

THE GEOMETRY OF A CLOSED FORM

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Abstract. It is proved that a closed r -form ω on a manifold M defines a cohomology (called ω -coeffective) on M . A general algebraic machinery is developed to extract some topological information contained in the ω -coeffective cohomology. The cases of 1-forms, symplectic forms, fundamental 2-forms on almost contact manifolds, fundamental 3-forms on G_2 -manifolds and fundamental 4-forms in quaternionic manifolds are discussed.

1. Introduction. As is well-known the existence of a particular geometric structure on a manifold M imposes often some topological properties on it. For instance, if ω is a symplectic form on a compact manifold M , then M has to be even dimensional, and in addition its even Betti numbers do not vanish. Moreover, a symplectic form defines a special subcomplex of the de Rham complex $(\Lambda^*(M), d)$ of forms on M : it consists of those forms α which are annihilated by ω , that is, $\alpha \wedge \omega = 0$. Since ω is closed, we have in fact a subcomplex of $(\Lambda^*(M), d)$ whose cohomology is called coeffective. This cohomology was introduced and studied by Bouché [6] and related with the truncated de Rham cohomology by the class $[\omega]$. For Kähler manifolds both cohomologies are isomorphic for $k \neq n$, $\dim M = 2n$, though in general they are different for non Kähler symplectic manifolds [1, 14]. A Nomizu's type theorem [28] was also proved in [13] and it was shown in [15] that the coeffective cohomology of a symplectic manifold of finite type is finite, so that we have introduced the coeffective numbers of the symplectic manifold and several

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inequalities relating them to the Betti numbers. Similar results were obtained in the context of almost contact [10, 16] and quaternionic manifolds [17].

The above results lead us to consider the following problem. Let ω be a closed r -form on a manifold M , and define the corresponding ω -coeffective cohomology. This can be done since the closedness of ω implies that the operator $L_\omega : \alpha \rightsquigarrow \alpha \wedge \omega$ and the exterior derivative d commute, so we obtain a subcomplex $(\mathcal{A}_\omega^*(M), d)$ of the de Rham complex of M , where $\mathcal{A}_\omega^k(M) = \ker \{L_\omega : \Lambda^k(M) \longrightarrow \Lambda^{k+r}(M)\}$. The resulting cohomology $H_\omega^*(\mathcal{A}(M))$ is called the ω -coeffective cohomology. The point now is to relate it to the de Rham cohomology, or, in general, to extract topological information from the existence of that closed form ω . In some sense, we are studying the geometry provided by ω . We will use a general technique which consists in considering the five-term exact sequences associated with the long exact sequence induced by passing to cohomology from the exact short sequence

$$0 \longrightarrow \ker L \longrightarrow \Lambda^*(M) \longrightarrow \text{Im } L \longrightarrow 0.$$

However, to go further, we need some particular properties of the form. In this paper we collect a list of results concerning 1-forms, symplectic forms and fundamental forms on almost contact manifolds, G_2 -manifolds and quaternionic manifolds. A more complete program in the Riemannian context will require the study of the fundamental forms in other G -structures according to Berger's list (see [2]). Another direction in a more general context will be to consider the coeffective cohomology defined for multisymplectic forms [8].

2. Coeffective cohomology. Let M be a real m -dimensional C^∞ differentiable manifold, $\mathfrak{X}(M)$ the Lie algebra of C^∞ vector fields on M and $\Lambda^k(M)$ the space of C^∞ k -forms on M .

If d denotes the exterior derivative on M , then we have the de Rham differential complex

$$\dots \longrightarrow \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \longrightarrow \dots$$

whose cohomology $H^*(M)$ is the de Rham cohomology of M .

