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INDUCED MAPPINGS OF HOMOLOGY DECOMPOSITIONS

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Abstract. We give conditions for a map of spaces to induce maps of the homology decompositions of the spaces which are compatible with the homology sections and dual Postnikov invariants. Several applications of this result are obtained. We show how the homotopy type of the $(n+1)$st homology section depends on the homotopy type of the nth homology section and the $(n+1)$st homology group. We prove that all homology sections of a co-H-space are co-H-spaces, all n-equivalences of the homology decomposition are co-H-maps and, under certain restrictions, all dual Postnikov invariants are co-H-maps. We give a new proof of a result of Berstein and Hilton which gives conditions for a co-H-space to be a suspension.

1. Introduction. The Postnikov decomposition of a 1-connected space has been extremely useful in homotopy theory. A basic property of this construction is the existence of induced maps, that is, a map between spaces induces maps between the Postnikov sections of the spaces which are compatible with all the data of the Postnikov decompositions [Wh, Chap. IX]. This can be used, for example, to show that the Postnikov sections of an H-space are H-spaces and the Postnikov invariants are H-maps. The Eckmann-Hilton dual of the Postnikov decomposition of a space is the homology decomposition of a space. This too has been a very useful way to describe a space. However, it has been known for some time that induced maps of homology decompositions do not always exist. In [Cu_1] Curjel gives necessary and sufficient conditions for a map of spaces to induce compatible maps of homology sections. Here we carry this one step further by giving conditions for the induced maps to be compatible with the dual Postnikov invariants. We derive several consequences of these results. We show that, with certain restrictions, the homotopy type of the homology sections of a space are determined by the homotopy type of the space. We also give conditions under which we can describe the homotopy type of $(n+1)$st homology sections with fixed nth homology section and fixed $(n+1)$st homology group.

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In the last section we consider the homology decomposition of a co-H-space X. We prove that the homology sections are co-H-spaces which are compatible with the co-H-structure of X and, if X is either 2-connected or has torsion-free homology, that the dual Postnikov invariants are co-H-maps. From this we obtain a new proof of the following result of Berstein and Hilton: a $(q-1)$-connected co-H-space of dimension $\leq 3q-3$ is equivalent to a suspension.

For the remainder of this section we present our notation and conventions. All spaces are 1-connected, based spaces of the based homotopy type of a CW-complex. We denote the base point of a space and the one point space by $*$, the constant map by 0 and the identity map or homomorphism by *id*. All maps preserve the base point and we do not distinguish notationally between a map and its homotopy class. Thus equality of maps means either homotopy of the maps or equality of their homotopy classes. The usual notation of homotopy theory will be in effect: $[A, B]$ for the set of homotopy classes $A \to B$, $f_* : [A, B] \to [A, B']$ for the function induced by $f : B \to B'$, Σ for the reduced suspension, C_g for the mapping cone of a map g, $K(G, n)$ for the Eilenberg-MacLane space of type (G, n) and $M(G, n)$ for the Moore space of type (G, n). We note that $M(G, n)$ can be regarded as a CW-complex of dimension $\leq n+1$ which has dimension $\leq n$ when G is free-abelian. The nth homotopy group $\pi_n(G; X)$ of X with coefficients in G is $[M(G, n), X]$. The nth cohomology group $H^n(X; G)$ of X with coefficients in G will often be taken to be $[X, K(G, n)]$.

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2. Basic classes and homology decompositions. We begin with a statement of the generalized Blakers-Massey Theorem which follows easily from [Hi$_2$, Thm. 1$'$].

THEOREM 2.1. *Let* $A \xrightarrow{i} Y \xrightarrow{p} C$ *be a cofibre sequence with* A *(m–1)-connected and* C *(n–1)-connected,* $m, n \geq 2$. *If* X *is a CW-complex of dimension* $\leq m+n-2$, *then the following sequence is exact*

$$[X, A] \xrightarrow{i_*} [X, Y] \xrightarrow{p_*} [X, C].$$

Now let B be an $(r-1)$-connected space with $H_r(B) = G$, for $r \geq 2$. Then the homomorphism of the universal coefficient theorem for cohomology $\eta : H^r(B; G) \to \mathrm{Hom}(H_r(B), G) = \mathrm{Hom}(G, G)$ is an isomorphism.

