\}$ where h is the Hurewicz homomorphism.

The paper is organized as follows. In §1 we consider the Lusternik–Schnirelmann category. In §2 we give a definition and background properties of (strict) category weight. In §3 we demonstrate multiplicative properties of (strict) category weight. In §4 we explain how to find elements of high category weight. In §5 we show how to use swgt for control of the Lusternik–Schnirelmann category and in §6 we apply this technique to manifolds. In particular, we prove special cases of the Ganea conjecture and the degree conjecture. In §7 we apply our technique to the famous Arnold conjecture about symplectic fixed points. In fact, we indicate how to prove it for closed symplectic manifolds (M, ω) with $\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}$.

1. Lusternik–Schnirelmann category

1.1. DEFINITION. (a) ([Fox], [Fet], [BG]) Given a map $\varphi : A \rightarrow X$, we say that $\text{cat } \varphi \leq k$ if there is a numerable covering U_1, \dots, U_{k+1} of A such that $\varphi|_{U_i}$ is inessential for every i . Then $\text{cat } \varphi = k$ if k is minimal with this property. Also, we set $\text{cat } \varphi = -1$ if $A = \emptyset$.

(b) If $i : A \rightarrow X$ is an inclusion then we set $\text{cat}_X A := \inf\{\text{cat } i\}$ where i runs over all the inclusions $i : U \rightarrow X$ of neighborhoods U of A .

(c) ([LS]) We define the *Lusternik–Schnirelmann category* $\text{cat } X$ of a space X by setting $\text{cat } X := \text{cat } 1_X = \text{cat}_X X$.

1.2. LEMMA, *Let $f : X \rightarrow Y$ be a map with Y connected, and let $x_0 \in X$ be an arbitrary point. If $\text{cat } f = k$ then there is a numerable covering $\{B_1, \dots, B_{k+1}\}$ such that $f|_{B_i}$ is inessential and $x_0 \in B$ for every i .*

PROOF. Let $\{A_1, \dots, A_{k+1}\}$ be a numerable covering of X with $f|_{A_i}$ inessential, and let $\{\varphi_1, \dots, \varphi_{k+1}\}$ be a partition of unity dominated by $\{A_1, \dots, A_{k+1}\}$. We set

$$C_i := \varphi_i^{-1} [1/(2k+2), 1]$$

and $B_i := C_i \cup \{x_0\}$. It is easy to see that $\{B_i\}$ is the desired covering. ■

1.3. PROPOSITION ([BG]). (i) *For every diagram $A \xrightarrow{\varphi} Y \xrightarrow{f} X$ we have $\text{cat } f\varphi \leq \min\{\text{cat } \varphi, \text{cat } f\}$. In particular, $\text{cat } f \leq \min\{\text{cat } X, \text{cat } Y\}$.*

(ii) *If $\varphi \simeq \psi : A \rightarrow X$ then $\text{cat } \varphi = \text{cat } \psi$.*

(iii) *If $h : Y \rightarrow X$ is a homotopy equivalence then $\text{cat } \varphi = \text{cat } h\varphi$ for every $\varphi : A \rightarrow X$.* ■

Notice that, in view of 1.3(iii), $\text{cat } X$ is a homotopy invariant of X . Also, it is easy to see that $\text{cat } X \leq \dim X$ for every connected CW -space X .

1.4. PROPOSITION ([B], [Fox]). *For any two connected CW-spaces X, Y we have*

$$\max\{\text{cat } X, \text{cat } Y\} \leq \text{cat}(X \times Y) \leq \text{cat } X + \text{cat } Y. \blacksquare$$

Let X be a connected space. Take a point $x_0 \in X$, set

$$PX = P(X, x_0) = \{\omega \in X^I \mid \omega(0) = x_0\}$$

and consider the fibration $p : PX \rightarrow X$, $p(\omega) = \omega(1)$ with the fiber ΩX . Given a natural number k , we use the short notation

$$(1.5) \quad p_k : P_k(X) \rightarrow X$$

for the map

$$\underbrace{p_X *_{X} \cdots *_{X} p_X}_{k \text{ times}} : \underbrace{PX *_{X} \cdots *_{X} PX}_{k \text{ times}} \longrightarrow X$$

where $*_X$ denotes the fiberwise join over X , see e.g. [J]. In particular, $P_1(X) = PX$.

1.6. THEOREM ([Sv, Theorems 3 and 19']). *Let $\varphi : A \rightarrow X$ be a map with X connected. Then $\text{cat } \varphi < k$ iff there is a map $\psi : A \rightarrow P_k(X)$ such that $p_k \psi = \varphi$. \blacksquare*

For future references we fix the following simple information, the proofs can be found e.g. in [R2].

- 1.7. PROPOSITION. (i) $P_k(X)$ is a fibration over X ;
(ii) If X has the homotopy type of a CW-space then $P_{-k}(X)$ does;
(iii) $\text{cat } P_k(X) \leq k - 1$.

1.8. PROPOSITION ([Sv, Th. 21]). *The map $p_2 : P_2(X) \rightarrow X$ is homotopy equivalent over X to the map $\varepsilon : S\Omega X \rightarrow X$. \blacksquare*

The concept of Lusternik–Schnirelmann category can be generalized as follows. Let \mathcal{T} be the category of pointed spaces and pointed maps. Let \mathcal{E} be the category whose objects are pairs (set, subset) and whose morphisms $(U, V) \rightarrow (U', V')$ are functions $f : U \rightarrow U'$ with $f(V) \subset V'$. Given a contravariant functor $F : \mathcal{T} \rightarrow \mathcal{E}$ with $F(X) = (U, V)$ for some $X \in \mathcal{T}$, the notation $u \in F(X)$ means that $u \in U$. We say that $u \in F(X) = (U, V)$ is *trivial* if $u \in V$. Given $u \in F(Y)$ and $f : X \rightarrow Y$, we write f^*u for $F(f)(u)$. Furthermore, given $A \subset X$, we denote i^*u by $u|_A$.