Consider now a closed r -form ω on M , and define an operator

$$L_\omega : \Lambda^k(M) \longrightarrow \Lambda^{k+r}(M)$$

by

$$L_\omega(\alpha) = \alpha \wedge \omega.$$

Then the space

$$\mathcal{A}_\omega^k(M) = \ker \{L_\omega : \Lambda^k(M) \longrightarrow \Lambda^{k+r}(M)\}$$

is called the subspace of ω -coeffective forms on M . Since ω is closed, L_ω and d commute, which implies that

$$\dots \longrightarrow \mathcal{A}_\omega^{k-1}(M) \xrightarrow{d} \mathcal{A}_\omega^k(M) \xrightarrow{d} \mathcal{A}_\omega^{k+1}(M) \longrightarrow \dots$$

is a differential subcomplex of the de Rham complex. Its cohomology $H_\omega^k(\mathcal{A}(M))$ is called ω -coeffective cohomology of M . If this cohomology is finite, we define the *coeffective numbers* (associated to ω) by $c_k(M, \omega) = \dim H_\omega^k(\mathcal{A}(M))$.

3. Exact sequences and coeffective cohomology. The aim of this section is to relate the coeffective cohomology to the de Rham cohomology by means of a long exact sequence in cohomology.

Consider the following natural short exact sequence for any degree k :

$$0 \longrightarrow \ker L_\omega = \mathcal{A}_\omega^k(M) \xrightarrow{i} \Lambda^k(M) \xrightarrow{L_\omega} \text{Im}^{k+r} L_\omega \longrightarrow 0 \quad (1)$$

Since L_ω and d commute, (1) becomes a short exact sequence of differential complexes:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \longrightarrow & \text{Im}^{k+r-1} L_\omega & \xrightarrow{d} & \text{Im}^{k+r} L_\omega & \xrightarrow{d} & \text{Im}^{k+r+1} L_\omega \longrightarrow \cdots \\
 & & \uparrow L_\omega & & \uparrow L_\omega & & \uparrow L_\omega \\
 \cdots & \longrightarrow & \Lambda^{k-1}(M) & \xrightarrow{d} & \Lambda^k(M) & \xrightarrow{d} & \Lambda^{k+1}(M) \longrightarrow \cdots \\
 & & \uparrow i & & \uparrow i & & \uparrow i \\
 \cdots & \longrightarrow & \mathcal{A}_\omega^{k-1}(M) & \xrightarrow{d} & \mathcal{A}_\omega^k(M) & \xrightarrow{d} & \mathcal{A}_\omega^{k+1}(M) \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Therefore, we can consider the associated long exact sequence in cohomology:

$$\cdots \longrightarrow H_\omega^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L_\omega)} H^{k+r}(\text{Im } L_\omega) \xrightarrow{C_{k+r}} H_\omega^{k+1}(\mathcal{A}(M)) \longrightarrow \cdots \quad (2)$$

where $H(i)$ and $H(L_\omega)$ are the homomorphisms induced in cohomology by i and L_ω , respectively, and C_{k+r} is the connecting homomorphism defined in the following way: if $[\alpha] \in H^{k+r}(\text{Im } L_\omega)$, then $C_{k+r}[\alpha] = [d\beta]$, for $\beta \in \Lambda^k(M)$ such that $L_\omega\beta = \alpha$.

4. Closed 1-forms. Assume that ω is a closed 1-form without zeros. Therefore, we have

$$H_\omega^0(\mathcal{A}(M)) \cong \{0\}.$$

Moreover, since $\alpha \wedge \omega = 0$ implies that $\alpha = \beta \wedge \omega$ we deduce that

$$\ker \{L_\omega : \Lambda^k(M) \longrightarrow \Lambda^{k+1}(M)\} = \text{Im} \{L_\omega : \Lambda^{k-1}(M) \longrightarrow \Lambda^k(M)\}$$

If M is of finite type, then the Betti numbers of M , $b_k(M) = \dim H^k(M)$ are finite. However, the coeffective cohomology groups are not finite dimensional in general. Indeed, consider the 1-form $\omega = dx$ on \mathbb{R}^3 with standard coordinates x, y, z . A direct computation shows that

$$H_\omega^1(\mathcal{A}(\mathbb{R}^3)) \cong C^\infty(\mathbb{R}).$$

Decompose the long exact sequence (2) in the following short exact sequences:

$$0 \rightarrow \text{Im } H(i) = \ker H(L_\omega) \xrightarrow{i} H^k(M) \xrightarrow{H(L_\omega)} \text{Im } H(L_\omega) \rightarrow 0.$$

Then we deduce the formula:

$$\dim \ker H(L_\omega) - b_k(M) + \dim \text{Im } H(L_\omega) = 0. \tag{3}$$

From (3) we obtain the following result.