DEFINITION 2.2. The *basic class* $b^r \in H^r(B; G)$ of B is defined by $\eta(b^r) = id$.

We can regard the basic class as a homotopy class $b^r : B \to K(G, r)$. Then b^r is an $(r+1)$-equivalence, i.e., on homotopy groups it induces an isomorphism in dimensions $\leq r$ and an epimorphism in dimension $r+1$. Thus by [Sp, Cor. 23, p. 405], if A is a CW-complex of dimension $\leq r$, then $b^r_* : [A, B] \to [A, K(G, r)] = H^r(A; G)$ is a bijection. Thus we have

PROPOSITION 2.3. *If* B *is an (r–1)-connected space,* $r \geq 2$ *and* A *is a CW-complex of dimension* $\leq r$, *then* $g \in [A, B]$ *is trivial if and only if* $g^*(b^r) = 0$.

We next consider homology decompositions.

Definition 2.4. Given a 1-connected space X. A *homology decomposition* of X consists of (i) a sequence of spaces X_n, $n \geq 2$ with $H_i(X_n) = 0$ for $i > n$ and maps $j_n : X_n \to X$ such that $j_{n*} : H_i(X_n) \to H_i(X)$ is an isomorphism for $i \leq n$ and (ii) maps $k_n : M(H_{n+1}(X), n) \to X_n$ with $M(H_{n+1}(X), n) \xrightarrow{k_n} X_n \xrightarrow{i_n} X_{n+1}$ a mapping cone sequence (i.e., X_{n+1} is the mapping cone of k_n with inclusion i_n). We require that $j_{n+1} i_n = j_n : X_n \to X$. We refer to the collection of spaces and maps $\{X_n;\, j_n, k_n, i_n;\, n \geq 2\}$ as the *homology decomposition* of X. The space X_n is called the *nth homology section* of X and the maps $k_n \in \pi_n(H_{n+1}(X); X_n)$ the *nth dual invariant.*

We make several comments about this definition.

(1) For a 1-connected space of the homotopy type of a CW-complex, a homology decomposition always exists [Hi_1, Chaps. 8, 10].

(2) The dual invariant k_n induces the trivial homomorphism on homology [Hi_1, p. 57].

(3) We can regard X_n as a CW-complex of dimension $\leq n+1$ [Hi_1, p. 57].

(4) If X is an N-dimensional CW-complex, then $j_N : X_N \to X$ is a homotopy equivalence and we can identify X_N with X.

(5) If $H_{n+1}(X) = 0$, then $M(H_{n+1}(X), n) = *$. Thus $k_n = 0$, $X_{n+1} = X_n$ and $i_n = id$. Note too that $X_2 = M(H_2(X), 2)$.

We conclude this section by defining basic classes for a homology decomposition. Let $\{X_n;\, j_n, k_n, i_n;\, n \geq 2\}$ be a homology decomposition of X. Consider

$$X_{r-1} \xrightarrow{j_{r-1}} X \xrightarrow{q_r} C_r,$$

where C_r is the cofibre of j_{r-1} and q_r is the projection. Then C_r is $(r-1)$-connected and $H_r(C_r) \approx H_r(X)$. Let $b^r \in H^r(C_r; H_r(X))$ be the basic class of C_r.

Definition 2.5. The element $h^r = q_r^*(b^r) \in H^r(X; H_r(X))$ is called the *rth basic class of the homology decomposition* $\{X_n;\, j_n, k_n, i_n;\, n \geq 2\}$.

3. Induced maps. Given two spaces X and X' with homology decompositions and a map $f : X \to X'$. We consider when f gives rise to induced maps, i.e., compatible maps of all the spaces of the homology decomposition of X into the corresponding spaces of the homology decomposition of X'.

