

THE COHOMOLOGY ALGEBRAS OF ORIENTABLE SEIFERT MANIFOLDS AND APPLICATIONS TO LUSTERNIK–SCHNIRELMANN CATEGORY

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Abstract. This note gives a complete description of the cohomology algebra of any orientable Seifert manifold with \mathbb{Z}/p coefficients, for an arbitrary prime p . As an application, the existence of a degree one map from an orientable Seifert manifold onto a lens space is completely determined. A second application shows that the Lusternik–Schnirelmann category for a large class of Seifert manifolds is equal to 3, which in turn is used to verify the Ganea conjecture for these Seifert manifolds.

1. Introduction. Throughout this paper M will denote an orientable Seifert fibred manifold and \tilde{M} its universal cover. Furthermore, Π will denote the fundamental group $\pi_1(M)$ of M and $R = \mathbb{Z}\Pi$ is its integral group ring. Then using the standard notation introduced in 1932 by Seifert [17], [18], either $M \cong (O, o, g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ or $M \cong (O, n, g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. In the first case M is orientable with orientable orbit surface V_g of genus $g \geq 0$, while in the second case M is orientable with non-orientable orbit surface W_g (a 2-sphere with g cross caps), $g \geq 1$. In both cases e is the Euler number, m is the number of singular fibres and, for each i , (a_i, b_i) is a pair of relatively prime integers that characterize the twisting of the i -th singular fibre. For other references on Seifert manifolds see [8], [12], [13], [16].

The cohomology algebra of the orientable Seifert manifolds with orbit surface S^2 and $\mathbb{Z}/2$ coefficients is described in [1]. The present paper provides a two-fold generalization of these results to the cohomology algebra of any orientable Seifert manifold with \mathbb{Z}/p coefficients, for any prime p . The techniques used are similar to those formulated in [1] and [2]. We proceed by identifying the cohomology of the Seifert manifold with the

1991 *Mathematics Subject Classification*: 57M25, 20F38, 20J05, 55M30.

Key words and phrases: Seifert manifolds, cohomology algebra, diagonal map, cup products, degree one maps, Lusternik–Schnirelmann category, cup product length.

The paper is in final form and no version of it will be published elsewhere.

group cohomology of its fundamental group, Π , which is assumed to be infinite. This can be done because such Seifert manifolds (apart from the two exceptions $\mathbb{R}P^3 \# \mathbb{R}P^3$ and $S^1 \times S^2$) are Eilenberg–MacLane spaces of type $K(\Pi, 1)$. This reduces the problem to a purely algebraic computation in group cohomology. The equivariant chain complex of the universal cover, \tilde{M} , of M can then be used to compute $H^*(\Pi; \mathbb{Z}/p)$. The calculation of the cup products can then be carried out from their definition by constructing a diagonal approximation for the equivariant chain complex. The assumption that $|\Pi| = \infty$ is not restrictive since there are relatively few Seifert manifolds with finite fundamental group and these are all well known. The construction of the equivariant chain complex, and the diagonal approximation, are stated in §2.

In §3, the diagonal approximation is used to calculate the \mathbb{Z}/p cup products and the \mathbb{Z}/p Bockstein maps. These results are then used in §4 to determine when an orientable Seifert manifold admits a degree one map onto a lens space $L(p, q)$, for a given prime p . The formulae and theorems in §2 and §3 are, for the most part, stated without proof. The complete details of the proofs of all of these theorems, which are somewhat lengthy and algebraically technical, will be given in [3].

The application to the Lusternik–Schnirelmann category (see Theorem 4.5) follows directly from the calculation of the cup products for the orientable Seifert manifolds and the well known result relating the cup product length to the Lusternik–Schnirelmann category. This theorem is proved in §4. The result on cup product length and the Künneth Theorem enable us to resolve the corresponding cases of the Ganea conjecture (cf. Theorem 4.6).

While §2 gives all the basic formulae necessary for the calculation of the cup products in §3 and the applications in §4, these formulae are algebraically complex. The reader, after looking at the notation and conventions in §2, may prefer to pass directly to the last two sections without examining the formulae $(R_{i,j})$, $(R_{i,j}^*)$ and Δ (Theorem 2.1) in detail, at least on first reading.

2. The equivariant chain complex and the diagonal approximation. In [2], the equivariant chain complex for the the universal cover \tilde{M} was constructed in the genus zero case where the associated orbit surface is S^2 , using results from [5] and [14]. In the general case, where the Seifert manifold is orientable and can have any orbit surface, this chain complex is modified using the same techniques given in [1] §2, [2] §4, and the resulting equivariant chain complex is now described in all possible cases.

(Oo): *The Seifert manifold and the associated orbit surface are both orientable.* Here

$$\Pi = \langle s_1, \dots, s_m, v_1, w_1, \dots, v_g, w_g, h \mid [s_j, h], s_j^{a_j} h^{b_j}, [v_j, h], [w_j, h], s_1 \dots s_m [v_1, w_1] \dots [v_g, w_g] h^{-e} \rangle.$$

Note that h is a central element in Π .

The equivariant chain complex \mathcal{C} for the universal cover \tilde{M} is formed from the free R -modules in dimensions 0,1,2,3 with free generators

$$\begin{aligned} (G_0) \quad 0: & \sigma_0^0, \dots, \sigma_m^0; \\ (G_1) \quad 1: & \sigma_1^1, \dots, \sigma_m^1; \rho_0^1, \dots, \rho_m^1; \nu_1^1, \omega_1^1, \dots, \nu_g^1, \omega_g^1; \eta_0^1, \dots, \eta_m^1; \end{aligned}$$

$$(G_2) \quad 2 : \sigma_1^2, \dots, \sigma_m^2; \rho_0^2, \dots, \rho_m^2; \nu_1^2, \omega_1^2, \dots, \nu_g^2, \omega_g^2; \mu_0^2, \dots, \mu_m^2; \delta^2;$$

$$(G_3) \quad 3 : \sigma_0^3, \dots, \sigma_m^3; \delta^3;$$

The definition of the boundary map, ∂ , of the chain complex C , requires the following conventions and definitions in the group ring R . First of all, in addition to the list of generators given in (G_1) , (G_2) adopt the notation $\sigma_0^1 = 0$, $\sigma_0^2 = 0$ and set $s_0 = h^{-e}$. Next, let $r_j = s_0 s_1 \dots s_j$ for $-1 \leq j \leq m$, ($r_{-1} = 1$) and $r_{m+j} := r_m \prod_{k=1}^j [v_k, w_k]$ for $1 \leq j \leq g$. Observe that $r_{m+g} = 1$. Given relatively prime integers $a_j > 0$, $b_j > 0$, choose integers $c_j > 0$, $d_j > 0$ so that $\left| \frac{a_j}{c_j} \frac{b_j}{d_j} \right| = 1$ and let $t_j = s_j^{c_j} h^{d_j}$. Then $s_j = t_j^{-b_j}$ and $h = t_j^{a_j}$. When $j = 0$ set $a_0 = 1$, $b_0 = e$, giving $s_0 = h^{-e}$, as before. Now define the Laurent polynomials