1.9. DEFINITION. Let $F : \mathcal{T} \rightarrow \mathcal{E}$ be a contravariant functor satisfying the following properties:

- (1) (weak homotopy property) If $f \simeq^\bullet g : X \rightarrow Y$ then f^*u is trivial if g^*u is.
- (2) (triviality property) $F(\text{pt})$ has the form (U, U) , i.e., every $u \in F(\text{pt})$ is trivial.

Let X be a connected pointed space. Given $u \in F(X)$, we define the *Lusternik–Schnirelmann category of u* , $\text{cat } u$, to be the minimal k with the following property: there is a numerable covering $\{A_1, \dots, A_{k+1}\}$ where each A_i is a pointed subspace of X and $u|_{A_i}$ is trivial for every $i = 1, \dots, k$.

1.10. EXAMPLES. (a) Given a pointed topological space Y , set $F(X) = T_Y(X) := (U, V)$ where U is the set of all pointed maps $X \rightarrow Y$ and V is the subset of all inessential

(under non-pointed homotopy) maps. (The action of F on morphisms is given by the composition.) Then, the definition above leads to the invariant $\text{cat } f$ defined in 1.1.

(b) Every pointed set (U, u_0) can be considered as the pair $(U, \{u_0\})$ of sets. Thus, every functor from \mathcal{T} to the category of pointed sets can be regarded as a functor $\mathcal{T} \rightarrow \mathcal{E}$.

(c) Every abelian group can be regarded as a pointed set (with base point 0). So, every functor from \mathcal{T} to the category of abelian groups can be treated as a functor $\mathcal{T} \rightarrow \mathcal{E}$.

(d) (Fary [F]) Because of (c), a reduced cohomology theory (not necessarily additive) $E^*(-)$ on \mathcal{T} yields the functor $X \mapsto F(X) := (E^i(X), \{0\})$. Then, given $u \in E^i(X)$, we have the invariant $\text{cat } u$, the Lusternik–Schnirelmann category of the class u .

(e) Given $X \in \mathcal{T}$, set $F(X) = (U, V)$ where U is the set of fiberwise homotopy equivalence classes of fibrations over X and V consists of fibrations which admit a section. The action of F on morphisms is given by passing to induced fibrations. If $u \in F(X)$ is the equivalence class of a fibration ξ , then $\text{cat } u$ is the *genus* of ξ , [Sv]; another name is the *sectional category*, $\text{secat } \xi$, [J].

Notice that the example (e) generalizes example (a) since $\text{cat } f = \text{secat } f^*\xi$ where $\xi = \{p_1 : PY \rightarrow Y\}$ for $f : X \rightarrow Y$.

1.11. PROPOSITION. *Let X be a connected pointed space.*

- (i) $\text{cat } f^*u \leq \min\{\text{cat } f, \text{cat } u\}$ for every $f : X \rightarrow Y$ in \mathcal{T} and every $u \in F(Y)$.
- (ii) If $f \simeq^{\bullet} g : X \rightarrow Y$ then $\text{cat } f^*u = \text{cat } g^*u$ for every $u \in F(Y)$.
- (iii) If $h : X \rightarrow Y$ is a pointed homotopy equivalence then $\text{cat } h^*u = \text{cat } u$ for every $u \in F(Y)$. ■

2. Category weight

2.1. DEFINITION. Let $F : \mathcal{T} \rightarrow \mathcal{E}$ be a functor as in 1.9, and let $u \in F(X)$. We do not require X to be connected.

(a) We define the *category weight* of u (denoted by $\text{cwgt } u$) by setting

$$\text{cwgt } u = \sup\{k \mid u|_A \text{ is trivial whenever } \text{cat}_X A < k\}$$

where A runs over all pointed closed subsets of X .

(b) We define the *strict category weight* of u (denoted by $\text{swgt } u$) by setting

$$\text{swgt } u = \sup\{k \mid \varphi^*u \text{ is trivial for every map } \varphi : A \rightarrow X \text{ in } \mathcal{T} \text{ with } \text{cat } \varphi < k\}.$$

Notice that $\text{cwgt } u = \infty = \text{swgt } u$ for every trivial element u .

In particular, the (strict) category weight of a map (cf. 1.10(a)) and a cohomology class (cf. 1.10(d)) is defined. Category weight was defined by Fadell and Husseini [FH] (for $F = H^*(-)$). Strict category weight was defined by Rudyak for $F = E^*(-)$ as in 1.10(d) and Strom for F as in 1.10(a), cf. [R2], [S2]; Strom calls it *essential category weight*. I must also note that Strom prefers to say that the (strict) category weight of a trivial element is not defined.

2.2. THEOREM (cf. [R2]). (i) $\text{swgt } u \leq \text{cwgt } u$, and $\text{cwgt } u \leq \text{cat } X$ provided u is not a trivial element. Furthermore, for every map $f : Y \rightarrow X$ in \mathcal{T} we have $\text{cat } f \geq \text{swgt } u$ provided f^*u is non-trivial. Finally, $\text{swgt } u \geq 1$ for every u .

(ii) For every inclusion $i : A \rightarrow X$ in \mathcal{T} we have $\text{cwgt } i^*u \geq \text{cwgt } u$.

- (iii) For every map $f : Y \rightarrow X$ in \mathcal{T} we have $\text{swgt } f^*u \geq \text{swgt } u$.
- (iv) If $f \simeq^\bullet g : X \rightarrow Y$ then $\text{cwgt } f^*u = \text{cwgt } g^*u$ and $\text{swgt } f^*u = \text{swgt } g^*u$.
- (v) $\text{swgt } u = \text{swgt } h^*u$ for every pointed homotopy equivalence h . ■

Notice that properties (iii) and (v) of 2.2 are not valid for cwgt even if $f = H^*(-)$, see [R2], [S2]. In other words, category weight is not a homotopy invariant, and this is the main motivation for introducing strict category weight. The following proposition gives us another description of strict category weight.

