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ON ARNOLD'S CONJECTURE FOR SYMPLECTIC FIXED POINTS

KAORU ONO

Department of Mathematics, Ochanomizu University, Otsuka, Tokyo 112, Japan
Current address: Hokkaido University, Sapporo, 060, Japan
E-mail: ono@math.sci.hokudai.ac.jp

1. Introduction. It is well-known that the number of critical points of a smooth function on a closed manifold is bounded below by the Lusternik-Schnirelmann category of the manifold. If all the critical points are non-degenerate, as a consequence of Morse theory, the number of critical points is at least as great as the sum of the Betti numbers and torsion numbers (namely, the minimal number of generators of the homology groups). Another origin of our story is the pioneering work of Poincaré, which is now called Poincaré's last geometric theorem or the Poincaré-Birkhoff fixed point theorem. Let ϕ be a self-diffeomorphism of an annulus $S^1 \times [0,1]$ which is area and orientation preserving. If ϕ satisfies the twisting condition, i.e., ϕ maps $S^1 \times \{0\}$ and $S^1 \times \{1\}$ to themselves and "rotates" them in opposite directions, then the theorem states that there are at least two fixed points of ϕ (see for instance [2]). Arnold proposed to study the symplectic analog of critical point theory and suggested to explore new fields, now collectively called symplectic topology.

A typical example of a symplectic manifold is the cotangent bundle T^*X of a smooth manifold X. For a function f on X, its differential df gives a lagrangian submanifold in T^*X, namely the graph Γ_{df}. The intersection of Γ_{df} and the zero section O_X can be identified with the critical point set of f, so the estimate mentioned above gives a lower bound on the number of intersection points. One can therefore think of intersections of lagrangian submanifolds as a generalization of critical point sets of functions on a certain space.

Let (M,ω) be a symplectic manifold and L a lagrangian submanifold. A diffeomorphism ϕ is called a symplectomorphism if ϕ preserves the symplectic structure, i.e., $\phi^*\omega = \omega$. If a diffeomorphism ϕ is the time-one map of a flow generated by a time dependent Hamiltonian vector field X_{H_t} (see §2), it is called an exact symplectomorphism.

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By the Cartan formula for the Lie derivative, exact symplectomorphisms are certainly symplectomorphisms. In the case of the graph Γ_{df} in T^*X, it is the image of the zero section by the time-one map of a Hamiltonian flow. For a closed lagrangian submanifold L in (M,ω), there exists a symplectomorphism from a tubular neighborhood of L in M to a tubular neighborhood of O_L in T^*L (Weinstein). Thus for a small exact deformation ϕ, $\phi(L)$ can be identified with the graph of an exact 1-form on L. Here a "small exact deformation" is an exact symplectomorphism whose Hamiltonian function H is C^2-small.

For an exact symplectomorphism ϕ, one may expect the number of intersection points of L and $\phi(L)$ is at least the minimal number of critical points of smooth functions on L and it is at least the minimal number of critical points of Morse functions on L provided that L and $\phi(L)$ intersect transversally. This is not true in general and we need some additional conditions to show such an estimate. For instance, a small circle on a surface can be separated from itself by an exact symplectomorphism.

On the other hand, this estimate is established in some situations. Hofer [14] and Laudenbach and Sikorav [18] established a somewhat weaker estimate for the case of a zero section in the cotangent bundle of a closed manifold. A breakthrough was made by Floer when he introduced a "middle infinite dimensional (co)homology theory" for the symplectic action functional which is now called Floer homology [7]. Floer's construction has now been extended by Oh [25]. Chekanov established an estimate for exact symplectomorphisms with small Hofer norm instead of assuming some condition for the lagrangian submanifold [5].

Since the graph Γ_ϕ of a symplectomorphism ϕ is a lagrangian submanifold in $(M \times M, -\omega \oplus \omega)$, we can consider an analogous problem for fixed points of ϕ. Note that the fixed point set of ϕ is identified with the intersection of Γ_ϕ and the diagonal set Δ_M and that Γ_ϕ is the image of Δ_M by $id \times \phi$. The following problem, called Arnold's conjecture for symplectic fixed points, is the theme of this survey. There are good survey articles [23], [29] up to 1990. We include here recent progress on the topic.