$$f_{l,j} = 1 + t_j + \dots + t_j^{l-1}, \quad l \geq 1; \quad f_{a_j,j} = F_j = \frac{t_j^{a_j} - 1}{t_j - 1},$$

$$g_{l,j} = t_j^{-1} + t_j^{-2} + \dots + t_j^{-l}, \quad l \geq 1; \quad g_{b_j,j} = G_j = \frac{1 - t_j^{-b_j}}{t_j - 1},$$

$$P_j = 1 + t_j^{-b_j} + \dots + t_j^{-b_j(c_j-1)},$$

$$Q_j = 1 + t_j^{a_j} + \dots + t_j^{a_j(d_j-1)}.$$

In particular, $F_0 = 1$ and $G_0 = (1 - h^{-e})/(h - 1)$. Finally, define the chains:

$$\pi_j^1 := r_{j-1} (\sigma_j^1 + \rho_j^1) - r_j \sigma_j^1,$$

$$\pi_{m+j}^1 := r_{m+j-1} (1 - v_j w_j v_j^{-1}) \nu_j^1 + (r_{m+j-1} v_j - r_{m+j}) \omega_j^1,$$

$$\pi_j^2 := -r_{j-1} (\sigma_j^2 + \rho_j^2) + r_j \sigma_j^2,$$

$$\pi_{m+j}^2 := r_{m+j-1} (v_j w_j v_j^{-1} - 1) \nu_j^2 + (r_{m+j} - r_{m+j-1} v_j) \omega_j^2.$$

The free resolution \mathcal{C} is given by the exact sequence

$$\mathcal{C} : \quad 0 \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

and the differentials are defined by

$$(R_{1,1}) \quad \partial \sigma_j^1 = \sigma_j^0 - \sigma_0^0, \quad 1 \leq j \leq m,$$

$$(R_{1,2}) \quad \partial \rho_j^1 = (s_j - 1) \sigma_j^0, \quad 0 \leq j \leq m,$$

$$(R_{1,3}) \quad \partial \eta_j^1 = (h - 1) \sigma_j^0, \quad 0 \leq j \leq m,$$

$$(R_{1,4}) \quad \partial \nu_j^1 = (v_j - 1) \sigma_0^0, \quad \partial \omega_j^1 = (w_j - 1) \sigma_0^0, \quad 1 \leq j \leq g,$$

$$(R_{2,1}) \quad \partial \sigma_j^2 = \eta_0^1 - \eta_j^1 + (h - 1) \sigma_j^1, \quad 1 \leq j \leq m,$$

$$(R_{2,2}) \quad \partial \rho_j^2 = (1 - s_j) \eta_j^1 + (h - 1) \rho_j^1, \quad 0 \leq j \leq m,$$

$$(R_{2,3}) \quad \partial \nu_j^2 = (1 - v_j) \eta_0^1 + (h - 1) \nu_j^1, \quad \partial \omega_j^2 = (1 - w_j) \eta_0^1 + (h - 1) \omega_j^1, \quad 1 \leq j \leq g,$$

$$(R_{2,4}) \quad \partial \delta^2 = \sum_{j=0}^m \pi_j^1 + \sum_{j=1}^g \pi_{m+j}^1,$$

$$(R_{2,5}) \quad \partial \mu_j^2 = F_j \cdot \rho_j^1 + G_j \cdot \eta_j^1, \quad 0 \leq j \leq m,$$

$$(R_{3,1}) \quad \partial\sigma_j^3 = \rho_j^2 + (1-t_j)\mu_j^2, \quad 0 \leq j \leq m,$$

$$(R_{3,2}) \quad \partial\delta^3 = (1-h)\delta^2 - \sum_{j=0}^m \pi_j^2 - \sum_{j=1}^g \pi_{m+j}^2.$$

REMARK 2.1. Observe that $\pi_0^1 = \rho_0^1$, $\pi_0^2 = -\rho_0^2$, and

$$\partial\pi_j^1 = (r_j - r_{j-1})\sigma_0^0, \quad \partial\pi_{m+j}^2 = (r_j - r_{j-1})\eta_0^1 + (1-h)\pi_{m+j}^1, \quad 0 \leq j \leq m+g.$$

(On): The Seifert manifold is orientable while the associated orbit surface is not orientable. In this case the cell complex of \tilde{M} does not include the cells ω_j^1, ω_j^2 listed in (G_1) and (G_2) but otherwise has the same generators as the (Oo) case. The fundamental group has the following presentation:

$$\Pi = \langle s_1, \dots, s_m, v_1, \dots, v_g, h \mid [s_j, h], s_j^{a_j} h^{b_j}, v_j h v_j^{-1} h, s_1 \dots s_m v_1^2 \dots v_g^2 h^{-e} \rangle.$$

Define the chains:

$$\begin{aligned} \pi_{m+j}^1 &:= r_{m+j-1} (1+v_j) \nu_j^1, \\ \pi_{m+j}^2 &:= r_{m+j-1} (h\nu_j - 1) \nu_j^2, \quad \text{where} \\ r_{m+j} &:= s_0 \dots s_m \prod_{k=1}^j v_k^2. \end{aligned}$$

Note that although h is not central here, as in the (O, o) case, it does commute with each s_j, v_i^2 , and hence also with $r_k, 0 \leq k \leq m+g$. The boundary relations $(R_{i,j})$ without $\nu_j, \omega_j, v_j, w_j$ are the same as in the case of the orientable surface. One has to replace the other $(R_{i,j})$ by the following $(R_{i,j}^*)$.

$$(R_{1,4}^*) \quad \partial\nu_j^1 = (v_j - 1)\sigma_0^0, \quad 1 \leq j \leq g,$$

$$(R_{2,3}^*) \quad \partial\nu_j^2 = (1+h\nu_j)\eta_0^1 + (h-1)\nu_j^1, \quad 1 \leq j \leq g.$$

The free resolution \mathcal{C} , given above, suffices to find the additive structure of $H^*(M; A)$. However, to find the ring structure (i.e. the cup products), make $\mathcal{C} \otimes \mathcal{C}$ into an R -chain complex by setting $\partial(x \otimes y) = \partial x \otimes y + (-1)^{\deg(x)} x \otimes \partial y$, and $(nu + mv)(x \otimes y) = n(ux \otimes uy) + m(vx \otimes vy)$ for $m, n \in \mathbb{Z}, u, v \in \Pi, x, y \in \mathcal{C}$. Then seek a diagonal approximation $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ such that

- (a) Δ is an R -chain map,
- (b) Δ preserves augmentation, that is, there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \otimes \mathcal{C} \\ \varepsilon \downarrow & & \downarrow \varepsilon \otimes \varepsilon \\ \mathbb{Z} & \xrightarrow{\approx} & \mathbb{Z} \otimes \mathbb{Z}. \end{array}$$

Such a diagonal map Δ must exist by the theorem of acyclic models (cf. Steenrod and Epstein [21] Chapter 5), but it must be found explicitly. One of the main difficulties in the construction of this diagonal map is that there are a variety of choices at each stage of the construction. Of course, by (a), it suffices to define Δ on the (free) generators of the complex \mathcal{C} .