2.3. PROPOSITION. *Let X be a connected pointed space. Then for every $u \in F(X)$ we have:*

- (i) $\text{swgt } u = \sup\{k \mid p_k^*(u) \text{ is trivial}\}$;
- (ii) If $\text{swgt } u = k$ then $\text{cwgt } p_{k+1}^*u = k$;
- (iii) $\text{swgt } u = \min_{f \in \mathcal{F}}\{\text{cwgt } f^*u\}$ where \mathcal{F} is the class of all maps $f : Y \rightarrow X$ in \mathcal{T} .

Proof. (i) See [R2, 1.8(v)].

(ii) Since $\text{swgt } u = k$, we conclude that, by (i), p_{k+1}^*u is non-trivial. Now

$$k = \text{swgt } u \leq \text{swgt } p_{k+1}^*u \leq \text{cwgt } p_{k+1}^*u \leq \text{cat } P_{k+1}X \leq k.$$

(iii) Clearly, $\text{swgt } u \leq \text{swgt } f^*u \leq \text{cwgt } f^*u$. Now, if $\text{swgt } u = k$ then, by (ii) $\text{swgt } u = \text{cwgt } p_{k+1}^*u$. ■

2.4. THEOREM. *Let X be a connected pointed space.*

(i) For every $f : X \rightarrow Y$ in \mathcal{T} and every $u \in F(Y)$ we have

$$\text{cat } f \geq (\text{swgt } u) \text{cat}(f^*u).$$

In particular, for every diagram $X \xrightarrow{f} Y \xrightarrow{g} Z$ we have $\text{cat } f \geq (\text{swgt } g) \text{cat}(gf)$.

(ii) For every $u \in F(X)$ we have

$$\text{cat}_X A \geq (\text{cwgt } u) \text{cat}(u|A).$$

In particular, for every map $f : X \rightarrow Y$ we have $\text{cat}_X A \geq (\text{cwgt } f) \text{cat}(f|A)$.

Formula (ii) was found by Strom [S2], formula (i) is an obvious analog of (ii). This theorem improves properties 1.11(i) and 2.2(i). Notice that the proof does not use properties (1) and (2) of 1.9.

Proof. We prove only (i). Let $\text{cat } f = n$, $\text{swgt } u = k > 0$. We must prove that $\text{cat}(f^*u) \leq p := [n/k]$. Let $\{A_1, \dots, A_{n+1}\}$ be a numerable covering of X such that $f|A_i$ is inessential for every i . Without loss of generality we can assume Y to be connected, and so, by 1.2, we can assume that each A_i contains the base point. We set

$$B_1 := A_1 \cup \dots \cup A_k, \quad B_2 = A_{k+1} \cup \dots \cup A_{2k}, \dots, \\ B_p = A_{(p-1)k+1} \cup \dots \cup A_{pk}, \quad B_{p+1} = A_{pk+1} \cup \dots \cup A_{n+1}.$$

Since $\text{cat}(f|B_j) < k$, we conclude that, by 2.2(i), $(f|B_j)^*u$ is trivial. So, $(f^*u)|B_j$ is trivial for every $j = 1, \dots, p+1$. Thus, $\text{cat } f^*u \leq p$. ■

2.5. COROLLARY ([S2]). *For every $f : X \rightarrow Y$ in \mathcal{T} with X connected and every $u \in F(Y)$ we have*

$$\text{cwtg } f^*u \geq (\text{cwtg } f) \text{ swgt } u, \text{ swgt } f^*u \geq (\text{swgt } f) \text{ swgt } u. \blacksquare$$

2.6. PROPOSITION ([R2], [S2]). *Let X be a CW-space, and let $f : X \rightarrow Y$, $X \in \mathcal{T}$ be an essential map such that $\pi_i(Y) = 0$ for $i > m$. Then $\text{cwtg } f \leq m$. In particular, if E is a spectrum with $\pi_i(E) = 0$ for $i > m$ then $\text{cwtg } u \leq q + m$ for every $u \in E^q(X)$, $u \neq 0$. \blacksquare*

3. Multiplicative properties. Let F be a functor as in 1.9. Given a Puppe sequence $X \xrightarrow{f} Y \rightarrow Cf$ in \mathcal{T} , we conclude, by 1.9(1), that the image of the composition

$$F(Cf) \rightarrow F(Y) \rightarrow F(X)$$

consists of trivial elements.

3.1. DEFINITION. A functor F as 1.9 is called *half-exact* if, for every pointed polyhedral pair (X, A) , the sequence

$$F(X \cup CA) \xrightarrow{j^*} F(X) \xrightarrow{i^*} F(A)$$

is “exact”, i.e., i^*u is trivial iff $u = j^*v$ for some $v \in F(X \cup CA)$. Here $i : A \rightarrow X$ and $j : X \rightarrow X \cup CA$ are the inclusions. Given $(U_1, V_1), \dots, (U_n, V_n) \in \mathcal{E}$, we set

$$(U_1, V_1) \times \cdots \times (U_n, V_n) := (U_1 \times \cdots \times U_n, V_1 \times \cdots \times V_n).$$

Let E, F be two half-exact functors, and let G be an arbitrary functor as in 1.9. Suppose that, for every two polyhedra X, Y (not necessarily finite), there is a natural transformation (where \times is as described in (3.2))

$$(3.3) \quad m : E(X) \times F(Y) \rightarrow G(X \wedge Y).$$

This transformation yields a transformation

$$\tau : E(X) \times F(Y) \xrightarrow{m} G(X \wedge Y) \xrightarrow{\lambda} G(X \times Y),$$

where λ shrinks the wedge $X \vee Y$. We set $u \boxtimes v := \tau(u, v)$ for $u \in E(X), v \in F(Y)$. If $X = Y$ and $\Delta : X \rightarrow X \times X$ is the diagonal, we set $u \bullet v := \Delta^*(u \boxtimes v) \in G(X)$.