PROPOSITION 4.1. *Let M be a differentiable manifold with a closed 1-form ω without zeros. Assume that the ω -coeffective cohomology is finite. Then we have*

$$b_k(M) \leq c_k(M, \omega) + c_{k+1}(M, \omega), \quad \text{for all } k.$$

PROBLEM 4.2. What are the coeffective numbers if ω satisfies some extra geometric properties, for instance, ω is parallel with respect to a Riemannian metric?

5. Closed 2-forms: symplectic and almost contact manifolds

5.1. *Symplectic manifolds.* Let M be a $2n$ -dimensional symplectic manifold with symplectic form ω . From Lepage’s decomposition [23] we deduce that L_ω is injective for $k \leq n - 1$ and surjective for $k \geq n - 1$. As a consequence we deduce that $\mathcal{A}_\omega^k(M) = \{0\}$, for $k \leq n - 1$, and hence $H_\omega^k(\mathcal{A}(M)) = \{0\}$, for $k \leq n - 1$. Also, we have $H^{k+2}(\text{Im } L_\omega) = H^{k+2}(M)$, for $k \geq n$.

Now, we decompose the long exact sequence (2) in 5-term exact sequences:

$$0 \rightarrow \ker H(i) = \text{Im } C_{k+1} \xrightarrow{i} H_\omega^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L_\omega)} H^{k+2}(M) \xrightarrow{C_{k+2}} \text{Im } C_{k+2} \rightarrow 0$$

THEOREM 5.1 [15]. *Let M be a symplectic $2n$ -dimensional manifold of finite type. Then the coeffective cohomology group $H_\omega^k(\mathcal{A}(M))$ has finite dimension, for $k \geq n + 1$. Furthermore,*

$$b_k(M) - b_{k+2}(M) \leq c_k(M, \omega) \leq b_k(M) + b_{k+1}(M).$$

REMARK 5.2. In general, the intermediate coeffective number $c_n(M, \omega)$ is not finite. A simple counterexample is the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n} / \mathbb{Z}^{2n}$ with its canonical Kähler form.

In [15] we have proved the following results.

THEOREM 5.3.

(i) *Let M be a compact Kähler manifold of dimension $2n$. Then we have*

$$c_k(M, \omega) = b_k(M) - b_{k+2}(M), \quad k \geq n + 1, \tag{4}$$

where ω is the Kähler form.

(ii) *Let M be a (non-compact) exact symplectic manifold of dimension $2n$ and of finite type. Then*

$$c_k(M, \omega) = b_k(M) + b_{k+1}(M), \quad \text{for } k \geq n + 1.$$

Proof. (i) follows since C_{k+2} identically vanishes, by using the Lepage’s decomposition for harmonic forms. To prove (ii) it is sufficient to notice that $H(L_\omega)$ identically vanishes. ■

Since the Betti numbers are topological invariants for compact manifolds, the coeffective cohomology groups are topological invariants for compact Kähler manifolds. Moreover, they measure the jumps of the Betti numbers.

In [15] we have proved, using the Moser Stability Theorem [27, 26, 36], that the coeffective cohomology does not depend on ω , but on the isotopy class of ω . We recall that two symplectic forms ω_0 and ω_1 are said to be isotopic if they can be joined by a continuous 1-parameter family of cohomologous symplectic structures ω_t , $0 \leq t \leq 1$. Actually, we can prove something stronger, that is, the coeffective cohomology depends only on the de Rham cohomology class of ω .

PROPOSITION 5.4. *Let (M, ω) be a symplectic manifold of finite type. Then the coeffective cohomology groups $H_\omega^k(\mathcal{A}(M))$ are invariants of the de Rham cohomology class of ω , for $k \geq n + 1$.*

Proof. We decompose the long exact sequence (2) in 5-term exact sequences,

$$0 \rightarrow \ker C_{k+1} = \text{Im } H(L_\omega^{k-1}) \xrightarrow{i} H^{k+1}(M) \xrightarrow{C_{k+1}} H_\omega^k(\mathcal{A}(M)) \xrightarrow{H^{(i)}} H^k(M) \xrightarrow{H(L_\omega)} \text{Im } H(L_\omega^k) \rightarrow 0$$

for $k \geq n + 1$. Then, we have

$$c_k(M, \omega) = b_k(M) + b_{k+1}(M) - \dim \text{Im } H(L_\omega^{k-1}) - \dim \text{Im } H(L_\omega^k). \blacksquare$$