Before stating the central result, we introduce the 1-chain $\tau_j^1 = P_j \rho_j^1 + t_j^{-b_j c_j} Q_j \eta_j^1$. Also, for convenience let $\bar{m} = m - 1$.

DIAGONAL APPROXIMATION THEOREM 2.2. *The diagonal approximation of the equivariant chain complex \mathcal{C} is defined on the generators of \mathcal{C} as follows:*

(Oo): $M \cong (O, o; g | e : (a_1, b_1), \dots, (a_m, b_m))$.

$$\begin{aligned}
\Delta(\sigma_j^0) &= \sigma_j^0 \otimes \sigma_j^0, & \Delta(\sigma_j^1) &= \sigma_j^1 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^1, \\
\Delta(\rho_j^1) &= s_j \sigma_j^0 \otimes \rho_j^1 + \rho_j^1 \otimes \sigma_j^0, & \Delta(\eta_j^1) &= h \sigma_j^0 \otimes \eta_j^1 + \eta_j^1 \otimes \sigma_j^0, \\
\Delta(\nu_j^1) &= \nu_j^1 \otimes \sigma_0^0 + v_j \sigma_0^0 \otimes \nu_j^1, & \Delta(\omega_j^1) &= \omega_j^1 \otimes \sigma_0^0 + w_j \sigma_0^0 \otimes \omega_j^1, \\
\Delta(\sigma_j^2) &= h \sigma_0^0 \otimes \sigma_j^2 - h \sigma_j^1 \otimes \eta_j^1 + \sigma_j^2 \otimes \sigma_j^0 + \eta_0^1 \otimes \sigma_j^1, \\
\Delta(\rho_j^2) &= \rho_j^2 \otimes \sigma_j^0 + s_j \eta_j^1 \otimes \rho_j^1 - h \rho_j^1 \otimes \eta_j^1 + h s_j \sigma_j^0 \otimes \rho_j^2, \\
\Delta(\nu_j^2) &= \nu_j^2 \otimes \sigma_0^0 + v_j \eta_0^1 \otimes \nu_j^1 - h \nu_j^1 \otimes \eta_0^1 + h v_j \sigma_0^0 \otimes \nu_j^2, \\
\Delta(\omega_j^2) &= \omega_j^2 \otimes \sigma_0^0 + w_j \eta_0^1 \otimes \omega_j^1 - h \omega_j^1 \otimes \eta_0^1 + h w_j \sigma_0^0 \otimes \omega_j^2, \\
\Delta(\mu_j^2) &= \mu_j^2 \otimes t_j^{-b_j} \sigma_j^0 + t_j^{a_j - b_j} \sigma_j^0 \otimes \mu_j^2 - \sum_{k=0}^{a_j-1} \sum_{l=-b_j}^{k-1} t_j^k \rho_j^1 \otimes t_j^l \tau_j^1 - \sum_{l=0}^{a_j-1} \sum_{k=l-b_j}^{a_j-b_j-1} t_j^k \tau_j^1 \otimes t_j^l \rho_j^1 \\
&\quad - \sum_{k=1-b_j}^{-1} \sum_{l=-b_j}^{k-1} t_j^k \eta_j^1 \otimes t_j^l \tau_j^1 + \sum_{l=1-b_j}^{-1} \sum_{k=a_j-b_j}^{a_j+l-1} t_j^k \tau_j^1 \otimes t_j^l \eta_j^1 - G_j \sum_{r=1}^{a_j-1} t_j^r \tau_j^1 \otimes f_{r,j} \tau_j^1 \\
&\quad - F_j \sum_{r=1}^{b_j} t_j^{-r} \tau_j^1 \otimes g_{r,j} \tau_j^1,
\end{aligned}$$

$$\Delta(\delta^2) = A + B,$$

where

$$\begin{aligned}
A &= \delta^2 \otimes s_0 \sigma_0^0 + r_{m-1} \sigma_0^0 \otimes \delta^2 + \pi_m^1 \otimes \pi_m^1 + \rho_0^1 \otimes \rho_0^1 + \pi_m^1 \otimes \rho_0^1 \\
&\quad - \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \pi_j^1 \otimes \pi_i^1 - \sum_{j=1}^m \pi_j^1 \otimes r_{j-1} \sigma_j^1 + \sum_{j=1}^m r_j \sigma_j^1 \otimes r_{j-1} \rho_j^1,
\end{aligned}$$

and

$$\begin{aligned}
B &= \sum_{j=1}^g \pi_{m+j}^1 \otimes \rho_0^1 + \sum_{j=1}^g \sum_{k=0}^{j-1} \pi_{m+k}^1 \otimes \pi_{m+j}^1 + \sum_{j=1}^g r_{\bar{m}+j} \nu_j^1 \otimes \pi_{m+j}^1 \\
&\quad + \sum_{j=1}^g r_{\bar{m}+j} v_j \omega_j^1 \otimes \pi_{m+j}^1 - \sum_{j=1}^g r_{\bar{m}+j} v_j \omega_j^1 \otimes r_{\bar{m}+j} \nu_j^1 \\
&\quad + \sum_{j=1}^g r_{\bar{m}+j} v_j w_j v_j^{-1} \nu_j^1 \otimes r_{m+j} \omega_j^1,
\end{aligned}$$

$$\begin{aligned}
\Delta(\sigma_j^3) &= \sigma_j^3 \otimes \sigma_j^0 + t_j^{a_j - b_j} \sigma_j^0 \otimes \sigma_j^3 - t_j \mu^2 \otimes t_j^{-b_j} \tau_j^1 - t_j^{a_j - b_j} \tau_j^1 \otimes t_j \mu_j^2 + t_j \mu_j^2 \otimes G_j \tau_j^1 \\
&\quad - \mu_j^2 \otimes G_j \tau_j^1 - t_j^{-b_j} P_j \mu_j^2 \otimes (\rho_j^1 + G_j \tau_j^1) - t_j^{a_j} (\rho_j^1 + G_j \tau_j^1) \otimes P_j \mu_j^2,
\end{aligned}$$