3.4. THEOREM (cf. [FH], [R2]). (i) *For every pointed connected polyhedron X and every $u \in E(X), v \in F(X)$ we have*

$$\text{cwtg}(u \bullet v) \geq \text{cwtg } u + \text{cwtg } v.$$

(ii) *For every pointed connected CW-space X and every $u \in E(X), v \in F(X)$ we have*

$$\text{swgt}(u \bullet v) \geq \text{swgt } u + \text{swgt } v.$$

Proof. (i) First, assume that $\text{cwtg } u = k < \infty, \text{cwtg } v = l < \infty$. Take a pointed closed subspace A of X with $\text{cat}_X A < k + l$. Then, clearly, $A \subset \bigcup_{i=1}^{k+l} V_i$ where each V_i is open and contractible in X . Since A is closed, there is a subdivision of X with the following property: every simplex e with $e \cap A \neq \emptyset$ is contained in some V_i , cf. [W, Theorem 35]. We let $A_i, i = 1, \dots, r_n$ be the union of all simplexes contained in V_i .

Then $A \subset A_1 \cup \dots \cup A_{k+l}$ where each subpolyhedron A_i is contractible in X . We set $K := A_1 \cup \dots \cup A_k \cup \{*\}$, $L := A_{k+1} \cup \dots \cup A_{k+l} \cup \{*\}$ and let

$$i_K : K \subset X, i_L : L \subset X, j_K : X \subset X \cup CK, j_L : X \subset X \cup CL, h : X \subset X \cup C(K \cup L)$$

be the inclusions. Consider the commutative diagram

$$\begin{array}{ccc} E(X \cup CK) \times F(X \cup CL) & \xrightarrow{\Delta^* m} & G(X \cup C(K \cup L)) \\ \downarrow j_K^* \times j_L^* & & \downarrow h^* \\ E(X) \times F(X) & \xrightarrow{\Delta^* \tau} & G(X), \end{array}$$

where, for the top row, we have used the canonical homotopy equivalence

$$(X \cup CX') \wedge (Y \cup CY') \simeq X \times Y \cup C((X \times Y') \cup (X' \times Y))$$

which holds for any polyhedral pairs (X, X') and (Y, Y') . Since both $i_K^* u$ and $i_L^* v$ are trivial, $u = j_K^* u', v = j_L^* v'$ for some $u' \in E(X \cup CK), v' \in F(X \cup CL)$. Clearly, $u \bullet v = h(u' \bullet v')$. Thus, $(u \bullet v)|A$ is trivial since $A \subset K \cup L$.

If, say, $\text{cwg}t u = \infty$ then we must prove that $u \bullet v|A$ is trivial if $\text{cat}_X A < \infty$. Arguing as above, we conclude that there is a pointed subpolyhedron $K \subset X$ with $A \subset K$ and $\text{cat}_X K < \infty$. Then $u|K$ is trivial, and hence $(u \bullet v)|K$ is (take $L = *$ in the above diagram).

(ii) Let $\text{swg}t(u \bullet v) = k$. Then, by 2.3(ii), $\text{cwg}t p_{k+1}^*(u \bullet v) = k$. Recall that every CW -space is homotopy equivalent to a polyhedron, and so, by 1.7(ii), there is a homotopy equivalence $\varepsilon : Y \rightarrow P_{k+1}(X)$ such that Y is a polyhedron. We put $f = p_{k+1} \varepsilon : Y \rightarrow X$. Then $f^*(u \bullet v) \neq 0$, and

$$k = \text{swg}t(u \bullet v) \leq \text{cwg}t f^*(u \bullet v) \leq \text{cat } Y \leq k.$$

Now

$$\begin{aligned} \text{swg}t(u \bullet v) &= \text{cwg}t f^*(u \bullet v) \geq \text{cwg}t f^* u + \text{cwg}t f^* v \geq \text{swg}t f^* u + \text{swg}t f^* v \\ &\geq \text{swg}t u + \text{swg}t v. \quad \blacksquare \end{aligned}$$

3.5. THEOREM ([R2, 1.14]). $\text{swg}t(u \boxtimes v) \geq \text{swg}t u + \text{swg}t v$. \blacksquare

Notice that the corresponding inequality for $\text{cwg}t$ is wrong, cf. 2.2(iii,v).

Certainly, we can consider not only two functors E, F but any finite number of functors equipped with a natural transformation like (3.3).

3.6. EXAMPLES. (a) Given n arbitrary spectra $E(1), \dots, E(n)$ and n pointed CW -spaces X_1, \dots, X_n , we have the homomorphism

$$\nu : E(1)^*(X_1) \otimes \dots \otimes E(n)^*(X_n) \rightarrow (E(1) \wedge \dots \wedge E(n))^*(X_1 \times \dots \times X_n),$$

see [Sw]. Furthermore, consider the correspondence (not a homomorphism!)

$$\begin{aligned} r : E(1)^*(X_1) \times \dots \times E(n)^*(X_n) &\rightarrow E(1)^*(X_1) \otimes \dots \otimes E(n)^*(X_n), \\ r(u_1, \dots, u_n) &= u_1 \otimes \dots \otimes u_n. \end{aligned}$$

So, we have a natural transformation

$$\tau := \nu r : E(1)^*(X_1) \times \dots \times E(n)^*(X_n) \rightarrow (E(1) \wedge \dots \wedge E(n))^*(X_1 \times \dots \times X_n).$$

Now, by 3.5

$$\text{swgt}(u_1 \boxtimes \cdots \boxtimes u_n) \geq \sum \text{swgt } u_i.$$

for every $u_i \in E(i)^*(X_i)$. Furthermore, given $u_i \in E(i)^*(X)$, we have the element $u_1 \bullet \cdots \bullet u_n \in (E(1) \wedge \cdots \wedge E(n))^*(X)$, and

$$\text{cwggt}(u_1 \bullet \cdots \bullet u_n) \geq \sum \text{cwggt } u_i, \quad \text{swgt}(u_1 \bullet \cdots \bullet u_n) \geq \sum \text{swgt } u_i$$

(in the first inequality X is assumed to be a polyhedron).