Theorem 3.1. *Let X and X' have homology decompositions $\{X_n;\, j_n, k_n, i_n\}$ and $\{X'_n;\, j'_n, k'_n, i'_n\}$, respectively, and let $f : X \to X'$ be a map.*

(1) *There is a map $f_n : X_n \to X'_n$ such that $j'_n f_n = f j_n$ if and only if $j_n^* f^*(h'^{n+1}) = 0$ in $H^{n+1}(X_n; H_{n+1}(X'))$, where h'^{n+1} is the $(n+1)$st basic class of the homology decomposition $\{X'_n;\, j'_n, k'_n, i'_n\}$.*

(2) *Assume that $j_r^* f^*(h'^{r+1}) = 0$ for $r = n, n+1$. Then there exists $f_r : X_r \to X'_r$ such that $j'_r f_r = f j_r$ for $r = n, n+1$ and $i'_n f_n = f_{n+1} i_n$.*

(3) *Assume that there exists $f_r : X_r \to X'_r$ for $r = n, n+1$ such that $i'_n f_n = f_{n+1} i_n$. If $H_{n+1}(X)$ is free-abelian or if X' is 2-connected, then there exists $\hat{f}_n : M(H_{n+1}(X), n) \to M(H_{n+1}(X'), n)$ such that $k'_n \hat{f}_n = f_n k_n$.*

Proof. (1) If there is an $f_n : X_n \to X'_n$ with $j'_n f_n = f j_n$, then $j_n^* f^*(h'^{n+1}) = f_n^* j'^*_n q'^*_{n+1}(b'_{n+1}) = 0$ since $q'_{n+1} j'_n = 0$. Now suppose that $j_n^* f^*(h'^{n+1}) = 0$. Then

$(q'_{n+1}fj_n)^*(b'^{n+1}) = 0$. By Proposition 2.3, $q'_{n+1}fj_n = 0$. Now apply Theorem 2.1 to the cofibration $X'_n \xrightarrow{j'_n} X' \xrightarrow{q'_{n+1}} C'_{n+1}$ to conclude that

$$[X_n, X'_n] \xrightarrow{j'_{n*}} [X_n, X'] \xrightarrow{q'_{n+1*}} [X_n, C'_{n+1}]$$

is exact. Since $q'_{n+1*}(fj_n) = 0$, there exists an $f_n \in [X_n, X'_n]$ such that $fj_n = j'_n f_n$.

(2) By (1), there exists $f_{n+1} : X_{n+1} \to X'_{n+1}$ such that $fj_{n+1} = j'_{n+1}f_{n+1}$. It suffices to prove that there is an $f_n : X_n \to X'_n$ such that $i'_n f_n = f_{n+1} i_n$. But this follows immediately from (1) by taking $\{X_r;\ i_n \cdots i_r, k_r, i_r;\ 2 \le r \le n\}$ and $\{X'_r;\ i'_n \cdots i'_r, k'_r, i'_r;\ 2 \le r \le n\}$ as homology decompositions of X_{n+1} and X'_{n+1}, respectively.

(3) For notational convenience we write $H_i = H_i(X)$ and $H'_i = H_i(X')$. Here we consider the cofibre sequence

$$M(H'_{n+1}, n) \xrightarrow{k'_n} X'_n \xrightarrow{i'_n} X'_{n+1}.$$

If H_{n+1} is free-abelian so that $\dim M(H_{n+1}, n) \le n$ or if X' is 2-connected so that X'_{n+1} is 2-connected, then by Theorem 2.1 the following sequence is exact

$$[M(H_{n+1}, n), M(H'_{n+1}, n)] \xrightarrow{k'_{n*}} [M(H_{n+1}, n), X'_n] \xrightarrow{i'_{n*}} [M(H_{n+1}, n), X'_{n+1}].$$

But $i'_{n*}(f_n k_n) = f_{n+1} i_n k_n = 0$. Thus there is an $\hat{f}_n \in [M(H_{n+1}, n), M(H'_{n+1}, n)]$ such that $k'_n \hat{f}_n = f_n k_n$. ∎

Remarks 3.2. (1) Part (1) of Theorem 3.1 was proved in [Cu$_1$], though we have given a different proof based on Theorem 2.1. It would be interesting to know if (3) holds under weaker hypotheses. We note that there is considerable freedom in the choice of $\hat{f}_n$ in (3), e.g., if $k'_n = 0$, then any map $M(H_{n+1}, n) \to M(H'_{n+1}, n)$ can be taken for $\hat{f}_n$.