Conjecture. Let M be a closed symplectic manifold and ϕ an exact symplectomorphism. Then the number of fixed points is at least the minimal number of critical points of functions on M. Moreover, if all the fixed points are non-degenerate, i.e., 1 is not an eigenvalue for $d\phi$ at any fixed points of ϕ, then the number is at least the minimal number of critical points of Morse functions on M.

Since this minorant is bounded from below by Lusternik-Schnirelmann category, and so also by the cup-length, in the general case and by the sum of Betti numbers and torsion numbers in the non-degenerate case, we also have a conjecture in terms of these topological invariants. For the estimate of fixed points of an exact symplectomorphism, it has been established in several cases. In the two dimensional case, this conjecture is proven. The result is due to Nikishin, Simon (2-sphere), Conley and Zehnder (torus) and Floer and Sikorav (oriented closed surfaces of higher genus). After these works, Floer gave an estimate for the case that $\pi_2(M) = 0$ using Floer homology. In the non-degenerate case, there is some progress. Floer himself generalized the result to the monotone case [8] and Hofer and Salamon and the author exteneded the argument to the weakly monotone case [15], [26]. Recently, K. Fukaya and the author, G. Liu and G. Tian, Y. Ruan, H.

Hofer and D. Salamon proved that the number is at least the sum of Betti numbers (rank of rational homology). A similar result for non-exact symplectomorphisms symplectic isotopic to the identity is also studied by Lê and the author [19].

For the degenerate case, there are results by Floer, Lê and the author, Schwarz for the cup-length estimate or quantum cup-length estimate [8], [20], [31]. However the conjecture has not yet been proven, even in its weak form, in general. During the present conference, [1] Rudyak and Oprea announced an estimate for the original form of the conjecture in cases including $\pi_2(M) = 0$.

2. Variational set-up. Let (M, ω) be a symplectic manifold and $h : M \to \mathbf{R}$ a smooth function. We define the Hamiltonian vector field X_h of h by $dh + i(X_h)\omega = 0$. For a time-dependent Hamiltonian function $H : M \times \mathbf{R} \to \mathbf{R}$, we have a time-dependent Hamiltonian vector field X_{H_t}, where $H_t(x) = H(x, t)$. Denote by ϕ the time-one map of the flow generated by X_{H_t}. For such a ϕ, the Hamiltonian function H can be chosen as a one-periodic function, i.e., $H(x, t + 1) = H(x, t)$. Then there is a one-to-one correspondence between fixed points of ϕ and one-periodic solutions of the equation

$$\dot{x}(t) = X_{H_t}.$$

This equation is the Euler-Lagrange equation for the following functional on the space of null-homotopic loops:

$$\mathcal{A}_H(x) = -\int_{D^2} u^*\omega + \int_0^1 H(x(t), t),$$

where $u : D^2 \to M$ is an extension of the loop $x : S^1 = \partial D^2 \to M$. The first term of the right hand side may depend on the homotopy class of u bounding the loop x, hence this functional is only defined on a certain covering space of the space of null-homotopic loops. However, we shall first restrict ourselves to the simple situation $\pi_2(M) = 0$ and leave the general case to later discussion. In such a case, $\mathcal{A}_H$ is defined on the space $\mathcal{L}M$ of null-homotopic loops.

If one can extend critical point theory, e.g., Morse theory, to $\mathcal{A}_H$, one may get some lower bound for the number of one-periodic solutions. We note a couple of difficulties arising in this project. The Hessian of $\mathcal{A}_H$ is a first order differential operator which has an infinite number of both positive and negative eigenvalues. Roughly, finite dimensional critical point theory studies the difference in topology of (sub)level sets of a function. However in our setting, "homotopy type" changes by attaching a handle of infinite index which is homotopically trivial when we cross a non-degenerate critical value, so one may not see this difference homotopically.

Another difference is that the equation for the gradient flow is ill-posed as an initial value problem and may not have solutions passing through given loops. Here is a rough comparison between Morse theory and Floer theory.