(b) Given a ring spectrum E with a multiplication $u : E \wedge E \rightarrow E$, for every n we have the iterated multiplication

$$u_n : \underbrace{E \wedge \cdots \wedge E}_{n \text{ times}} \rightarrow E$$

which yields a function

$$\tau : E^*(X_1) \times \cdots \times E^*(X_n) \rightarrow (E \wedge \cdots \wedge E)^*(X_1 \times \cdots \times X_n) \xrightarrow{(u_n)^*} E^*(X_1 \times \cdots \times X_n)$$

where the first arrow is τ from (a). In this case the element $u_1 \bullet \cdots \bullet u_n \in E^*(X)$ is usually denoted by $u_1 \cdots u_n$, and we have

$$(3.7) \quad \text{cat } X \geq \text{cwggt}(u_1 \cdots u_n) \geq \sum \text{cwggt } u_i \geq n$$

if $u_1 \cdots u_n \neq 0$. Certainly, this refines the cup-length estimation $\text{cat } X \geq n$.

3.8. EXAMPLE ([RO]). A closed connected symplectic manifold is a pair (M^{2n}, ω) where M is a connected closed smooth $2n$ -dimensional manifold and ω is a closed non-degenerate 2-form. Notice that in this case ω^n is a volume form for M . In particular, ω yields a non-trivial de Rham cohomology class which we denote also by ω . Since $\omega^n \neq 0$, we conclude that $\text{cat } M \geq n$. Rudyak–Oprea [RO] proved that $\text{swgt } \omega = 2$ provided $\omega|_{\pi_2(M)} = 0$. Thus, if (M^{2n}, ω) is a symplectic manifold with $\omega|_{\pi_2(M)} = 0$ then, by (3.7),

$$\text{cat } M \geq \text{swgt } \omega^n \geq n \text{swgt } \omega = 2n,$$

and hence $\text{cat } M = 2n$ (because $\text{cat } M \leq \dim M$).

3.9. EXAMPLES. (a) ([S2]) Given two pointed spaces A, B , set $E(-) = T_A(-)$, $F(-) = T_B(-)$ and $G(-) = T_{A \wedge B}(-)$ (see 1.10(a)). We define $m : E(X) \times F(Y) \rightarrow G(X \wedge Y)$ by setting $m(f, g) = f \wedge g$, and 3.4 is applicable to this case.

(b) According to 1.10(e), the category weight $\text{cwggt } \xi$ of a fibration ξ is defined. It turns out that, for any fibrations ξ, η over the same base X ,

$$\text{cwggt}(\xi *_X \eta) \geq \text{cwggt } \xi + \text{cwggt } \eta.$$

This can be deduced from 3.4, but we indicate a direct proof. Let A, K, L be as in the proof of 3.4, and let s_1 (resp. s_2) be a section of ξ over K (resp of η over L). Take a function $\varphi : A \rightarrow [0, 1]$ such that $\varphi^{-1}(0, 1] \in K$ and $\varphi^{-1}[0, 1) \in L$. We set

$$s(x) := \langle s_1(x), \varphi(x), s_2(x) \rangle \in \{\text{the total space of } \xi *_X \eta\}, \quad x \in A.$$

Then s is a desired section over A . Now, we can formulate (and, probably, exploit) the obvious analog of (3.7).

The following result shows how *Massey products* help to estimate $\text{cat } X$. Let $H^*(-; R)$ denote singular cohomology with coefficients in a commutative ring R . Given a matrix V over $H^*(X; R)$, we set $\text{cwg} V = \min\{\text{cwg} v\}$ where v runs over all entries of V .

3.10. THEOREM ([R2]). *Given $X \in \mathcal{T}$, let V_1, \dots, V_n be matrices over $H^*(X; R)$. Suppose that the matrix Massey product $\langle V_1, \dots, V_n \rangle$ is defined. If $0 \notin \langle V_1, \dots, V_n \rangle$ then*

$$\text{cat } X \geq \min_i \{\text{cwg} V_{2i}\} + \min_i \{\text{cwg} V_{2i+1}\}. \blacksquare$$

My feeling is that this result is somehow related to (3.7), but I can't say how explicitly.

4. Elements of high category weight. The results of the previous section show that it makes sense to search for (indecomposable) elements of high category weight. The first example of this kind was found by Fadell–Husseini [FH]. Namely, they proved that, for every odd prime p and connected X ,

$$(4.1) \quad \text{cwg} \beta P^n u \geq 2 \text{ if } u \in H^{2n+1}(X; \mathbb{Z}/p).$$

(Actually, they proved that $\text{swgt} \beta P^n u \geq 2$.)

4.2. THEOREM. *Let $X \in \mathcal{T}$, and let V_1, \dots, V_n be matrices over $H^*(X; R)$ (singular cohomology with coefficients in a commutative ring). Suppose that the matrix Massey product $\langle V_1, \dots, V_n \rangle$ is defined. Then, for every $V \in \langle V_1, \dots, V_n \rangle$ and every entry u of V , we have $\text{cwg} u \geq 2$. \blacksquare*

This theorem was explicitly formulated in [R2]. Actually, it follows from 1.6, 1.8, and the result of Gugenheim–May [GM] that $\text{Ker}\{\varepsilon^* : H^*(X) \rightarrow H^*(S\Omega X)\}$ contains all the matrix Massey products. Note that 4.2 implies (4.1) since $\beta P^n u \in \langle u, \dots, u \rangle$ (p times) for every $u \in H^{2n+1}(X; \mathbb{Z}/p)$, Kraines [K].