$$\Delta(\delta^3) = A' + B',$$

where

$$\begin{aligned} A' &= \delta^3 \otimes s_0 \sigma_0^0 + h r_{m-1} \sigma_0^0 \otimes \delta^3 - h \delta^2 \otimes s_0 \eta_0^1 - r_{m-1} \eta_0^1 \otimes \delta^2 - \rho_0^2 \otimes \rho_0^1 + h \rho_0^1 \otimes \rho_0^2 \\ &\quad + \pi_m^2 \otimes \pi_m^1 - h \pi_m^1 \otimes \pi_m^2 + \pi_m^2 \otimes \rho_0^1 + h \pi_m^1 \otimes \rho_0^2 - \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \pi_j^2 \otimes \pi_i^1 \\ &\quad + \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} h \pi_j^1 \otimes \pi_i^2 - \sum_{j=1}^m \pi_j^2 \otimes r_{j-1} \sigma_j^1 - \sum_{j=1}^m h \pi_j^1 \otimes r_{j-1} \sigma_j^2 \\ &\quad - \sum_{j=1}^m r_j \sigma_j^2 \otimes r_{j-1} \rho_j^1 + \sum_{j=1}^m h r_j \sigma_j^1 \otimes r_{j-1} \rho_j^2, \end{aligned}$$

and

$$\begin{aligned} B' &= \sum_{j=1}^g \pi_{m+j}^2 \otimes \rho_0^1 + \sum_{j=1}^g \sum_{k=0}^{j-1} \pi_{m+k}^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^g \sum_{k=0}^{j-1} h \pi_{m+k}^1 \otimes \pi_{m+j}^2 \\ &\quad - \sum_{j=1}^g r_{\bar{m}+j} \nu_j^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^g h r_{\bar{m}+j} \nu_j^1 \otimes \pi_{m+j}^2 - \sum_{j=1}^g h r_{\bar{m}+j} v_j \omega_j^1 \otimes \pi_{m+j}^2 \\ &\quad - \sum_{j=1}^g h r_{\bar{m}+j} v_j \omega_j^1 \otimes r_{\bar{m}+j} \nu_j^2 + \sum_{j=1}^g r_{\bar{m}+j} v_j \omega_j^2 \otimes r_{\bar{m}+j} \nu_j^1 - \sum_{j=1}^g r_{\bar{m}+j} v_j \omega_j^2 \otimes \pi_{m+j}^1 \\ &\quad - \sum_{j=1}^g r_{\bar{m}+j} v_j w_j v_j^{-1} \nu_j^2 \otimes r_{m+j} \omega_j^1 + \sum_{j=1}^g h r_{\bar{m}+j} v_j w_j v_j^{-1} \nu_j^1 \otimes r_{m+j} \omega_j^2 \\ &\quad + \sum_{j=1}^g h \pi_{m+j}^1 \otimes \rho_0^2. \end{aligned}$$

This completes the construction of the diagonal approximation for the case when both the Seifert manifold and the orbit surface are orientable.

(On): $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. In this case the diagonal on the equivariant chain complex \mathcal{C} is defined as in the (Oo) case, except on the generators ω_j^1, ω_j^2 , which do not occur in this case, and on the generators $\nu_j^2, \delta^2, \delta^3$. On these generators, the diagonal is given by:

$$\Delta(\nu_j^2) = \nu_j^2 \otimes \sigma_0^0 - h v_j \eta_0^1 \otimes \nu_j^1 - h \nu_j^1 \otimes \eta_0^1 + h v_j \sigma_0^0 \otimes \nu_j^2 + h v_j \eta_0^1 \otimes h v_j \eta_0^1,$$

$$\Delta(\delta^2) = A + B,$$

where

$$\begin{aligned} A &= \delta^2 \otimes s_0 \sigma_0^0 + r_{m-1} \sigma_0^0 \otimes \delta^2 + \pi_m^1 \otimes \pi_m^1 + \rho_0^1 \otimes \rho_0^1 + \pi_m^1 \otimes \rho_0^1 \\ &\quad - \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \pi_j^1 \otimes \pi_i^1 - \sum_{j=1}^m \pi_j^1 \otimes r_{j-1} \sigma_j^1 + \sum_{j=1}^m r_j \sigma_j^1 \otimes r_{j-1} \rho_j^1, \end{aligned}$$

and

$$B = \sum_{j=1}^g \pi_{m+j}^1 \otimes \rho_0^1 + \sum_{j=1}^g \sum_{k=0}^{j-1} \pi_{m+k}^1 \otimes \pi_{m+j}^1 + \sum_{j=1}^g r_{\overline{m+j}} \nu_j^1 \otimes \pi_{m+j}^1 \\ + \sum_{j=1}^g r_{\overline{m+j}} \nu_j^1 \otimes r_{\overline{m+j}} \nu_j^1,$$

$$\Delta(\delta^3) = A' + B',$$

where

$$A' = \delta^3 \otimes s_0 \sigma_0^0 + h r_{m-1} \sigma_0^0 \otimes \delta^3 - h \delta^2 \otimes s_0 \eta_0^1 - r_{m-1} \eta_0^1 \otimes \delta^2 - \rho_0^2 \otimes \rho_0^1 + h \rho_0^1 \otimes \rho_0^2 \\ + \pi_m^2 \otimes \pi_m^1 - h \pi_m^1 \otimes \pi_m^2 + \pi_m^2 \otimes \rho_0^1 + h \pi_m^1 \otimes \rho_0^2 - \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} \pi_j^2 \otimes \pi_i^1 \\ + \sum_{j=2}^{m-1} \sum_{i=1}^{j-1} h \pi_j^1 \otimes \pi_i^2 - \sum_{j=1}^m \pi_j^2 \otimes r_{j-1} \sigma_j^1 - \sum_{j=1}^m h \pi_j^1 \otimes r_{j-1} \sigma_j^2 \\ - \sum_{j=1}^m r_j \sigma_j^2 \otimes r_{j-1} \rho_j^1 + \sum_{j=1}^m h r_j \sigma_j^1 \otimes r_{j-1} \rho_j^2,$$

and

$$B' = \sum_{j=1}^g \pi_{m+j}^2 \otimes \rho_0^1 + \sum_{j=1}^g \sum_{k=0}^{j-1} \pi_{m+k}^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^g \sum_{k=0}^{j-1} h \pi_{m+k}^1 \otimes \pi_{m+j}^2 \\ - \sum_{j=1}^g r_{\overline{m+j}} \nu_j^2 \otimes \pi_{m+j}^1 - \sum_{j=1}^g h r_{\overline{m+j}} \nu_j^1 \otimes \pi_{m+j}^2 + \sum_{j=1}^g h r_{\overline{m+j}} \nu_j^2 \otimes r_{\overline{m+j}} \nu_j^1 \\ - \sum_{j=1}^g h r_{\overline{m+j}} \nu_j^1 \otimes h r_{\overline{m+j}} \nu_j^2 + \sum_{j=1}^g h r_{\overline{m+j}} \nu_j^2 \otimes h r_{\overline{m+j}} \nu_j^1 \\ + \sum_{j=1}^g r_{m+j} \eta_0^1 \otimes h r_{\overline{m+j}} \nu_j^2 + \sum_{j=1}^g h \pi_{m+j}^1 \otimes \rho_0^2.$$