COROLLARY 5.5. *If the mapping L_ω is also surjective in cohomology, that is, the mapping $H(L_\omega) : H^{k-1}(M) \rightarrow H^{k+1}(M)$ is surjective for $k \geq n + 1$, then*

$$c_k(M, \omega) = b_k(M) - b_{k+2}(M),$$

and in such a case, $b_k(M) \geq b_{k+2}(M)$.

COROLLARY 5.6. $c_k(M, \omega) \geq b_{k+1}(M) - b_{k-1}(M)$, for $k \geq n + 1$.

As is well-known, a compact Kähler manifold is formal [11]. On the other hand, its coeffective numbers are topological invariants and satisfy (4). Moreover, a compact Kähler manifold satisfies the Hard Lefschetz theorem. Lupton and Oprea have suggested [24] the following problem.

PROBLEM 5.7. Let M be a compact symplectic manifold which is formal or satisfies the Hard Lefschetz theorem. Is (4) satisfied then?

The answer to Problem 5.7 in the case of compact symplectic manifolds satisfying the Hard Lefschetz theorem is affirmative. Concretely, for a compact symplectic manifold (M^{2n}, ω) satisfying the Hard Lefschetz theorem, that is,

$$H(L_\omega)^r : H^{n-r}(M) \rightarrow H^{n+r}(M)$$

is an isomorphism, we observe that the mapping $H(L_\omega) : H^{k-1}(M) \rightarrow H^{k+1}(M)$ is surjective for $k \geq n + 1$, therefore, from Corollary 5.5 the equality (4) is satisfied.

Some evidence for Problem 5.7 (in the formal case) is given by the following results.

i) Complex projective spaces. As we know, $\mathbb{C}\mathbb{P}^n$ is formal [34]. On the other hand, the cohomology ring of $\mathbb{C}\mathbb{P}^n$ is $H^*(\mathbb{C}\mathbb{P}^n) = \mathbb{R}[y]/\langle y^{n+1} \rangle$, where the degree of y is two.

Then, for any symplectic structure ω , the mapping $H(L_\omega)$ is always surjective. Therefore, Corollary 5.5 implies (4).

ii) Compact locally symmetric spaces. A compact locally symmetric space is formal [20, 34]. Moreover, we have the following.

PROPOSITION 5.8. *Let (M, g) be a compact Riemannian (locally) symmetric space with harmonic symplectic structure. Then*

$$c_k(M, \omega) = b_k(M) - b_{k+2}(M) \quad \text{for } k \geq n + 1,$$

therefore, $b_k(M) \geq b_{k+2}(M)$.

PROOF. Since the wedge product of harmonic forms is harmonic [20], using Hodge theorem and the surjectivity of L_ω , we obtain that the mapping $H(L_\omega)$ is surjective, so we conclude by Corollary 5.5. ■

We propose to study the equality (4) for the following formal spaces:

- iii) homogeneous spaces of maximal rank [34],*
- iv) H -spaces (for example, Lie groups) [34].*

Also, it is interesting the study of simply connected symplectic manifolds, which is related to the Lupton-Oprea conjecture [25, 34]: any simply connected symplectic manifold is formal.

REMARK 5.9. Bouché [6] has considered the de Rham cohomology groups truncated by the class of the symplectic form ω , that is,

$$\tilde{H}^k(M) = \{a \in H^k(M) \mid a \wedge [\omega] = 0\} = \ker H(L_\omega).$$

For a compact Kähler manifold of dimension $2n$, there exists an isomorphism

$$H^k(\mathcal{A}(M)) \cong \tilde{H}^k(M), \quad k \geq n + 1. \quad (5)$$

Thus, Bouché [6] proposed the following problem: is the isomorphism (5) satisfied for any compact symplectic manifold? We answered this question in the negative by constructing a counterexample [1, 13]. A crucial point here was the proof of a Nomizu's type theorem [13] to compute the coeffective cohomology of a compact symplectic nilmanifold at its Lie algebra level.