[1] The author is grateful to J. Oprea and A. Tralle, the organizers of the workshop "Homotopy and Geometry" held at the Banach Center in June 1997, for a stimulating and enjoyable atmosphere.

$f : M \to \mathbf{R}$ Morse function	$\mathcal{A}_H : \mathcal{L}M \to \mathbf{R}$
$Crit(f)$ the critical point set	1-periodic solutions of $\dot{x} = X_{H_t}(x)$
Morse index	Conley-Zehnder index
gradient flow lines connecting critical points	connecting orbits for $\mathcal{A}_H$ between one-periodic solutions
Morse homology	Floer homology

This table is just the beginning of the theory and one may hope for more comparisons (e.g., [11]).

Conley-Zehnder [6] used finite dimensional reduction for $\mathcal{A}_H$ in the case of a torus $\mathbf{R}^{2n}/\mathbf{Z}^{2n}$ and proved the conjecture (see also [4]). The theory of pseudo-holomorphic curves was invented by Gromov who gave many applications in symplectic topology [13]. One of them is the existence of a fixed point of an exact symplectomorphism in the case that $\pi_2(M) = 0$. Floer combined these ideas and constructed "middle infinite dimensional homology theory", which is now called Floer homology. We shall sketch the argument in the case that $\pi_2(M) = 0$.

In order to define the gradient of a function, one needs a Riemannian metric. We need a little preparation to define a metric on $\mathcal{L}M$. Since the unitary group $U(n)$ is a maximal compact subgroup of the group of linear symplectic transformations on a $2n$-dimensional symplectic vector space, the structure group of the tangent bundle of (M, ω) reduces to $U(n)$. Hence, M admits an almost complex structure J.

An almost complex structure J on (M, ω) is called compatible or ω-calibrated, if the bilinear form given by $g_J(u, v) = \omega(u, Jv)$ is a Riemannian metric. By reduction of the structure group, there exist compatible almost complex structures unique up to homotopy. Pick a compatible almost complex structure J, where g_J is a Riemannian metric which induces an L^2-inner product on the tangent space of the loop space. In this way, we get a "metric" on $\mathcal{L}M$.

Computing the gradient of $\mathcal{A}_H$ with respect to this metric, we get

$$grad\ \mathcal{A}_H(x) = J\dot{x} + \nabla\ H_t(x),$$

where $\nabla\ H_t$ is the gradient of H_t with respct to g_J. We also compute formally its Hessian, which is a first order elliptic ordinary differential operator. A critical point is called non-degenerate, if its Hessian is non-degenerate.

In our case, non-degenerate critical points of $\mathcal{A}_H$ are exactly one-periodic solutions such that 1 is not a Floquet multiplier, i.e., 1 is not an eigenvalue of $d\phi$ at any fixed point, where ϕ is the time-one map of the corresponding Hamiltonian flow. In Morse theory, the index of a critical point is the number of negative eigenvalues of Hessian matrix. In our case, there are infinitely many negative eigenvalues (as well as positive eigenvalues). However, the relative index, i.e., the difference of indices at one-periodic solutions, can be defined by the spectral flow [1] of Hessian operators along a path joining two one-periodic solutions. Moreover there is a so-called Conley-Zehnder index [30] so that the relative index is the difference of Conley-Zehnder indices.

Formally, a minus gradient trajectory γ of $\mathcal{A}_H$ is a path $\gamma : \mathbf{R} \to \mathcal{L}M$ satisfying

$$\dot{\gamma} = -grad\ \mathcal{A}_H(\gamma).$$

Interpret the map $\gamma : \mathbf{R} \to \mathcal{L}M$ as $u : \mathbf{R} \times S^1 \to M$. Then a gradient trajectory corresponds to a map u satisfying

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla\ H_t(u) = 0. \tag{1}$$

So far, these computations are just formal. From now on, we take this as the definition of gradient trajectories of $\mathcal{A}_H$, which we call connecting orbits for $\mathcal{A}_H$. Denote by $\widetilde{\mathcal{M}}(x^-, x^+)$ the space of connecting orbits joining x^- to x^+. The equation (1) is invariant under translations in the s variable, hence $\mathbf{R}$ acts on $\widetilde{\mathcal{M}}(x^-, x^+)$. This action is free unless $x^- = x^+$.