3. The cohomology algebra. The cohomology groups of the Seifert manifolds $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ and $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ are well known and can be determined by first calculating their homology from a CW-decomposition (or by abelianizing $\pi_1(M)$) and then applying Poincaré duality. For our purposes it is most convenient to calculate them directly from the cochain complex $\text{Hom}_R(\mathcal{C}, \mathbb{Z}/p)$, where \mathbb{Z}/p is the trivial R -module. Specifically,

$$H^*(\text{Hom}_R(\mathcal{C}, \mathbb{Z}/p)) \cong H^*(\Pi; \mathbb{Z}/p) \cong H^*(M; \mathbb{Z}/p),$$

since M is a $K(\Pi, 1)$ space (cf. [11] Theorem 11.5). Any description of the cohomology algebra of a space, such as Theorems 3.1–3.7 below, is imprecise unless the generators of the various classes involved are also specified. The description of these generators

is given in the Appendix in terms of specific cocycles in $\text{Hom}_R(C, \mathbb{Z}/p)$, and can be referred to when necessary. The proofs of all the results in this section are actually quite straightforward and follow directly from the results in §2. They are, however, omitted here (full details are given in [3]).

In order to describe the cohomology of these manifolds the following notational conventions are made. For any prime p , assume without loss of generality that $a_1, \dots, a_{n_p} \equiv 0 \pmod{p}$ and $a_{n_p+1}, \dots, a_m \not\equiv 0 \pmod{p}$. In this case there exist integers a'_1, \dots, a'_{n_p} , such that $a_1 = pa'_1, \dots, a_{n_p} = pa'_{n_p}$. Choose integers c_i, d_i so that $a_i d_i - b_i c_i = 1$. Then, for $1 \leq i \leq n_p$, $b_i, c_i \not\equiv 0 \pmod{p}$. When $n_p = 0$, that is when $a_i \not\equiv 0 \pmod{p}$ for all $1 \leq i \leq m$, let $b_1, \dots, b_r \equiv 0 \pmod{p}$, $b_{r+1}, \dots, b_m \not\equiv 0 \pmod{p}$. Then there exist b'_1, \dots, b'_r such that $b_1 = pb'_1, \dots, b_r = pb'_r$. Let $A = \prod_{i=1}^m a_i$, $A_i = a_i^{-1}A \in \mathbb{Z}$ and $C = \sum b_i A_i$. Observe that $A \not\equiv 0 \pmod{p}$. Finally note that the same symbol will be used to denote an integer s and its mod p reduction. The context will remove any ambiguity.

The calculation of these cohomology groups gives the following results:

For $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$,

- (i) If $n_p > 0$ then $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^{2g+n_p-1}$.
- (ii) If $n_p = 0$, then
 - (a) $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^{2g}$, if $Ae + C \not\equiv 0 \pmod{p}$ and
 - (b) $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^{2g+1}$, if $Ae + C \equiv 0 \pmod{p}$.

In the case when $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ then similarly

- (i) If $n_p > 0$ then $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^{g+n_p-1}$.
- (ii) If $n_p = 0$, then
 - (a) $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^g$, if $Ae + C \not\equiv 0 \pmod{p}$ and
 - (b) $H^1(M; \mathbb{Z}/p) \approx H^2(M; \mathbb{Z}/p) \approx (\mathbb{Z}/p)^{g+1}$, if $Ae + C \equiv 0 \pmod{p}$.

The remainder of this section will be devoted to the description of the cup products. Again we remind the reader that the generating classes in cohomology are described in the Appendix.

THEOREM 3.1. *Let $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ and let δ_{jk} denote the Kronecker delta.*

If $n_p > 0$, then as a graded vector space,

$$H^*(M; \mathbb{Z}/p) = \mathbb{Z}/p \{1, \alpha_i, \theta_l, \theta'_l, \beta_i, \varphi_l, \varphi'_l, \gamma \mid 2 \leq i \leq n_p, 1 \leq l \leq g\}.$$

with $\alpha_i, \theta_l, \theta'_l$ in degree 1, $\beta_i, \varphi_l, \varphi'_l$ in degree 2, and γ in degree 3. Now set $\beta_1 = -\sum_{i=2}^{n_p} \beta_i$. Then, the non-trivial cup products in $H^(M; \mathbb{Z}/p)$ are given by:*

- (i) *For $p = 2$, let $2 \leq i, j \leq n_2$. Then*

$$\alpha_i \cdot \alpha_j = \binom{a_1}{2} \beta_1 + \delta_{ij} \binom{a_i}{2} \beta_i.$$

Furthermore, if $2 \leq k \leq n_2$ as well, then

$$\alpha_i \cdot \alpha_j \cdot \alpha_k = \binom{a_1}{2} \gamma \text{ if } i \neq j \text{ or } j \neq k, \quad \alpha_i^3 = \left[\binom{a_1}{2} + \binom{a_i}{2} \right] \gamma.$$

(ii) For any prime p , let $2 \leq i \leq n_p$ and $1 \leq l \leq g$. Then,

$$\alpha_i \cdot \beta_i = -\gamma, \quad \theta_l \cdot \varphi'_l = \theta'_l \cdot \varphi_l = \gamma.$$

Additionally, the mod p Bockstein, B_p , on $H^1(M; \mathbb{Z}/p)$ is given by

$$\begin{aligned} B_p(\alpha_i) &= -a'_i c_i \beta_i + a_1' c_1 \beta_1 \in H^2(M; \mathbb{Z}/p), \\ B_p(\theta_l) &= B_p(\theta'_l) = 0. \end{aligned}$$

REMARKS 3.2. (i) For $p = 2$, $B_2(\alpha_i) = Sq^1(\alpha_i) = \alpha_i^2$. Thus the formula given in (i) for α_i^2 agrees with the formula for $B_2(\alpha_i)$ given in (ii).

(ii) In case $n_p = 1$, there are no α_i or β_i classes. Furthermore, if $g = 0$ there are no θ_l , θ'_l , ϕ_l , or ϕ'_l classes.

(iii) Since $n_p > 0$ only when there exists an a_i that is divisible by p , it follows that $n_p = 0$ for all but a finite number of primes p .

