HOMOTOPY AND GEOMETRY
BANACH CENTER PUBLICATIONS, VOLUME 45
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

HOMOTOPICAL DYNAMICS OF GRADIENT FLOWS

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In this paper we will be interested in results surrounding the following basic question: *what are the homotopy properties that one can extract from a gradient flow?* We approach this question by decomposing it into three parts:

1. Identify what are the homotopical objects that are provided by the flow (e.g. critical points, Conley indexes).

2. Discover what are the relations that have to be satisfied by these objects (e.g. Morse inequalities, Lusternik-Schnirelmann type inequalities).

3. (The Realizability Problem.) Given some homotopy objects that satisfy the relations from 2., is there a corresponding flow? Is this flow unique up to some equivalence relation?

We will consider only gradient flows induced by functions with *isolated critical points*, restrict our discussion to the finite dimensional, compact context and concentrate on the non-Morse case. One way to look at these questions in this case is to first treat numerical invariants, then local invariants, and finally, pairwise invariants (that concern pairs of consecutive critical points)... At each level, we can look at the points 1.,2.,3. above.

1. Numerical and local data. Let us fix some notation first. Let W be a smooth, riemannian, compact, n-manifold with boundary and let $f : W \longrightarrow \mathbf{R}$ be smooth, maximal, regular and constant on ∂W. Consider $W' = f^{-1}[a, b]$ with a and b regular values, $V_0 = f^{-1}(a)$. Assume f has a single critical point in W' which is denoted by P. If P is reasonable let $A_f(P) \simeq f^{-1}(-\infty, f(P)] \cap S(P, \epsilon)$, where $S(P, \epsilon)$ is a sphere of radius ϵ around P. Its homotopy type does not depend on the different choices made if ϵ is small enough. The Conley index [5,31] of the critical point P with respect to the flow induced by $-\nabla f$ is denoted $C_f(P)$; it is the homotopy type of the quotient W'/V_0. We denote by CX the cone over the space X; $Cat(X)$ is the strong Lusternik-Schnirelmann category

1991 *Mathematics Subject Classification*: 58F09, 58F25, 55P25, 55P42.
The paper is in final form and no version of it will be published elsewhere.

of X (it is the minimal number of cone attachments needed to construct a space of the homotopy type of X starting from a point [22]) and $cat(X)$ is the Lusternik-Schnirelmann category of X [23,25].

It is convenient to group known facts about our subject in a table below. We start with the well understood Morse case and proceed towards that of general critical points. In between, one has the "reasonable" critical points [8], a class that contains all the isolated critical points of locally analytic functions. More information about the contents of the table follows the table itself.

	Morse	Reasonable	General
Numerical inv.	no. crit pts. of index k $= crit_k(f)$	no. crit. pts. $= crit(f)$	$crit(f)$
Relations	$crit_k(f) \geq b_k(W)$	$crit(f) \geq Cat(W) + 1$	$crit(f) \geq cat(W) + 1$
Realization	$\pi_1 W = 0$ $n \geq 6$ $\exists f$ perfect	$\pi_1 W = 0$, $\pi_2 W = 0$ $\pi_1 \partial W = 0$, $W \simeq X$ $dim(X) < n/2$ $\exists g, crit(g) = Cat(W) + 1$	same $\exists g, crit(g) = cat(W) + 2$
Local inv.	index (P)	$C_f(P)$	$C_f(P)$
Relations	$W' = V_0 \bigcup D^{k-1}$ $C_f(P) = S^k$	$W' = V_0 \bigcup CA_f(P)$ $C_f(P) = \Sigma A_f(P)$	$C_f(P) =$ co-H-space
Realization	$\forall k$ $\exists f, P$ index (P)$= k$	$\forall X$ $\exists f, P$ $C_f(P) = \Sigma X$	Open

The information contained in the table is classical in the case of Morse functions. This case appears in the first vertical column. Recall that a Morse function is perfect if it has the minimal possible number of critical points allowed by homological restrictions. The second column deals with the functions having only reasonable critical points and the third with that of functions with general critical points. The results in these two last columns follow from [8]. The top inequality in the general case is the classical motivation for the introduction by Lusternik and Schnirelmann of the notion of category [25].