(2) There are a few cases in which induced maps always exist, i.e., when (1) of Theorem 3.1 holds. We mention two of these: (i) If X' is a rational space, then $H_r(X')$ is a rational vector space for all r. Thus $\mathrm{Ext}(H_r(X_r), H_{r+1}(X')) = 0$ for all r. But $j_r^* f^*(h'^{r+1}) \in \mathrm{Ext}(H_r(X_r), H_{r+1}(X'))$. Hence in this case there are maps $f_n : X_n \to X'_n$ which satisfy (1) and (2) of Theorem 3.1 for all n. (ii) If $f : X \to X$ is such that $f^* = id : H^{n+1}(X; H_{n+1}(X)) \to H^{n+1}(X; H_{n+1}(X))$, then $j_n^* f^*(h^{n+1}) = j_n^* q_{n+1}^*(b^{n+1}) = 0$. Thus there is an $f_n : X_n \to X_n$ such that $j_n f_n = f j_n$.

Next we give a concrete example to show that induced maps do not always exist.

Example 3.3. Let T be a non-trivial finite abelian group and F a non-trivial free-abelian group of finite rank. Let $n \ge 3$, $M_1 = M(T, n-1)$, $M_2 = M(F, n)$ and $X = X' = M_1 \vee M_2$. Let $\lambda_s : M_s \to X$ be the inclusions and $\pi_r : X \to M_r$ the projections, $r, s = 1, 2$. A map $f : X \to X$ is completely determined by the 4-tuple $(f_{11}, f_{12}, f_{21}, f_{22})$, where $f_{rs} : M_s \to M_r$ is defined by $f_{rs} = \pi_r f \lambda_s$ (see for example [A-M, §4]). A homology decomposition for X is obtained by setting $X_2 = \cdots = X_{n-2} = *$, $X_{n-1} = M_1$ and $X_n = X = M_1 \vee M_2$. Then $i_{n-1} : X_{n-1} \to X_n$ is the inclusion $\lambda_1 : M_1 \to M_1 \vee M_2$. Suppose that f induces $f' : M_1 \to M_1$ such that $\lambda_1 f' = f \lambda_1$. If $x \in M_1$,

$$(f'(x), *) = \lambda_1 f'(x) = f\lambda_1(x) = (f_{11}(x), f_{21}(x)).$$

Hence $f_{11} = f'$ and $f_{21} = 0$. Thus if $f : X \to X$ is a map such that $f_{21} \neq 0$, then there can be no map $f' = f_{n-1} : X_{n-1} \to X_{n-1}$ such that $i_{n-1} f_{n-1} = f i_{n-1}$. But $f_{21} \in [M_1, M_2] = \pi_{n-1}(T; M(F, n)) \approx \mathrm{Ext}(T, F) \neq 0$ since T and F are non-trivial. Therefore we can choose $f_{21} \neq 0$. For example, if $T = \mathbb{Z}_m$ and $F = \mathbb{Z}$, then f_{21} can be taken to be the projection $M(\mathbb{Z}_m, n-1) \to S^n$. We then form $f = (f_{11}, f_{12}, f_{21}, f_{22})$ which admits no induced map of homology decompositions. ∎

We conclude this section with a number of simple results which are a direct consequence of the existence of induced maps.

It is known that for a fixed n, the homotopy type of the nth homology section of a homology decomposition of X is not determined by X. More precisely, an example is given in [B-C, §3] of two spaces X and X' with nth homology sections X_n and X'_n such that X and X' have the same homotopy type but X_n and X'_n do not. We next give a condition which ensures that this does not happen. This generalizes Theorems 3.3 and 3.4 of [B-C].