As we noticed, we can compare the difference of indices of Hessian operator along a path u joining two one-periodic solutions x^- and x^+. The Atiyah-Patodi-Singer index theorem implies that the spectral flow equals the index of the corresponding elliptic operator on the cylinder with a certain boundary condition. In our situation, the spectral flow equals the index of the linearization of (1) with asymptotic convergence to $x^\pm$ as s tends to $\pm\infty$. Hence the dimension of the moduli space of solutions of (1) is given by the difference of Conley-Zehnder indices.

The gradient flow of f is called of Morse-Smale type if stable manifolds and unstable manifolds intersect transversally. Suppose that a function f is a Morse function and its gradient flow is of Morse-Smale type. The Morse complex of f is generated by the critical point set as modules and its differential is defined by counting, with signs, gradient trajectories joining two critical points of index difference 1. The resulting homology group is called Morse homology, which is isomorphic to the ordinary homology. Details can be found in [32].

The Floer complex is generated by one-periodic solutions. The Morse-Smale condition of the gradient flow is interpreted as the surjectivity of the linearization operator of (1). Let $C_i(H,J)$ be the free module generated by one periodic solutions with Conley-Zehnder index $\mu = i$. Denote by $\mathcal{P}(H)$ the set of one-periodic solutions which are null homotopic. The boundary homomorphism $\partial : C_i(H,J) \to C_{i-1}(H,J)$ is defined by:

$$\partial x = \sum_{\mu(y)=i-1} m(x,y)y,$$

where $m(x,y)$ is the cardinality of $\mathcal{M}(x,y) = \widetilde{\mathcal{M}}(x,y)/\mathbf{R}$ counted with sign. By the Gromov compactness argument, if a sequence in $\mathcal{M}(x,y)$ is uniformly bounded up to the first derivative, it converges locally uniformly (after reparametrization). In this case, the limit may be a tuple $(u_1, \dots, u_k) \in \mathcal{M}(x, x_1) \times \cdots \times \mathcal{M}(x_{k-1}, y)$. Here $u_1, \dots, u_k$ are not constant paths in the loop space. For a generic pair (H,J), the linearization of (1) is surjective and $\widetilde{\mathcal{M}}(x,y)$ is a manifold of dimension $\mu(x) - \mu(y)$.

Suppose that $\mu(x) - \mu(y) = 1$. Since $\mathcal{M}(x,y)$ has positive dimension unless $x = y$, one of $\mathcal{M}(x_i, x_{i+1})$ has negative dimension, hence is empty, if $k \geq 2$. So the sequence is convergent. If the derivatives of the sequence are not uniformly bounded, the Gromov compactness argument implies that J-holomorphic bubbles appear from the sequence. In the case of $\pi_2(M) = 0$, no J-holomorphic sphere exists. Hence $\mathcal{M}(x, y)$ consists of finitely many points. This manifold also carries a natural orientation (see [8], [9], [12]). When $\mu(x) - \mu(y) = 1$, we compare this orientation with the action of $\mathbf{R}$. If these orientations are the same, we count the point by $+1$, if not, -1. In this way, the definition of the boundary homomorphism is established.

We can show that $\partial^2 = 0$ by studying $\mathcal{M}(x, z)$ with $\mu(x)-\mu(z) = 2$. This manifold may not be compact and its ends correspond to $\mathcal{M}(x, y) \times \mathcal{M}(y, z)$ with $\mu(y) = \mu(x) - 1 = \mu(z) + 1$. The signed number of points in the boundary of a compact 1-dimensional manifold must be zero and this implies that $\partial^2 = 0$.

The resulting homology group is called Floer homology $HF_*(H, J)$ and we can show that the groups $HF_*(H_i, J_i)$ are isomorphic for generic pairs (H_i, J_i), $i = 1, 2$.

To prove this, we need to study the solution space of the following equation:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla\ H_{s,t}(u) = 0, \tag{2}$$

where H_s is a path in the space of one-periodic Hamiltonians such that $H_s = H_1$ for $s < -R$ and $H_s = H_2$ for $s > R$ for some constant $R > 0$. The equation (2) is an s-dependent analog of the equation (1) and not invariant under translations in the s-variable.