In the "realization" lines we mention results providing functions and critical points realizing the restrictions indicated. For example, in the Morse case, the existence of a perfect Morse function on any simply connected manifold is a key classical result equivalent to the h-cobordism theorem. A further example is provided by the last entry in the reasonable column. This "realization" result claims that any finite suspension can be realized as the Conley index of some reasonable critical point.

The construction of the function g in the reasonable case implies the result in general because $cat(X) \leq Cat(X) \leq cat(X) + 1$ [33]. In fact g can be assumed to be self-indexed in the sense that its critical points P_i, $-1 \leq i \leq Cat(X)$ can be indexed such that $f(P_i) = i$, the hessian of f has index i at P_i for $i > -1$ and P_{-1} is a minimum. Modulo the h-cobordism theorem this is the direct "critical points" translation of the homotopy fact [6,7] that $Cat(X)$ equals the cone-length $Cl(X) = \min\{n : \exists\ \Sigma^i Z_i \longrightarrow X_i \longrightarrow X_{i+1}$ cofibration sequences $, 0 \leq i < n, X_0 \simeq *, X_n \simeq X\}$.

The relationship between the Lusternik-Schnirelmann category and the cone length plays an important role here. Its study was initiated by the papers [18] and [24]. In particular a conjecture of Lemaire and Sigrist claimed the equality of Cl and cat for rational spaces. The conjecture is now known to hold for rational Poincaré duality spaces [11]. In spite of the fact that strong category and cone length agree (even geometrically), the conjecture has been disproved by an example of Dupont [15] who constructed a rational space Y with $Cl(Y) = 4$ but $cat(Y) = 3$. Rationally, if $cat(Y) = 2$ then $Cl(Y) = 2$ also [19].

The fact that the Conley index of any isolated critical point is a co-H-space appears in non-explicit ways in the literature (e.g. see [14]). It was made explicit, in a more general setting, in [8] and independently in [28]. There are *no* examples of critical points with a Conley index different from a suspension.

2. Pairwise data of consecutive critical points and stable homotopy. Assume $W'' = f^{-1}[a, c]$ with a as above and $c > b$ regular value such that f has a single critical point in $W'' - W'$ that we denote by Q. We also assume that P and Q are both reasonable critical points.

We have two cofibration sequences $A_f(P) \longrightarrow V_0 \longrightarrow W'$ and $A_f(Q) \longrightarrow W' \longrightarrow W''$. This provides a relative attaching map $k(Q, P) : A_f(Q) \longrightarrow \Sigma A_f(P)$. We also have the same construction for the function $-f$. We get another relative attaching map $k(P, Q)$: $A_{-f}(P) \longrightarrow \Sigma A_{-f}(Q)$. The homotopy classes of these two maps are the invariants that we propose in the pairwise (consecutive) case.

The relation in this case is that the two maps are Spanier-Whitehead duals up to some twisting coming from the stable normal bundle of W [9]. In the stably parallelizable Morse-Smale case this was proved by Franks [20]. The homological consequence of this duality result follows also from a result of McCord [26]. Again in the Morse-Smale case, but for general manifolds, the stable difference between $k(P, Q)$ and $k(Q, P)$ can be explicitly estimated as a function of certain Hopf invariants and the stable normal bundle of the manifold. In particular, it belongs to the image of the obvious factor J^q : $\pi_*\Omega^q\Sigma^q\mathbf{SO} \longrightarrow \pi_*^S$ of the classical J-homomorphism ($q = min\{index(P), index(Q)\}$). In turn, this can be viewed as an obstruction used to detect non-smoothable Poincaré duality spaces [13].