Let E, F be two spectra, and let $\theta : E \rightarrow F$ be a (stable) cohomology operation. Without loss of generality we can assume that $E = \{E_n\}$ and $F = \{F_n\}$ are Ω -spectra, i.e., that there are weak homotopy equivalences $\sigma_n : E_n \rightarrow \Omega E_{n+1}$, etc. Then θ yields a family $\theta_n : E_n \rightarrow F_n$ of maps such that the diagram

$$\begin{array}{ccc} E_n & \xrightarrow{\theta_n} & F_n \\ \sigma_n^E \downarrow & & \downarrow \sigma_n^F \\ \Omega E_{n+1} & \xrightarrow{\Omega \theta_{n+1}} & \Omega F_{n+1} \end{array}$$

commutes up to homotopy. Following Strom [S2], set

$$d = d(\theta) = \inf\{k \mid \theta_k \text{ is an essential map}\}$$

4.3. THEOREM ([S2]). *$\text{swgt} \theta_n = 1$ for $n > d$, and $\text{swgt} \theta_d \geq 2$. \blacksquare*

Notice that $d(\theta) = 2n + 1$ for $\theta = \beta P^n$. This gives us another proof of (4.1). Indeed, if $u \in H^{2n+1}(X; \mathbb{Z}/p)$ then

$$\text{swgt} \beta P^n(u) = \text{swgt} \theta_{2n+1}(u) \geq \text{swgt}(\theta_{2n+1}) \text{swgt } u \geq 2 \text{swgt } u \geq 2.$$

4.4. THEOREM ([S3]). *Let G be a discrete group, and let E be a spectrum such that $\pi_i(E) = 0$ for $i < m$. Then $\text{swgt } u \geq k + m$ for every $u \in E^k(BG)$. \blacksquare*

For $k = 2$ this result goes back to Fadell–Husseini [FH]. Notice that, by 2.6, $\text{swgt } u = k$ for every $u \in H^k(BG), u \neq 0$. As an application of 4.4, we note the following corollary.

4.5. COROLLARY. *Let M be a smooth manifold such that the structure group of its tangent bundle reduces to a discrete group G . If $w_I(M) \neq 0$ then $\text{cat } M \geq |I|$. Here $I = \{i_1, \dots, i_k\}$, w_I is the Stiefel–Whitney class $w_{i_1} \cdots w_{i_k}$ and $|I| = \sum i_k$.*

PROOF. It suffices to prove that $\text{swgt } w_I \geq |I|$. But this follows from 4.3 and 2.2(iii), since w_I is induced from BG . ■

The last example of elements of high category weight is as follows.

4.6. THEOREM. *Let X be a simply connected rational space, and let (Λ, d) be the minimal Sullivan model for X . If a cohomology class $x \in H^*(X; \mathbb{Q}) = H^*(\Lambda, d)$ has the form $x = [a_1 \cdots a_k]$ for some $a_i \in \Lambda, \dim a_i \geq 0$, then $\text{swgt } x \geq k$.*

Notice that the a_i are not assumed to be cocycles and, hence, we can't apply (3.6).

PROOF. This follows from the result of Felix–Halperin [FeH, §3] that $p_k^* : H^*(X; \mathbb{Q}) \rightarrow H^*(P_k(X); \mathbb{Q})$ annihilates all elements of the form $[a_1 \cdots a_k]$. ■

5. Detecting elements

5.1. DEFINITION. Let F be as 1.9. An element $u \in F^*(X)$ is called a *detecting element* for X if $\text{swgt } u = \text{cat } X$.

We formulate the above definition for the general situation, but really we will apply it to the case when F is a cohomology theory, as in 1.9(d).

5.2. THEOREM ([R2]). *Let X, Y be two connected pointed CW-spaces and let E, G be two spectra. Suppose that there are detecting elements $u \in E^*(X), v \in G^*(Y)$. If $0 \neq u \boxtimes v \in (E \wedge G)^*(X \times Y)$ then $\text{cat}(X \times Y) = \text{cat } X + \text{cat } Y$, and $u \boxtimes v$ is a detecting element for $X \times Y$.*

PROOF. By 2.2(i), 3.5 and 1.4,

$$\text{cat}(X \times Y) \geq \text{swgt}(u \boxtimes v) \geq \text{swgt } u + \text{swgt } v = \text{cat } X + \text{cat } Y \geq \text{cat}(X \times Y). \quad \blacksquare$$

5.3. COROLLARY. *If a pointed CW-space X possesses a detecting element then $\text{cat}(X \times S^n) = \text{cat } X + 1$ for every $n \geq 0$. In other words, the Ganea conjecture holds for X .*

PROOF. We take a detecting element $u \in E^*(X)$ and apply 5.2 to the case $Y = S^n, G^n(Y) = \Pi^n(S^n)$, and v is given by the identity map $S^n \rightarrow S^n$. One can prove that $u \boxtimes v \neq 0$ (see e.g. [R2]), and the result follows. ■

For example, Strom [S1] proved that a $(q - 1)$ -connected CW-space X possesses a detecting element if $\text{cat } X = [\dim X/q]$, and so such an X satisfies the Ganea conjecture.