THEOREM 3.3. Let $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$ and suppose that for some prime $p \geq 2$, $n_p = 0$. In case $Ae + C \not\equiv 0 \pmod{p}$ $H^*(M; \mathbb{Z}/p)$ has generators θ_l , θ'_l , φ_l , φ'_l , $1 \leq l \leq g$. If $Ae + C \equiv 0 \pmod{p}$, then

$$H^*(M; \mathbb{Z}/p) = \mathbb{Z}/p \{1, \alpha, \theta_l, \theta'_l, \beta, \varphi_l, \varphi'_l, \gamma \mid 1 \leq l \leq g\},$$

where $\deg(\alpha) = 1$ and $\deg(\beta) = 2$.

The non-trivial cup products are given by:

(i) For $p = 2$, if $Ae + C \equiv 0 \pmod{2}$

$$\alpha^2 = \left[q + \frac{1}{2}(Ae + C) \right] \beta,$$

where q is defined to be the number of b_i , $1 \leq i \leq r$, which are congruent to 2 (mod 4).

(ii) If $Ae + C \equiv 0 \pmod{p}$, for any p , then for $1 \leq l \leq g$,

$$\begin{aligned} \alpha \cdot \theta_l &= \varphi_l, & \alpha \cdot \theta'_l &= \varphi'_l, \\ \theta_l \cdot \theta'_l &= \beta, & \alpha \cdot \beta &= -\gamma. \end{aligned}$$

(iii) In either case, for $1 \leq m \leq g$ (and for any p),

$$\theta_l \cdot \varphi'_l = \theta'_l \cdot \varphi_l = \gamma.$$

Furthermore, the mod p Bockstein on $H^1(M; \mathbb{Z}/p)$ is given by:

$$\begin{aligned} B_p(\alpha) &= -A^{-1} \left[\sum_{i=1}^r b'_i A_i + \frac{Ae + C}{p} \right] \beta \in H^2(M; \mathbb{Z}/p), \\ B_p(\theta_l) &= B_p(\theta'_l) = 0. \end{aligned}$$

In the case when the Seifert manifold has a non-orientable orbit surface homeomorphic to $V_g = \mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$, the cohomology algebra is now given.

THEOREM 3.4. Let $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. If for any prime p , $n_p > 0$, then as a graded vector space,

$$H^*(M; \mathbb{Z}/p) = \mathbb{Z}/p \{1, \alpha_i, \theta_l, \beta_i, \varphi_l, \gamma \mid 2 \leq i \leq n, 1 \leq l \leq g\}.$$

In this case let $\beta_1 = -\sum_{i=2}^n \beta_i - 2\sum_{l=1}^g \varphi_l$. The non-trivial cup products in $H^*(M; \mathbb{Z}/p)$ are given by:

(i) For $p = 2$ and $2 \leq i, j \leq n_2$,

$$\alpha_i \cdot \alpha_j = \binom{a_1}{2} \beta_1 + \delta_{ij} \binom{a_i}{2} \beta_i.$$

Moreover, if $2 \leq k \leq n_2$ as well, then

$$\alpha_i \cdot \alpha_j \cdot \alpha_k = \binom{a_1}{2} \gamma \text{ if } i \neq j \text{ or } j \neq k, \quad \alpha_i^3 = \left[\binom{a_1}{2} + \binom{a_i}{2} \right] \gamma.$$

(ii) For any prime p , let $2 \leq i, j \leq n_p$ and $1 \leq l \leq g$. Then,

$$\alpha_j \cdot \beta_j = -\gamma \quad \text{and} \quad \theta_l \cdot \varphi_l = -\gamma.$$

In addition, the mod p Bockstein on $H^1(M, \mathbb{Z}/p)$ is given by:

$$\begin{aligned} B_p(\alpha_j) &= -a'_j c_j \beta_j + a'_1 c_1 \beta_1, \\ B_p(\theta_l) &= -2a'_1 c_1 \beta_1. \end{aligned}$$

REMARK 3.5. (i) When $p = 2$, $B_p(\alpha_i) = \alpha_i^2$ as in Remark 3.2 (i) and the formulas given in Theorem 3.4 (i) and (ii) agree.

(ii) When $n_p = 1$, there are no α_i or β_i classes.

(iii) Note that in the (On) case $g \geq 1$ always holds.

THEOREM 3.6. Let $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. Suppose, for some $p > 2$, $n_p = 0$. Then as a graded vector space

$$H^*(M; \mathbb{Z}/p) = \mathbb{Z}/p \{1, \theta_l, \varphi_l, \gamma \mid 1 \leq l \leq g\}.$$

In this case

$$\theta_l \cdot \varphi_l = -\gamma, \quad \text{for } 1 \leq l \leq g, \quad B_p(\theta_l) = 0,$$

while all other cup products are zero.

THEOREM 3.7. Let $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. Suppose that $p = 2$ and $n_2 = 0$. If $Ae + C \not\equiv 0 \pmod{2}$, then $H^1(M; \mathbb{Z}/2)$ is generated by the elements $\theta_1, \dots, \theta_g$, and $H^2(M; \mathbb{Z}/2)$ has generators $\varphi_1, \dots, \varphi_g$. In case $Ae + C \equiv 0 \pmod{2}$ there are two additional generators $\alpha \in H^1(M; \mathbb{Z}/2)$ and $\beta \in H^2(M; \mathbb{Z}/2)$. The non-trivial cup products are:

(i) $\theta_l \cdot \varphi_l = \gamma$.

(ii) If $Ae + C \equiv 0 \pmod{2}$, then

$$\begin{aligned} \alpha^2 &= \left[q + \frac{1}{2} (Ae + C) \right] \beta, & \theta_l^2 &= \beta, & \alpha \cdot \theta_l &= \varphi_l, \\ \alpha^3 &= \left[q + \frac{1}{2} (Ae + C) \right] \gamma, & \alpha \cdot \beta &= \gamma. \end{aligned}$$

4. Applications to degree one maps and Lusternik–Schnirelmann category.

By [7] Theorem 2.2, for given p , a degree one map from an orientable 3-manifold M onto a lens space $L(p, q)$ exists, for some q , if and only if there exists an element $x \in H^1(M; \mathbb{Z}/p)$ such that the linking number $x \odot x = \langle x \cdot B_p(x), [M] \rangle \neq 0$. The following result determines

all possible non-zero values for $x \cdot B_p(x)$ in any orientable Seifert manifold M , and is a direct corollary of the previous results. Observe that when $p = 2$, $L(2, q) = L(2, 1) = \mathbb{R}P^3$ and $x \cdot B_p(x) = x^3$ so that the description of degree one maps given in Corollary 4.2 agrees with the description in Theorem 4.14 [2], which is restated in Corollary 4.3. It is worth noting that in the cases where a degree one map $M \rightarrow L(p, q)$ exists, the value of q is not arbitrary. By [7] Theorem 2.2, if the linking number $x \odot x = [r/p] \in \mathbb{Q}/\mathbb{Z}$, then $qr \equiv 1 \pmod{p}$.