A similar argument as that in the proof of $\partial^2 = 0$ gives a chain homomorphism $HF_*(H_1, J_1) \to HF_*(H_2, J_2)$. In fact, it is an isomorphism (see, for instance, [23]).

For computation, we take a C^2-small Morse function f and consider a Hamiltonian $H(x, t) = f(x)$. If the solutions of (1) are t-independent, hence correspond to gradient trajectories of f, then $HF_*(f, J) \cong H_{*+n}(M)$. Once the surjectivity of the linearization is established, the conclusion follows easily. For surjectivity, see [10], [26]. For another approach to computation, see [27].

So far, we have considered the case $\pi_2(M) = 0$. In general, we need to work on a covering space of $\mathcal{L}M$. Namely, we introduce

$$\widetilde{\mathcal{L}}M = \{(x, u) | x : S^1 \to M, u : D^2 \to M \text{ such that } u|_{\partial D^2} = x\}/\sim,$$

where $(x, u) \sim (y, v)$ if and only if $x = y$ and evaluations of ω and $c_1(M)$ with the 2-spherical cycle $u\sharp(-v)$ are zero. Here $-v$ is the mapping from the disk with opposite orientation. Then $p : \widetilde{\mathcal{L}}M \to \mathcal{L}M$ is a covering space with covering transformation group $\Gamma = \pi_2(M)/Ker\ I_\omega \cap Ker\ I_{c_1(M)}$. I_ω and $I_{c_1(M)}$ are homomorphisms from $\pi_2(M)$ to $\mathbf{R}$ corresponding to the cohomology classes $[\omega]$ and $c_1(M)$.

The action functional $\mathcal{A}_H$ is well-defined on $\widetilde{\mathcal{L}}M$. Write

$$\widetilde{\mathcal{P}}(H) = p^{-1}(\mathcal{P}(H)) \subset \widetilde{\mathcal{L}}M.$$

The Floer chain complex is generated by $\widetilde{\mathcal{P}}(H)$, on which the Conley-Zehnder index is well-defined. The boundary homomorphism is defined, in the same way as before, by counting solutions of (1). We call a solution u a connecting orbit from (x^-, u^-) to (x^+, u^+) if the evaluations of ω and $c_1(M)$ on the 2-spherical cycle $u^-\sharp u\sharp(-u^+)$ are zero. Note that the number of end points (x^+, u^+) of connecting orbits, with index difference 1, starting from (x^-, u^-) can be infinite. However the boundary homomorphism is well-defined on the completion of the free module generated by $\widetilde{\mathcal{P}}(H)$ with respect to $\mathcal{A}_H$.

Compactness of $\mathcal{M}(x,y)$ with $\mu(x) - \mu(y) = 1$ is established once bubbling off of J-holomorphic spheres is avoided. This follows from transversality of evaluation maps for connecting orbits and J-holomorphic spheres (see [15]). For a weakly monotone symplectic manifold (M,ω), Hofer and Salamon constructed Floer homology for a generic pair (H,J). Here (M,ω) is called weakly monotone (or semi-positive), if the following condition holds: $\omega \cdot A < 0$ for $A \in \pi_2(M)$ satisfying $c_1(M) \cdot A < 3 - m$, where m is half of the dimension of M.

Denote by $\widetilde{\mathcal{M}}_J(A)$ the space of J-holomorphic spheres representing the homology class $A \in H_2(M;\mathbf{Z})$. Write $\mathcal{M}_J(A) = \widetilde{\mathcal{M}}_J(A)/PSL(2,\mathbf{C})$ the moduli space of J-holomorphic spheres representing the homology class $A \in H_2(M;\mathbf{Z})$. Suppose that (M,ω) is weakly monotone. If a sequence f_k is not convergent in $\mathcal{M}_J(A)$, it converges locally uniformly outside of finitely many points. J-holomorphic bubbles appear around these points after a rescaling procedure. So the limit of f_k consists of finitely many J-holomorphic spheres $g_1, \ldots, g_l$ satisfying the following conditions.