5.4. THEOREM ([R2]). *Let R be a ring spectrum, and let E be an arbitrary R -module spectrum. Let M^n, N^n be two closed connected $H\mathbb{Z}$ -orientable PL manifolds, and let $f : N \rightarrow M$ be a map of degree ± 1 and such that N is R -orientable. If M possesses a detecting element $u \in E^*(M)$ then $\text{cat } f = \text{cat } M$. In particular, $\text{cat } N \geq \text{cat } M$.*

PROOF. It is easy to see that $f^* : E^*(M) \rightarrow E^*(N)$ is monic. So, $f^*u \neq 0$, and hence, by 2.2(i), $\text{cat } f \geq \text{swgt } u = \text{cat } M$. Hence, by 1.3, $\text{cat } f = \text{cat } M$. ■

The results above show that it is useful and important to know whether a space possesses detecting elements. Consider the Puppe sequence

$$P_m(X) \xrightarrow{p_m} X \xrightarrow{j_m} C_m(X) := C(p_m)$$

where $p_m : P_m(X) \rightarrow X$ is the fibration (1.5) and $C(p_m)$ is the cone of p_m .

5.5. THEOREM ([R2]). *Let X be a pointed connected CW-space with $\text{cat } X = k < \infty$. If j_k is stably essential (i.e., the stable homotopy class of j_k is non-zero) then X possesses a detecting element. ■*

In fact, the stable homotopy class of j_m can be treated as a universal (among cohomology functors) detecting element. Also, we note that, for every X with $\text{cat } X = k$, the fibration $p_k : P_k(X) \rightarrow X$ is a detecting element (see 1.10(e)). However, unfortunately, it is difficult to apply analogs of 5.2–5.4 to this case.

6. Manifolds. Given a PL manifold M , we denote by ν_M the stable normal bundle of M .

6.1. THEOREM ([R2]). *Let M^n , $n = \dim M \geq 4$, be a closed $(q-1)$ -connected PL manifold, $q \geq 1$. Suppose that there is a natural number m such that $\nu_M|_{M^{(m)}}$ is trivial and*

$$n \leq \min\{2q \text{ cat } M - 4, m + q \text{ cat } M - 1\}.$$

Then M possesses a detecting element. ■

Putting $m = n + 1$ and $m = 1$, we get the following corollary.

6.2. COROLLARY. (i) *Let M be a closed $(q-1)$ -connected stably parallelizable PL manifold, $q \geq 1$. Suppose that $4 \leq \dim M \leq 2(q \text{ cat } M - 2)$. Then M possesses a detecting element.*

(ii) *Let $q \geq 1$, and let M^n , $n = \dim M \geq 4$ be a closed orientable $(q-1)$ -connected PL manifold such that $q \text{ cat } M = n$. Then M possesses a detecting element. Moreover, there exists a detecting element $u \in H^n(M; \pi_n(C_n(M)))$. ■*

6.3. COROLLARY. *Let M be as in 6.1. Then $\text{cat}(M \times S^{m_1} \times \cdots \times S^{m_n}) = \text{cat } M + n$ for any natural numbers m_1, \dots, m_n . ■*

Other results about the Ganea conjecture for manifolds can be found in [R1], [R2], [S1]. Based on 6.2 and 5.4, we get the following theorem.

6.4. THEOREM. (i) *Let M^n be as in 6.2(i), and let $f : N^n \rightarrow M^n$ be a map of degree ± 1 where N is a stably parallelizable PL manifold. Then $\text{cat } f = \text{cat } M$. In particular, $\text{cat } N \geq \text{cat } M$.*

(ii) *Let $f : N \rightarrow M$ be a map of degree ± 1 of closed H \mathbb{Z} -orientable PL manifolds. If $\text{cat } M = \dim M$ then $\text{cat } f = \text{cat } M = \text{cat } N$. ■*

6.5. COROLLARY. *Let M be an oriented PL manifold with $\text{cat } M = \dim M$, and let $f : X \rightarrow M$ be a map of an arbitrary topological space such that*

$$f^* : H^n(M; \pi_n(C_n(M))) \rightarrow H^n(M; \pi_n(C_n(M)))$$

is a monomorphism. Then $\text{cat } f = \text{cat } M$.

Proof. By 6.2, there is a detecting element $u \in H^n(M; \pi_n(C_n(M)))$, and $f^*u \neq 0$. Thus,

$$\text{cat } f \geq \text{swgt } u = \text{cat } M.$$

But, by 1.3(i), $\text{cat } f \leq \text{cat } M$. ■

7. Applications to the Arnold conjecture. In [A, Appendix 9] Arnold proposed a beautiful conjecture concerning the relation between the number of fixed points of certain self-diffeomorphisms of a closed symplectic manifold (M, ω) and the minimum number of critical points of any smooth ($= C^\infty$) function on M .

Let (M^{2n}, ω) be a closed symplectic manifold. A symplectomorphism $\phi : M \rightarrow M$ (i.e., a diffeomorphism with $\phi^*\omega = \omega$) is called *Hamiltonian* (or *exact*) if it belongs to the flow of a time-dependent Hamiltonian vector field on M . See [HZ] or [MS] for details. We define $\text{Arn}(M, \omega)$ to be the minimum number of fixed points for any Hamiltonian symplectomorphism of M . The Arnold conjecture claims that the following inequality holds for every closed symplectic manifold (M, ω) :

$$\text{Arn}(M, \omega) \geq \text{Crit } M.$$

The conjecture, usually (but not universally) weakened by replacing $\text{Crit } M$ by the cup-length of M , has been proved under various hypotheses for various classes of manifolds ([CZ], [H], [F11], [F12]). Here the following theorem (formulated explicitly in [R2] and based on Floer's approach) plays the crucial role.

7.1. THEOREM. *Let (M, ω) be a closed connected symplectic manifold such that $\omega|_{\pi_2(M)} = 0 = c_1(M)|_{\pi_2(M)}$. Then there exists a map $f : X \rightarrow M$ with the following properties:*

- (i) X is a compact metric space;
- (ii) $1 + \text{cat } f \leq \text{Arn}(M, \omega)$;
- (iii) The homomorphism $f^* : H^n(M; G) \rightarrow H^n(X; G)$ is a monomorphism for every coefficient group G . ■

The following theorem is proved in [R3] and [RO].