THEOREM 4.1. *For any prime p , and any orientable Seifert manifold M let x be an element of $H^1(M; \mathbb{Z}/p)$. Then all possible non-zero values of $x \cdot B_p(x)$ are given by:*

- (i) *For the Seifert manifold $M \cong (O, o, g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, when $p \geq 2$ and $n_p \geq 2$,*

$$\alpha_i \cdot B_p(\alpha_i) = (a'_i c_i + a'_1 c_1) \gamma, \quad 2 \leq i \leq n_p.$$

- (ii) *For $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, with $n_p = 0$ and $Ae + C \equiv 0 \pmod{p}$,*

$$\alpha \cdot B_p(\alpha) = A^{-1} \left(\sum_{i=1}^r b'_i A_i + \left(\frac{Ae + C}{p} \right) \right) \gamma.$$

- (iii) *For $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, when $p > 2$ and $n_p > 0$,*

$$\begin{aligned} \alpha_i \cdot B_p(\alpha_i) &= (a'_i c_i + a'_1 c_1) \gamma, \quad 2 \leq i \leq n_p, \\ \theta_l \cdot B_p(\theta_l) &= -4a'_1 c_1 \gamma. \end{aligned}$$

- (iv) *For $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, when $p = 2$, $n_2 = 0$, and $Ae + C \equiv 0 \pmod{2}$,*

$$\alpha^3 = \alpha \cdot B_2(\alpha) = \left(q + \frac{Ae + C}{2} \right) \gamma,$$

where q is defined in Theorem 3.3 (i).

COROLLARY 4.2. *Let $p > 2$ be a fixed prime. Then any orientable Seifert manifold M admits a degree one map to $L(p, q)$, for some q , precisely in the following cases.*

- (i) $M = (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, $n_p \geq 3$, and a_1, \dots, a_{n_p} not all divisible by p^2 ,
- (ii) M as in (a) with $n_p = 2$, and $p^{-1}(a_1 c_1 + a_2 c_2) \not\equiv 0 \pmod{p}$,
- (iii) M as in (a) with $n_p = 0$, $Ae + C \equiv 0 \pmod{p}$, and $p^{-1}(\sum_{i=1}^r b_i A_i + Ae + C) \not\equiv 0 \pmod{p}$,
- (iv) $M = (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$, $n_p \geq 1$, and a_1, \dots, a_{n_p} not all divisible by p^2 .

Outline of proof. The proofs of (ii), (iii), and (iv) are self evident from Theorem 4.1. To prove (i), suppose there is no degree one map $M \rightarrow L(p, q)$, that is, $x \cdot B_p(x) = 0$, for all $x \in H^1(X; \mathbb{Z}/p)$. First consider $x = \alpha_i$ for $2 \leq i \leq p$ and secondly let $x = \alpha_j - \alpha_i$ for $2 \leq i < j \leq n_p$ (this choice is possible because $n_p \geq 3$). A short calculation then shows that $0 = a'_i c_i \in \mathbb{Z}/p$, for $1 \leq i \leq n_p$ and this implies that $a'_i \equiv 0 \pmod{p}$ (since $c_i \not\equiv 0 \pmod{p}$). Thus $a_i \equiv 0 \pmod{p^2}$. Taking the contrapositive gives the result. ■

COROLLARY 4.3. *For $p = 2$ and M an orientable Seifert manifold as above (of type (O, o) or (O, n)), M admits a degree one map to RP^3 in precisely the following cases.*

- (i) $n_2 \geq 2$, and $\frac{(a_i + a_j)}{2} \equiv 1 \pmod{2}$ for some i, j , $1 \leq i, j \leq m$,
- (ii) $n_2 = 0$, $Ae + C \equiv 0 \pmod{2}$, and $q + \frac{(Ae + C)}{2} \equiv 1 \pmod{2}$.

This corollary is also of interest in relativity theory (cf. [19]).

Consider the set, \mathcal{L} , of all open coverings of a space X in which each open set of the covering $C \in \mathcal{L}$ is contractible in X . Let $|C|$ be the cardinality of the covering $C \in \mathcal{L}$. The Lusternik–Schnirelmann category of X is defined to be $\min\{|C| : C \in \mathcal{L}\}$. Let $\text{cat}(X)$ denote the normalized Lusternik–Schnirelmann category, which is defined to be one less than the category defined above. Thus if X is contractible, $\text{cat}(X) = 0$. For a comprehensive review of this subject, see James' survey article [10]. It should be remarked though, that James does not use the normalized version of category. For further references on this subject see Fox [6], Eilenberg and Ganea [4], and Rudyak [15]. A second application of the results presented in §3 determines the normalized Lusternik–Schnirelmann category of a class of Seifert manifolds, by finding the cup product length of the Seifert manifold M (that is, the longest sequence of elements u_1, \dots, u_k in cohomology of positive degree such that $u_1 \smile u_2 \smile \dots \smile u_k \neq 0$).

REMARK 4.4.

- (i) *Let X be a CW-complex of dimension k . Then $\text{cat}(X) \leq k$ (cf. [6], [10]).*
- (ii) *Let X be a topological space with $\text{cat}(X) \leq k$ and let u_1, \dots, u_{k+1} be any $k + 1$ cohomology classes of X . Then $u_1 \smile u_2 \smile \dots \smile u_{k+1} = 0$ (cf. [21] Lemma 2.2 and [20] p. 279).*
- (iii) $\text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y)$ (cf. [10]).

It follows immediately from Remark 4.4 (ii) that if X has cup product length k , then $\text{cat}(X) \geq k$. Henceforth the cup product length of a space X will be denoted by $\text{cup}(X)$.

The next theorem is obtained by simply applying Remark 4.4 to each case. The notation is that used in §3.

THEOREM 4.5. (i) *Let $M \cong (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. If $n_2 \geq 2$ and $a_i \equiv 2 \pmod{4}$ for some i , $1 \leq i \leq n_2$, or if $n_p = 0$ and $Ae + C \equiv 0 \pmod{p}$, then $\text{cat}(M) = 3$.*

(ii) *Let $M \cong (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. If $n_2 \geq 2$ and $a_i \equiv 2 \pmod{4}$ for some i , $1 \leq i \leq n$, or if $n_2 = 0$ and $Ae + C \equiv 0 \pmod{2}$, then $\text{cat}(M) = 3$.*

Proof. (i) By Theorem 3.1 (i) $\text{cup}(M) = 3$ when $a_i \equiv 2 \pmod{4}$, for some i , $1 \leq i \leq n_2$. Furthermore, Theorem 3.3 (ii) shows that when $n_p = 0$ and $Ae + C \equiv 0 \pmod{p}$, $\theta_i \cdot \theta'_i = \beta$ and $\alpha \cdot \beta = \gamma$. Thus $\text{cup}(M) = 3$ in this case as well.