Proposition 3.4. *Let* $\{X_n; j_n, k_n, i_n\}$ *and* $\{X'_n; j'_n, k'_n, i'_n\}$ *be homology decompositions of* X *and* X' *respectively. If* $f : X \to X'$ *is a homotopy equivalence and* $\mathrm{Ext}(H_n(X), H_{n+1}(X')) = 0$, *then there exists a homotopy equivalence* $f_n : X_n \to X'_n$ *such that* $f j_n = j'_n f_n$. *If in addition,* $\mathrm{Ext}(H_{n+1}(X), H_{n+2}(X')) = 0$ *and either* X' *is 2-connected or* $H_{n+1}(X)$ *is free-abelian, then there exists a homotopy equivalence* $\hat{f}_n : M(H_{n+1}(X), n) \to M(H_{n+1}(X'), n)$ *such that* $k'_n \hat{f}_n = f_n k_n$.

Proof. Since $H^{n+1}(X_n; H_{n+1}(X')) \approx \mathrm{Ext}(H_n(X), H_{n+1}(X')) = 0$, $j_n^* f^*(h'_{n+1}) = 0$. Thus there exists $f_n : X_n \to X'_n$ with $f j_n = j'_n f_n$. It follows that f_n induces an isomorphism of homology, and so is a homotopy equivalence.

Similarly the condition $\mathrm{Ext}(H_{n+1}(X), H_{n+2}(X')) = 0$ implies the existence of a homotopy equivalence $f_{n+1} : X_{n+1} \to X'_{n+1}$ such that $f_{n+1} i_n = i'_n f_n$. Also the condition X' is 2-connected or $H_{n+1}(X)$ is free-abelian implies there exists $\hat{f}_n : M(H_{n+1}(X), n) \to M(H_{n+1}(X'), n)$ with $k'_n \hat{f}_n = f_n k_n$. It follows that the diagram

$$\begin{array}{ccccc} M(H_{n+1}(X), n) & \xrightarrow{k_n} & X_n & \xrightarrow{i_n} & X_{n+1} \\ \downarrow{\scriptstyle \hat{f}_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ M(H_{n+1}(X'), n) & \xrightarrow{k'_n} & X'_n & \xrightarrow{i'_n} & X'_{n+1} \end{array}$$

commutes. Thus $\hat{f}_n$ is a homotopy equivalence. ∎

We next determine, under suitable restrictions, the homotopy types of all $(n+1)$st homology sections with fixed nth homology section. We first introduce some notation. If A and B are spaces, then define an equivalence relation on the set $[A, B]$ as follows: if $f, g \in [A, B]$, then f is equivalent to g means that there exist homotopy equivalences $a : A \to A$ and $b : B \to B$ such that $g = bfa$. We let $[[A, B]]$ denote the set of equivalence classes. We consider the set of homotopy types of mapping cones of maps from a Moore space to a homology section.

Proposition 3.5. *Let* B *be a space such that* $H_i(B) = 0$ *for* $i > n$, $n \geq 2$, *and consider the collection of maps* $f : M(G, m) \to B$ *for a fixed abelian group* G *and integer*

$m \geq n$ (if $m = n$, we require that $f_* = 0$ in homology). Then the set of homotopy types of mapping cones C_f for all such f is in one-one correspondence with the set $[[M(G,m),B]]$ provided $\operatorname{Ext}(H_m(B),G) = 0$ and either B is 2-connected or G is free-abelian.

Proof. We have a homology decomposition of $X = C_f$ with $X_m = B$, $k_m = f : M(G,m) \to B$ and $X_{m+1} = X$. Let $X' = C_g$ be another such mapping cone with analogous homology decomposition and assume that C_f and C_g have the same homotopy type. By Proposition 3.4, there exist homotopy equivalences $a : M(G,m) \to M(G,m)$ and $b : B \to B$ such that $bf = ga$. Thus f is equivalent to g.