7.2. THEOREM. *Let (M^{2n}, ω) be a closed connected symplectic manifold with*

$$\omega|_{\pi_2(M)} = 0 = c_1|_{\pi_2(M)}.$$

Then $\text{Arn}(M, \omega) \geq \text{Crit } M$, i.e., the Arnold conjecture holds for M .

Proof. First, note that, by 3.8, $\text{cat } M = 2n = \dim M$. Hence,

$$1 + \text{cat } M \leq \text{Crit } M \leq 1 + \dim M = 1 + \text{cat } M$$

(the last inequality is a theorem of Takens [T]). So, $1 + \text{cat } M = \text{Crit } M$. Thus, in view of 7.1, it remains to prove that $\text{cat } f \geq \text{cat } M$ where f is the map from 7.1. We give two proofs of this inequality.

First proof. This follows from 6.5.

Second proof. Rudyak–Oprea [RO] proved that $\text{swgt } \omega = 2$. Thus, since $f^*\omega^n \neq 0$,

$$\text{cat } f \geq \text{swgt } \omega^n \geq 2n = \text{cat } M. \quad \blacksquare$$

References

- [A] V. I. Arnold, *Mathematical Methods in Classical Mechanics*, Springer, Berlin 1989.
- [B] A. Bassi, *Su alcuni nuovi invarianti delle varietà topologiche*, Annali di Math. 16 (1935), 275–297.
- [BG] I. Bernstein and T. Ganea, *The category of a map and of a cohomology class*, Fund. Math. 50 (1961/2), 265–279.
- [CZ] C. Conley and E. Zehnder, *The Birkhoff–Lewis Fixed Point Theorem and a Conjecture of V.I. Arnold*, Invent. Math. 73 (1983), 33–49.
- [FH] E. Fadell and S. Husseini, *Category weight and Steenrod operations*, Boletín de la Sociedad Matemática Mexicana 37 (1992), 151–161.
- [F] I. Fary, *Sur la catégorie des classes de homologie d’un espace*, Proc. Intern. Congr. Math. 1954, 2, 215, North–Holland, Amsterdam 1957.
- [FeH] Y. Felix and S. Halperin, *Rational L.-S. category and its applications*, Trans. Amer. Math. Soc. 273 (1982), 1–38.
- [Fet] A. I. Fet, *A connection between the topological properties and the number of extremals on a manifold* (Russian), Doklady AN SSSR 88 (1953), 415–417.
- [Fl1] A. Floer, *Cuplength estimates on Lagrangian intersections*, Comm. Pure Appl. Math. 42 (1989), 335–356.
- [Fl2] A. Floer, *Symplectic fixed points and holomorphic spheres*, Commun. Math. Phys. 120 (1989), 575–611.
- [Fox] R. Fox, *On the Lusternik–Schnirelmann category*, Ann. of Math. 42 (1941), 333–370.
- [FE] S. Froloff and L. Elsholz, *Limite inférieure pour le nombre des valeurs critiques d’une fonction, donné sur une variété*, Math. Sbornik 42, 5 (1935), 637–643.
- [G] T. Ganea, *Some problems on numerical homotopy invariants*, Symposium in Algebraic Topology, Seattle 1971, 23–30, Lecture Notes in Mathematics 249, Springer, Berlin 1971.
- [GM] V. K. A. M. Gugenheim and J. P. May, *On the Theory and Applications of Differential Torsion Products*, Memoirs Amer. Math. Soc. 142, AMS, Providence, Rhode Island 1974.
- [H] H. Hofer, *Lusternik–Schnirelmann theory for Lagrangian intersections*, Annales de l’inst. Henri Poincaré– analyse non linéaire 5 (1988), 465–499.
- [HZ] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel 1994.
- [I] N. Iwase, *Ganea’s conjecture on Lusternik–Schnirelmann category*, preprint 1997.
- [J] I. M. James, *On category, in the sense of Lusternik–Schnirelmann*, Topology 17 (1978), 331–349.
- [K] D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. 124 (1966), 431–449.
- [LS] L. A. Lusternik and L. G. Schnirelmann, *Méthodes topologiques dans les problèmes variationnels*, Hermann, Paris 1934.
- [MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Clarendon Press, Oxford 1995.
- [R1] Yu. B. Rudyak, *On the Ganea conjecture for manifolds*, Proc. Amer. Math. Soc. 125 (1997), 2511–2512.
- [R2] Yu. B. Rudyak, *On category weight and its applications*, Topology 38 (1999), 37–55.
- [R3] Yu. B. Rudyak, *On analytical applications of stable homotopy (the Arnold conjecture, critical points)*, Math. Zeitschrift, to appear.
- [RO] Yu. B. Rudyak and J. Oprea, *On the Lusternik–Schnirelmann Category of Symplectic Manifolds and the Arnold Conjecture*, Math. Zeitschrift, to appear.

- [S1] J. Strom, *Two Special Cases of Ganea's Conjecture*, Trans. Amer. Math. Soc., to appear.
- [S2] J. Strom, *Essential category weight*, preprint, 1997.
- [S3] J. Strom, *Essential category weight and classifying spaces*, preprint, 1997.
- [Sv] A. Svarc, *The genus of a fiber space*, Amer. Math. Soc. Translations 55 (1966), 49–140.
- [Sw] R. W. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer, Berlin 1975.
- [T] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik–Schnirelmann category*, Invent. Math. 6 (1968), 197–244.
- [To] G. Toomer, *Lusternik–Schnirelmann Category and the Moore Spectral Sequence*, Math. Z. 138 (1974), 123–143.
- [W] J. H. C. Whitehead, *Simplicial spaces, nuclei, and m -groups*, Proc. London Math. Soc. 45 (1939), 243–327,