(ii) As in (i), Theorem 3.4 (i) shows that $\text{cup}(M) = 3$ when $a_i \equiv 2 \pmod{4}$, for some i , $1 \leq i \leq n_2$. Finally when $n_2 = 0$ and $Ae + C \equiv 0 \pmod{2}$, it follows from Theorem 3.7 (ii) that $\theta_i^2 = \beta$ and $\alpha \cdot \beta = \gamma$. So again $\text{cup}(M) = 3$. ■

It should be noted that Theorem 4.5 can also be deduced from the results of Eilenberg and Ganea [4], who considered Eilenberg–MacLane spaces in general.

Recall that for every finite CW-complex and every integer $m > 0$ the Ganea conjecture states that $\text{cat}(X \times S^m) = \text{cat}(X) + 1$. This conjecture has recently been disproved in general by Iwase [9]. However, the next result describes a new class of spaces for which the Ganea conjecture is true.

THEOREM 4.6. *Let M be an orientable Seifert manifold satisfying either of the conditions in Theorem 4.5 (i) or (ii). Then,*

$$\text{cat}(M \times S^{m_1} \times \dots \times S^{m_k}) = \text{cat}(M) + k = 3 + k.$$

Proof. By Remark 4.4 (i) $\text{cat}(M) \leq \dim(M) = 3$. By Theorem 4.5 and Remark 4.4 (ii) $\text{cup}(M) \geq \text{cat}(M) = 3$. Thus $\text{cup}(M) = \text{cat}(M)$ (for the appropriate coefficients described in Theorem 4.5). Furthermore observe that for any coefficients, $\text{cup}(S^{m_1} \times \dots \times S^{m_k}) = k$. Thus (for the appropriate coefficients) the Künneth formula and Remark 4.4 (ii), (iii) give:

$$\begin{aligned} 3 + k &\leq \text{cup}(M \times S^{m_1} \times \dots \times S^{m_k}) \\ &\leq \text{cat}(M \times S^{m_1} \times \dots \times S^{m_k}) \\ &\leq \text{cat}(M) + \text{cat}(S^{m_1} \times \dots \times S^{m_k}) \leq 3 + k. \end{aligned}$$

Therefore $\text{cat}(M \times S^{m_1} \times \dots \times S^{m_k}) = 3 + k$. ■

Observe that the proof of 4.6 is valid for any n -manifold M^n such that $\text{cup}(M) = n$.

If M is a connected PL manifold that satisfies $\dim M \leq 2 \text{cat} M - 3$ then by [15] Theorem G, $\text{cat}(M \times S^{m_1} \times \dots \times S^{m_k}) = \text{cat}(M) + k$. Since any 3-manifold is a PL manifold (cf. [8]), Theorem 4.6 can also be deduced from [15] Theorem G and Theorem 4.5, by noting that $\dim M \leq 2 \text{cat} M - 3 = 3$.

Appendix. Consider the equivariant chain complex $\mathcal{C} := \{C_i\}$ of §2. For any generator α of C_i , let $\hat{\alpha}$ denote the dual generator of $\text{Hom}_R(C_i; \mathbb{Z}/p)$; that is, $\hat{\alpha}(\alpha) = 1$, $\hat{\alpha}(\beta) = 0$ for any other generator β of C_i , for $i = 0, 1, 2, 3$.

THEOREM A.1. *Let $M = (O, o; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. Then its \mathbb{Z}/p cohomology groups have the following generators:*

CASE (1): *Let $a_1, \dots, a_{n_p} \equiv 0 \pmod{p}$, $1 \leq n_p \leq m$, and $a_{n_p+1}, \dots, a_m \equiv 1 \pmod{p}$. Then*

$$\left\{ \begin{array}{ll} 1 = \left[\sum_{j=0}^m \hat{\sigma}_j^0 \right], & \text{degree } 0, \\ \alpha_j := [\hat{\rho}_j^1 - \hat{\rho}_1^1], \theta_k := [\hat{\nu}_k^1], \theta'_k := [\hat{\omega}_k^1], \quad 2 \leq j \leq n_p, 1 \leq k \leq g, & \text{degree } 1, \\ \beta_j := [\hat{\sigma}_j^2] = [b_j \hat{\mu}_j^2], \varphi_k := [\hat{\nu}_k^2], \varphi'_k := [\hat{\omega}_k^2], \quad 2 \leq j \leq n_p, 1 \leq k \leq g, & \text{degree } 2, \\ \gamma := [\hat{\delta}^3] = [-\hat{\sigma}_0^3] = \dots = [-\hat{\sigma}_m^3], & \text{degree } 3. \end{array} \right.$$

CASE (2): *Assume that $n_p = 0$, $b_i \equiv 0 \pmod{p}$ for $1 \leq i \leq r$, $b_i \not\equiv 0 \pmod{p}$ for $r+1 \leq i \leq m$. Then*

$$\left\{ \begin{array}{ll} 1 = \left[\sum_{j=0}^m \hat{\sigma}_j^0 \right], & \text{degree 0,} \\ \theta_k := [\hat{\nu}_k^1], \theta'_k := [\hat{\omega}_k^1], 1 \leq k \leq g, & \text{degree 1,} \\ \varphi_k := [\hat{\nu}_k^2], \varphi'_k := [\hat{\omega}_k^2], 1 \leq k \leq g, & \text{degree 2,} \\ \alpha := \left[\sum_{j=0}^m \hat{\eta}_j^1 - \sum_{j=r+1}^m b_j a_j^{-1} \hat{\rho}_j^1 - e \hat{\rho}_0^1 \right], \theta_k, \theta'_k, 1 \leq k \leq g, Ae + C \equiv 0 \pmod{p}, & \text{degree 1,} \\ \beta := [\hat{\delta}^2] = [\hat{\mu}_0^2] = \dots = [\hat{\mu}_m^2], \varphi_k, \varphi'_k, 1 \leq k \leq g, Ae + C \equiv 0 \pmod{p}, & \text{degree 2,} \\ \gamma := [\hat{\delta}^3] = [-\hat{\sigma}_0^3] = \dots = [-\hat{\sigma}_m^3], & \text{degree 3.} \end{array} \right.$$

THEOREM A.2. *Let $M = (O, n; g \mid e : (a_1, b_1), \dots, (a_m, b_m))$. Then the \mathbb{Z}/p cohomology groups have the same generators that are given in Theorem A.1, in both cases, except that the classes θ'_k and ϕ'_k are deleted.*

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