Conversely, if $f, g : M(G,m) \to B$ are equivalent, then it is easily seen that C_f and C_g have the same homotopy type. ∎

Corollary 3.6. *Let B be a space such that $H_i(B) = 0$ for $i > n$ and G an abelian group. Suppose $\operatorname{Ext}(H_n(B),G) = 0$ and either B is 2-connected or G is free-abelian. Then the set of homotopy types of $(n+1)$st homology sections with nth homology section B and $(n+1)$st homology group G is in one-one correspondence with the equivalence classes of homologically trivial maps in $[[M(G,n),B]]$.*

Remark 3.7. Corollary 3.6 generalizes Theorem 4.2 of [B-C]. The dual result for Postnikov sections is true without any restrictions [Ar_1, 5.2, p. 197].

4. Co-H-spaces. We first recall the definitions of co-H-space and co-H-map (for more details, see [Ar_2]). If X is a space, then $\phi : X \to X \vee X$ is called a *comultiplication* if $q_i\phi = id : X \to X$, where $q_i : X \vee X \to X$ are the projections, $i = 1, 2$. The pair (X, ϕ) is then called a *co-H-space*. If (A, ψ) and (X, ϕ) are co-H-spaces, and $f : A \to X$ is a map, then f is called a *co-H-map* if $\phi f = (f \vee f)\psi$. We then write $f : (A,\psi) \to (X,\phi)$. If f is a co-H-map and a homotopy equivalence, we say that f is a *co-H-equivalence* and that the co-H-spaces A and X are *co-H-equivalent*.

The following lemma will be useful.

Lemma 4.1. *Given spaces A and X, maps $f : A \to X$, $\phi' : A \to A \vee A$ and $\phi : X \to X \vee X$ and projections $p_i : A \vee A \to A$, $i = 1, 2$. Suppose (X,ϕ) is a co-H-space, $(f \vee f)\phi' = \phi f : A \to X \vee X$ and $p_i\phi' : A \to A$ are homotopy equivalences. Then there exists a comultiplication $\psi : A \to A \vee A$ such that $f : (A,\psi) \to (X,\phi)$ is a co-H-map.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{\phi'} & A \vee A & \xrightarrow{p_i} & A \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f\vee f} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\phi} & X \vee X & \xrightarrow{q_i} & X. \end{array}$$

Let $a_i = p_i\phi' : A \to A$ and let $\overline{a}_i : A \to A$ be the homotopy inverse of a_i. Then $fa_i = q_i\phi f = f$ and so $f = f\overline{a}_i$. Now define $\psi = (\overline{a}_1 \vee \overline{a}_2)\phi' : A \to A \vee A$. Then

$$p_i\psi = \overline{a}_i p_i \phi' = \overline{a}_i a_i = id$$

for $i = 1, 2$. Thus ψ is a comultiplication of A. Finally,

$$(f \vee f)\psi = (f\overline{a}_1 \vee f\overline{a}_2)\phi' = (f \vee f)\phi' = \phi f$$

and so $f : (A,\psi) \to (X,\phi)$ is a co-H-map. ∎

Next we consider a space X with homology decomposition $\{X_n;\, j_n, k_n, i_n\}$. It is clear that $\{X_n \vee X_n;\, j_n \vee j_n, k_n \vee k_n, i_n \vee i_n\}$ is a homology decomposition of $X \vee X$. We express the basic classes of this homology decomposition of $X \vee X$ in terms of the basic classes of the homology decomposition of X. Let $q_1, q_2 : X \vee X \to X$ be projections and let $i_1, i_2 : X \to X \vee X$ be inclusions. For any space A, denote by $i_{r**} : H^k(A; H_l(X)) \to H^k(A; H_l(X \vee X))$, $r = 1, 2$, the coefficient homomorphism induced by $i_{r*} : H_l(X) \to H_l(X \vee X)$. Let $h^n \in H^n(X; H_n(X))$ be the nth basic class of the homology decomposition of X. Then it is straightforward to show that

$$\chi^n = i_{1**}q_1^*(h^n) + i_{2**}q_2^*(h^n) \in H^n(X \vee X; H_n(X \vee X))$$

is the nth basic class of the homology decomposition of $X \vee X$.

Now we consider the homology decomposition of a co-H-space.