In relationship with Bouché's problem we have the following result which is a direct consequence of the long exact sequence in cohomology (2).

PROPOSITION 5.10. *Let (M, ω) be a symplectic manifold of dimension $2n$ and of finite type. Then the following assertions are equivalent:*

- (i) $H^k(\mathcal{A}(M)) \cong \tilde{H}^k(M)$, for $k \geq n + 1$;*
- (ii) $H(L_\omega) : H^{k-1}(M) \rightarrow H^{k+1}(M)$ is surjective for $k \geq n + 1$.*

Moreover, in such a case, the equality (4) is satisfied.

REMARK 5.11. Although originally the coeffective cohomology was introduced for symplectic structures, it can be also defined for locally conformal symplectic (l.c.s.) structures [35]. Moreover, in both cases, $(\text{Im } L_\omega, d)$ is a differential complex too. The difference

between both situations is that for l.c.s. structures the short exact sequence (1) is not a short exact sequence of differential complexes. We have proved the following result.

PROPOSITION 5.12. *Let (M, g, J) be an almost Hermitian manifold of dimension $2n$ with Kähler form ω . If $(\text{Im } L_\omega, d)$ is a differential complex, then M is symplectic or l.c.s.*

REMARK 5.13. For indefinite Kähler manifolds a coeffective Dolbeault cohomology was introduced and discussed in [21].

5.2. *Almost contact manifolds.* An almost contact metric structure (ϕ, ξ, η, g) on a $(2n+1)$ -dimensional manifold M is given by a Riemannian metric g and an almost contact structure (ϕ, ξ, η) such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

The associated fundamental 2-form Φ is defined by

$$\Phi(X, Y) = g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

It follows that $\Phi^n \wedge \eta \neq 0$. If $d\Phi = d\eta = 0$, it defines an *almost cosymplectic structure*, and if $\Phi = d\eta$, it defines a *contact structure* [3].

Let (ϕ, ξ, η, g) be an almost contact metric structure on a $(2n + 1)$ -dimensional manifold M . It is called [3, 29]:

- i) *normal* if $[\phi, \phi] + 2d\eta \otimes \xi = 0$.
- ii) *cosymplectic* if it is normal and almost cosymplectic,
- iii) *quasi-Sasakian* if it is normal and Φ is closed,
- iv) *Sasakian* if it is contact and normal.

It was shown in [9] that L_Φ is injective for $k \leq n - 1$, and surjective for $k \geq n$. Therefore, if Φ is closed, we deduce that $\mathcal{A}_\Phi^k(M) = \{0\}$, for $k \leq n - 1$, and then $H_\Phi^k(\mathcal{A}(M)) = \{0\}$, for $k \leq n - 1$. Also, $H^{k+2}(\text{Im } L_\Phi) = H^{k+2}(M)$, for $k \geq n + 1$.

PROPOSITION 5.14. *Let M be a $(2n + 1)$ -dimensional almost contact metric manifold with closed fundamental 2-form Φ and of finite type. Then*

- (i) *the coeffective numbers $c_k(M, \Phi)$ are finite dimensional, for $k \geq n + 2$;*
- (ii) *the coeffective cohomology groups are invariants of the de Rham cohomology class of Φ , for degrees $k \geq n + 2$.*

THEOREM 5.15 [16].

- (i) *For an almost contact manifold of dimension $2n + 1$ with closed fundamental form, say $d\Phi = 0$, and of finite type, we have*

$$b_k(M) - b_{k+2}(M) \leq c_k(M, \Phi) \leq b_k(M) + b_{k+1}(M), \quad \text{for } k \geq n + 2.$$

- (ii) *If M is a compact cosymplectic manifold, then*

$$c_k(M, \Phi) = b_k(M) - b_{k+2}(M), \quad \text{for } k \geq n + 2. \tag{6}$$

- (iii) *If M is contact, then*

$$c_k(M, \Phi) = b_k(M) + b_{k+1}(M), \quad \text{for } k \geq n + 2.$$

(iv) Let M be a (non-compact) exact almost cosymplectic manifold of dimension $2n+1$. Then

$$c_k(M, \Phi) = b_k(M) + b_{k+1}(M), \quad \text{for } k \geq n+2.$$

Therefore, the coeffective cohomology groups are topological invariants for compact cosymplectic manifolds. Moreover, since

$$b_k(M) \geq b_{k+2}(M), \quad k \geq n+1,$$

for compact cosymplectic manifolds [10], then the coeffective numbers measure the jumps of the Betti numbers.