3. More stable ideas. Let $q : \mathbf{R}^m \longrightarrow \mathbf{R}$ be a quadratic function of index k. One of the key ingredients of both the duality result above and of the construction of the function g in the table is that, given a flow γ on some manifold M, the flow $\gamma \times -\nabla q$ on $M \times \mathbf{R}^m$ suspends all the homotopical information k-times [9]. Understanding the behaviour of flows under suspension is also motivated by the "stable" Morse Lusternik-Schnirelmann theory of Eliashberg and Gromov [16], developed in the context of Lagrangian intersections.

Stable ideas are also relevant to this same subject in a different way. Consider the finite dimensional setting for the Arnold conjecture discussed in [16]. In this case, one needs to evaluate the minimal number of critical points of a function that is quadratic at infinity on $M \times R^m$. As shown in [10] this number is estimated by $cat(M, M \times S^{k-1})$ (if the index of the quadratic part is k). Here, $cat(X, A)$ is the relative category introduced by Fadell [17].

The Arnold conjecture suggests that, in general, the inequality

$$cat(X, X \times S^k) \geq cat(X) + 1 \tag{1}$$

can be expected to hold.

A stable version of this inequality has been recently proven by Moyaux [27]. He shows that $cat(X, X \times S^{k-1})$ is bigger than the least n for which the k-th suspension of the projection $p_n : B_n\Omega X \longrightarrow X$ admits a section (this number has been introduced and studied by Vanderbrouk [34] who called it the σ^k-category of X; the existence of a section for p_n is equivalent to $cat(X) \leq n$ [22]; p_n is the projection at the n-th stage of Milnor's classifying construction applied to ΩX).

As shown in [10], the inequality (1) implies the Ganea conjecture

$$cat(X \times S^k) = cat(X) + 1.$$

Iwase has recently disproved the Ganea conjecture, so (1) cannot be valid in full generality. Finding a universal, unstable lower bound for $cat(X, X \times S^n)$ is an interesting open question. Properties of the relative category in relation to counting critical points have also been investigated in [2] especially in the equivariant context. For other interesting applications see also [1].

Stable ideas have also been used efficiently by Rudyak [29] and Strom [32] in the homotopical study of the Lusternik-Schnirelmann category in relation to the Ganea conjecture. Rudyak has also applied his ideas to the Arnold conjecture context and, in particular, proved together with Oprea [30] that the category of each symplectic manifold (M, ω) with $[\omega]|_{\pi_2(M)} = 0$ is equal to $dim(M)$. It also follows from [30] and [27] that, in this case, $cat(M, M \times S^k) = cat(M) + 1$.

4. Final comments. In what ways do the homotopical objects above—the Conley indexes of the critical points and pairwise data for critical points—characterize the flow on W or, at least, the space W itself ?

In the Morse-Smale case this is quite well understood. Pairwise, *consecutive* information is insufficient to determine W even if, as indicated above, the stable difference between $k(P, Q)$ and $k(Q, P)$ is highly relevant for the stable normal structure. At the same time, the moduli spaces of connecting flow lines for *all the pairs of critical points together with their compactifications* provide the homeomorphism type of the manifold [3].

In the case of functions with isolated but general critical points much less is clear. Essentially, one only knows how to use the maps induced in homology by the relative attaching maps to determine the homology of W. This is the technique of connection matrices of Franzosa [21]. Homological results abound in Conley index theory and they apply to general flows, not only to gradient ones. However, at the geometric level, in this same setting of general flows, the only available general results are extensions of the suspension and duality statements mentioned before [12].

In going beyond homology in reconstructing the homotopy type of the manifold, the discussion above suggests that only stable global information can be expected to be recovered from pairwise flow data in the non-Morse context. For the reconstruction of the stable homotopy type, it is likely that one would need to use techniques similar to those of [4].

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