THEOREM 4.2. *If (X, ϕ) is a co-H-space and $\{X_n;\, j_n, k_n, i_n\}$ a homology decomposition of X, then there are comultiplications $\phi_n : X_n \to X_n \vee X_n$ such that $j_n : (X_n, \phi_n) \to (X, \phi)$ and $i_n : (X_n, \phi_n) \to (X_{n+1}, \phi_{n+1})$ are co-H-maps. If in addition X is 2-connected or $H_{n+1}(X)$ is free-abelian, then $k_n : (M(H_{n+1}(X), n), \mu_n) \to (X_n, \phi_n)$ is a co-H-map, where μ_n is the canonical comultiplication of the Moore space $M(H_{n+1}(X), n)$.*

Proof. We verify (1) of Theorem 3.1 for the map ϕ and basic class χ^{n+1}. We have

$$\begin{aligned} j_n^*\phi^*(\chi^{n+1}) &= j_n^*\phi^*(i_{1**}q_1^*(h^{n+1}) + i_{2**}q_2^*(h^{n+1})) \\ &= i_{1**}j_n^*(q_1\phi)^*q_{n+1}^*(b^{n+1}) + i_{2**}j_n^*(q_2\phi)^*q_{n+1}^*(b^{n+1}) \\ &= i_{1**}j_n^*q_{n+1}^*(b^{n+1}) + i_{2**}j_n^*q_{n+1}^*(b^{n+1}) = 0 \end{aligned}$$

since $q_{n+1}j_n = 0$. Thus there exists $\phi_n' : X_n \to X_n \vee X_n$ such that $(j_n \vee j_n)\phi_n' = \phi j_n$ and $(i_n \vee i_n)\phi_n' = \phi_{n+1}i_n$. Then with $p_r : X_n \vee X_n \to X_n$ the projections,

$$j_n p_r \phi_n' = q_r(j_n \vee j_n)\phi_n' = q_r\phi j_n = j_n.$$

But $j_{n*} : H_i(X_n) \to H_i(X)$ is a monomorphism for all i. Therefore $p_r\phi_n'$ is a homotopy equivalence. By Lemma 4.1, there exists a comultiplication $\phi_n : X_n \to X_n \vee X_n$ such that $(j_n \vee j_n)\phi_n = \phi j_n$. From the construction of ϕ_n in Lemma 4.1, it immediately follows that $(i_n \vee i_n)\phi_n = \phi_{n+1}i_n$.

Now assume that X is 2-connected or $H_{n+1}(X)$ is free-abelian and write M_n for $M(H_{n+1}(X), n)$. Then there exists $\hat{\phi}_n : M_n \to M_n \vee M_n$ such that $(k_n \vee k_n)\hat{\phi}_n = \phi_n k_n$. Let $r_i : M_n \vee M_n \to M_n$, $p_i : X_n \vee X_n \to X_n$ and $q_i : X_{n+1} \vee X_{n+1} \to X_{n+1}$ be projections and consider the commutative diagram

$$\begin{array}{ccccc} M_n & \xrightarrow{k_n} & X_n & \xrightarrow{i_n} & X_{n+1} \\ \downarrow{\scriptstyle r_i\hat{\phi}_n} & & \downarrow{\scriptstyle p_i\phi_n} & & \downarrow{\scriptstyle q_i\phi_{n+1}} \\ M_n & \xrightarrow{k_n} & X_n & \xrightarrow{i_n} & X_{n+1}. \end{array}$$

Since $p_i\phi_n = id$ and $q_i\phi_{n+1} = id$, it follows that $r_i\hat{\phi}_n$ is a homotopy equivalence. By Lemma 4.1, there exists a comultiplication μ_n of M_n such that $k_n : (M_n, \mu_n) \to (X_n, \phi_n)$ is a co-H-map. Then μ_n is the canonical comultiplication of M_n (see Remark 4.4 (2)). ∎

COROLLARY 4.3. *If X is a co-H-space and $\{X_n;\, j_n, k_n, i_n\}$ is a homology decomposition of X, then there is a comultiplication on each nth homology section X_n such that*

$j_n : X_n \to X$ and $i_n : X_n \to X_{n+1}$ are co-H-maps. If either X is 2–connected or X has no torsion in its homology, then all dual invariants $k_n : M(H_{n+1}(X), n) \to X_n$ are co-H-maps.