PROBLEM 5.16. Since a compact cosymplectic manifold is formal [9] and satisfies the Hard Lefschetz theorem [9], we can ask again about the relation of the equality (6) to the formality and the Hard Lefschetz theorem.

REMARK 5.17. The relation between the coeffective cohomology and the truncated de Rham cohomology was discussed in [10, 13].

6. Closed 3-forms: calibrated G_2 -manifolds. A G_2 -manifold is a 7-dimensional Riemannian manifold with a 2-fold vector cross product [7]. Such a manifold M has a nowhere vanishing differential 3-form φ , which is called the fundamental 3-form of M . A G_2 -manifold M is said to be: *calibrated* if φ is closed (i.e. $d\varphi = 0$), *cocalibrated* if φ is coclosed (i.e. $d*\varphi = 0$), *nearly parallel* if $d*\varphi = 0$ and $d\varphi = k*\varphi$, where $k \in \mathbb{R}$, and *parallel* if $\nabla\varphi = 0$ (i.e. $d\varphi = 0, d*\varphi = 0$).

Fernández and Gray [12] have proved that $\Lambda^k(M)$ splits orthogonally into components $\Lambda_l^k(M)$ of dimension l , which are irreducible under the action of G_2 :

$$\Lambda^1(M), \quad \Lambda^2(M) = \Lambda_7^2(M) \oplus \Lambda_{14}^2(M), \quad \Lambda^3(M) = \Lambda_1^3(M) \oplus \Lambda_7^3(M) \oplus \Lambda_{27}^3(M),$$

and the others are obtained from these using the Hodge star operator $*$. These spaces are:

$$\begin{aligned} \Lambda_7^2(M) &= \{*(\alpha \wedge *\varphi) \mid \alpha \in \Lambda^1(M)\}, \\ \Lambda_{14}^2(M) &= \{\beta \in \Lambda^2(M) \mid \beta \wedge *\varphi = 0\}, \\ \Lambda_1^3(M) &= \{f\varphi \mid f \in C^\infty(M)\}, \\ \Lambda_7^3(M) &= \{*(\alpha \wedge \varphi) \mid \alpha \in \Lambda^1(M)\}, \\ \Lambda_{27}^3(M) &= \{\gamma \in \Lambda^3(M) \mid \gamma \wedge *\varphi = \gamma \wedge \varphi = 0\}. \end{aligned}$$

For some classes of G_2 -manifolds one can define coeffective complexes (for example, with respect to φ and $*\varphi$) and other related complexes [32, 31, 18, 19]. In this and the following section we will recall some results on these complexes.

Now, let M be a calibrated G_2 -manifold. Then one can consider the coeffective complex with respect to the closed 3-form φ . The spaces of φ -coeffective forms on M are:

$$\begin{aligned} \mathcal{A}_\varphi^i(M) &= \{0\}, \quad i = 0, 1, 2, \\ \mathcal{A}_\varphi^3(M) &= \Lambda_1^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{A}_\varphi^4(M) &= \Lambda_7^4(M) \oplus \Lambda_{27}^4(M), \\ \mathcal{A}_\varphi^i(M) &= \Lambda^i(M), \quad i = 5, 6, 7. \end{aligned}$$

Also, we can consider the complementary complex $(\mathcal{B}(M), \overset{\vee}{d})$, where $\overset{\vee}{d}$ is the orthogonal projection of d , so that we have the short exact sequence of differential complexes

$$0 \longrightarrow (\mathcal{A}(M), d) \xrightarrow{i} (\Lambda(M), d) \xrightarrow{p} (\mathcal{B}(M), \overset{\vee}{d}) \longrightarrow 0$$

where i and p denote the natural inclusion and the orthogonal projection, respectively. Therefore we have also the long exact sequence in cohomology (for a more complete description see [19]).