Remarks 4.4. (1) The first assertion of Corollary 4.3 was originally proved in [Cu_2, Lem. 2.3] (see also [B-H_1] and [G-K]). Moreover, Berstein and Hilton proved in [B-H_1, §3] the analogous result for spaces of cat $\leq n$. The second assertion of Corollary 4.3 for 2–connected spaces was proved by Golasiński and Klein in [G-K, Cor. 3] by different methods.

(2) A Moore space $M(G, n)$ has a unique comultiplication for $n \geq 3$. However, the comultiplications on $M(G, 2)$ are in one-one correspondence with $\mathrm{Ext}(G, G \otimes G)$. Thus if G is free-abelian, $M(G, 2)$ has a unique comultiplication. For more details on comultiplications on Moore spaces, see [A-G].

(3) It would be interesting to know if the second assertion of Corollary 4.3 is true with a weaker hypothesis or even without any restrictions. In this connection we note that the first dual invariant of a co-H-space is always a co-H-map. Because this result is limited and the proof is long, we just state it: Let (X, ϕ) be a co-H-space, $\{X_n;\, j_n, k_n, i_n\}$ a homology decomposition of X and $\phi_n : X_n \to X_n \vee X_n$ the induced comultiplication. Then there exists a comultiplication ψ_2 on $M(H_3(X), 2)$ such that $k_2 : (M(H_3(X), 2), \psi_2) \to (X_2, \phi_2)$ is a co-H-map.

We conclude the paper by giving a new proof of a basic result on co-H-spaces which is due to Berstein and Hilton. We base our proof on Theorem 4.2 and another result of Berstein and Hilton which we now state.

Theorem B [B-H_2, Thm. B]. *If A and B are spaces such that dimension $A \leq 3q - 2$ and B is $(q-1)$-connected, $q \geq 1$, then every co-H-map $\Sigma A \to \Sigma B$ is a suspension.*

The following theorem appears in [B-H_2, Thm. A].

Theorem 4.5. *If X is a $(q-1)$-connected CW-complex of dim $\leq 3q - 3$, $q \geq 1$, and ϕ is a comultiplication of X, then (X, ϕ) is co-H-equivalent to a suspension.*

Proof. For notational convenience we write H_i for $H_i(X)$. The case $q = 1$ is trivial and so we first consider the case $q = 2$. Then X is a 1-connected complex of dimension ≤ 3. Thus H_3 is free-abelian and so by Theorem 4.2, $k_2 : M(H_3, 2) \to X_2 = M(H_2, 2)$ is a co-H-map (this also follows from the result stated in Remark 4.4 (3)). By Theorem B above, k_2 is a suspension, and so $X = X_3$ is co-H-equivalent to a suspension. Now assume $q \geq 3$ so that X is 2-connected. We let $\{X_n;\, j_n, k_n, i_n\}$ be a homology decomposition for X with $X_{q-1} = *$, $X_q = M(H_q, q)$ and $X_{3q-3} = X$. Then by Corollary 4.3, all X_n are co-H-spaces and all j_n, i_n and k_n are co-H-maps. We prove by induction on i that X_i is a suspension, $i = q, \ldots, 3q-3$. Clearly this is true for $i = q$. Now suppose that $X_i = \Sigma X'_i$ for some space X'_i and consider $k_i : \Sigma M(H_{i+1}, i-1) \to \Sigma X'_i$. If $i < 3q-4$, then $\dim M(H_{i+1}, i-1) \leq i \leq 3q-5$. Now let $i = 3q-4$. Then $H_{i+1} = H_{3q-3}$ is free-abelian, and so $\dim M(H_{3q-3}, 3q-5) \leq 3q-5$. Thus $\dim M(H_{i+1}, i-1) \leq 3q-5$ for all $i \leq 3q-4$. Therefore we apply Theorem B to conclude that k_i is a suspension and so X_{i+1} is co-H-equivalent to a suspension. This completes the induction. ∎

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