We have the following results.

PROPOSITION 6.1 [19].

- (i) The coeffective complex $(\mathcal{A}_\varphi(M), d)$ is elliptic in degree k (for $k \neq 2$);
- (ii) the complementary complex $(\mathcal{B}(M), \overset{\vee}{d})$ is elliptic in degree k (for $k \neq 3$);
- (iii) in the compact case, the cohomology groups of both complexes are finite dimensional in the ellipticity degrees;
- (iv) $c_i(M, \varphi) = 0$ ($i = 0, 1, 2, 3$) and $c_i(M, \varphi) = b_i(M)$ ($i = 5, 6$);

THEOREM 6.2 [19]. If M is a compact parallel G_2 -manifold, then

$$c_4(M, \varphi) = b_4(M) - b_7(M), \quad c_5(M, \varphi) = b_5(M).$$

REMARK 6.3. In general, for any G_2 -manifold the sequences $(\mathcal{A}_\varphi(M), d)$ and $(\mathcal{B}(M), \overset{\vee}{d})$ are differential complexes if and only if M is a locally conformal calibrated G_2 -manifold (see [19]).

7. Closed 4-forms: cocalibrated G_2 and quaternionic manifolds

7.1. *Cocalibrated G_2 -manifolds.* Let M be a cocalibrated G_2 -manifold. Then one can consider the coeffective complex with respect to the 4-form $*\varphi$. The spaces of $(*\varphi)$ -coeffective forms are:

$$\begin{aligned} \mathcal{A}_{*\varphi}^i(M) &= \{0\}, \quad i = 0, 1, \\ \mathcal{A}_{*\varphi}^2(M) &= \Lambda_{14}^2(M), \\ \mathcal{A}_{*\varphi}^3(M) &= \Lambda_7^3(M) \oplus \Lambda_{27}^3(M), \\ \mathcal{A}_{*\varphi}^i(M) &= \Lambda^i(M), \quad i = 4, 5, 6, 7. \end{aligned}$$

The coeffective complex is the complementary of the differential complex defined by S. Salamon [32] and whose ellipticity was studied by R. Reyes [31]:

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{\overset{\vee}{D}} \Lambda^1(M) \xrightarrow{\overset{\vee}{D}} \Lambda_7^2(M) \xrightarrow{\overset{\vee}{D}} \Lambda_1^3(M) \longrightarrow 0$$

where the differential $\overset{\vee}{D}$ is the orthogonal projection of d . Again, we can consider the associated exact sequences (for a complete description see [18]).

We have the following results.

PROPOSITION 7.1 [18].

- (i) $c_i(M, *\varphi) = 0$, ($i = 0, 1$) and $c_i(M, *\varphi) = b_i(M)$, ($i \geq 4$);

- (ii) if M is a compact nearly-parallel non-parallel G_2 -manifold, then $c_2(M, *\varphi) = b_2(M)$;
- (iii) if M is a compact parallel G_2 -manifold we get $c_2(M, *\varphi) = b_2(M) - b_1(M)$, $c_3(M, *\varphi) = b_3(M) - b_0(M) \geq b_1(M)$, and then

$$b_2(M) \geq b_1(M), \quad b_3(M) \geq b_0(M) + b_1(M).$$

REMARK 7.2. The above complexes can even be defined for integrable G_2 -manifolds, that is, locally conformal cocalibrated G_2 -manifolds (see [18]).

7.2. *Quaternionic manifolds.* A $4n$ -dimensional manifold M is said to be *almost quaternionic* if there is a subbundle \mathcal{Q} of $EndTM$ such that for each $x \in M$ there is a neighborhood U over which $\mathcal{Q}|_U$ has a basis J_1, J_2, J_3 satisfying

$$J_1^2 = J_2^2 = -1 \quad \text{and} \quad J_3 = J_1J_2 = -J_2J_1.$$

A Riemannian metric g on M is said to be compatible with \mathcal{Q} if $g(AX, AY) = g(X, Y)$, for each $X, Y \in T_xM$ and $A \in \mathcal{Q}_x$ such that $A^2 = -1$. A compatible metric g can be constructed from any Riemannian metric g' by putting

$$g(X, Y) = \frac{1}{4}(g'(X, Y) + g'(J_1X, J_1Y) + g'(J_2X, J_2Y) + g'(J_3X, J_3Y)).$$

An almost quaternionic manifold with a compatible metric is called a *quaternionic Hermitian manifold*.

We may embed \mathcal{Q} isometrically in Λ^2T^*M by mapping an element $A \in \mathcal{Q}$ to the 2-form ω_A defined by

$$\omega_A(X, Y) = g(X, AY), \text{ for } X, Y \in T_xM.$$

This enables us to define a global, non-degenerate 4-form Ω , the fundamental 4-form [22, 4], by

$$\Omega = \omega_{J_1} \wedge \omega_{J_1} + \omega_{J_2} \wedge \omega_{J_2} + \omega_{J_3} \wedge \omega_{J_3},$$

where J_1, J_2, J_3 is a local basis as above. A quaternionic Hermitian manifold M is said to be:

- i) *quaternionic almost Kähler* if $d\Omega = 0$,
- ii) *quaternionic Kähler* if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection.

In dimension 4 both conditions are trivially satisfied. Furthermore, for any dimension $4n$, the condition $\nabla\Omega = 0$ implies the condition $d\Omega = 0$, and Swann [33] has proved that the converse also holds for dimension $4n \geq 12$.

A quaternionic Kähler manifold is said to be *hyperkähler* if the almost complex structures J_1, J_2, J_3 are globally defined and

$$d\omega_{J_1} = d\omega_{J_2} = d\omega_{J_3} = 0.$$

Hitchin (see [33]) has shown that this implies that J_1, J_2, J_3 are integrable, and hence they define three Kähler structures on M . If M is a quaternionic Kähler manifold with zero scalar curvature it can be shown that M is locally hyperkähler.

Let (M, \mathcal{Q}, g) be a quaternionic almost Kähler manifold of dimension $4n$ with fundamental 4-form Ω .

In [5, 17] it was shown that the mapping $L_\Omega : \Lambda^k(M) \rightarrow \Lambda^{k+4}(M)$ is injective for $k \leq 2n - 2$ and surjective for $k \geq 2n - 2$. Moreover, its adjoint Λ with respect to the Riemannian metric g is injective for $k \geq 2n + 2$ and surjective for $k \leq 2n + 2$. As a consequence, we have $H^k(\mathcal{A}(M)) = \{0\}$, for $k \leq 2n - 2$.

Since $L_\Omega(\Lambda^k(M)) = \Lambda^{k+4}(M)$, for $k \geq 2n - 2$ we deduce that

$$H^{k+4}(\text{Im } L) = H^{k+4}(M),$$

for $k \geq 2n - 1$. Then proceeding as in the precedent sections we obtain the following result.

PROPOSITION 7.3. *Let M be a $4n$ -dimensional quaternionic almost Kähler manifold of finite type. Then the coeffective numbers $c_k(M, \Omega)$ are finite, for $k \geq 2n$. Also, the coeffective numbers are invariants of the de Rham cohomology class of Ω .*

Moreover, we also deduce

THEOREM 7.4.

(i) *Let M be a $4n$ -dimensional quaternionic almost Kähler of finite type. Then*

$$b_k(M) - b_{k+4}(M) \leq c_k(M, \Omega) \leq b_k(M) + b_{k+3}(M), \quad \text{for } k \geq 2n. \quad (7)$$

(ii) *If M is a compact quaternionic Kähler manifold, then*

$$c_k(M, \Omega) = b_k(M) - b_{k+4}(M), \quad \text{for } k \geq 2n.$$

(iii) *If M is a (non-compact) exact almost quaternionic Kähler manifold of finite type, then*

$$c_k(M, \Omega) = b_k(M) + b_{k+3}(M), \quad \text{for } k \geq 2n.$$

Therefore, the coeffective numbers are bounded by upper and lower limits depending on the Betti numbers of the manifold. A first consequence of (7) is that $c_k(M, \Omega) = b_k(M)$, for $k = 4n - 2, 4n - 1$ and $4n$.

REMARK 7.5. The relationship between the coeffective cohomology and the truncated de Rham cohomology by the fundamental 4-form was discussed in [17].

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