1. $\sum_{i=1}^{l}[g_i] = A$.
2. The union of images of g_i is connected.

If the linearization operator of the J-holomorphic curve equation at g_i is surjective, then $\mathcal{M}_J([g_i])$ is a manifold near g_i by the implicit function theorem and its dimension is given by the Atiyah-Singer index formula.

$$\dim \mathcal{M}_J([g_i]) = 2m + 2c_1(TM)[g_i] - 6.$$

Here 6 is the dimension of the automorphi sm group $PSL(2,\mathbf{C})$ of $\mathbf{C}P^1$. If evaluation maps for $[g_i]$ are transver sal, then the dimension of the set of limit points of $\mathcal{M}_J(A)$ is less than the dimension of $\mathcal{M}_J(A)$ at least by 2. This insures the existence of a good compac tification. We explain this fact in the case $l = 2$ for simplicity. The general case is also treated in the same spirit, but is more complicated.

Let $ev_i : \widetilde{\mathcal{M}}_J([g_i]) \to M$ be the evaluation map at the point $[1,0]$ of $\mathbf{C}P^1$. If ev_1 and ev_2 are transversal, then the dimension of $(ev_1 \times ev_2)^{-1}(\Delta_M)$ is $2m + 2c_1(TM)[g_1] + 2m + 2c_1(TM)[g_2] - 2m = 2m + 2c_1(A)$, where Δ_M is the diagonal set in $M \times M$. This is the space of pairs of J-holomorphic spheres representing $[g_i]$ respectively and coinciding at $[1,0]$. Then the product of two copies of the automorphism group of $(\mathbf{C}P^1,[1,0])$ acts on it locally freely if $[g_i] \neq 0$ for $i = 1,2$.

This group is $2 \times 4 = 8$ dimensional and the moduli space of such pairs is $2m + 2c_1(TM)[A] - 8$, which is less than $\dim \mathcal{M}_J(A)$ by 2. McDuff proved surjectivity of the linearization operator for somewhere injective J-holomorphic curves with respect to a generic J. Here a map f is called somewhere injective if there exists a point x in the domain such that df is injective at x and $f^{-1}(f(x)) = \{x\}$. For transversality of evaluation maps, see [24]. For connecting orbits and J-holomorphic bubbles, a similar argument holds [15].

If a J-holomorphic map f from a closed Riemann surface is not somewhere injective, it factors into a branched covering between closed Riemann surfaces and a somewhere injective map h. If (M,ω) is weakly monotone, we have $c_1(TM)[f] \geq c_1(TM)[h]$. So if a multi-covered curve f appears as g_l, say, then the union of the image of $g_1, \ldots, g_{l-1}, h$ is also connected.

This condition enables us to compactify $\mathcal{M}_J(A)$ by attaching lower dimensional strata. We encounter a problem when a multi-covered curve f satisfies $c_1(TM)[f] < c_1(TM)[h]$, i.e., $c_1(TM)[h] < 0$. We call such a problem a "negative multiple cover problem". We shall explain the way to overcome it in the next section.

3. Kuranishi structure and multi-valued perturbation. This section is a survey on joint work with Kenji Fukaya concerning Gromov-Witten invariants and the Arnold conjecture for general symplectic manifolds [12]. There are several mathematicians who have discussed this problem independently. J. Li and G. Tian constructed Gromov-Witten invariants for algebraic manifolds and also for general symplectic manifolds. G. Liu and G. Tian used their method to prove the Arnold conjecture. There are also works by Y. Ruan, B. Siebert, and H. Hofer and D. Salamon. Gromov-Witten invariants for algebraic manifolds were also constructed by K. Behrend [3].

To deal with compactification of the moduli space of J-holomorphic curves and the moduli space of connecting orbits, we introduce the notion of stable maps due to Kontsevich [16]. It is widely known that singular Riemann surfaces with at most double points are necessary to compactify the moduli space of Riemann surfaces. We need to consider the moduli space of Riemann surfaces with marked points, namely the space of $(C, x_1, \ldots, x_k)$ where C is a Riemann surface and $x_1, \ldots, x_k$ are points on C, i.e., marked points. The genus of a singular Riemann surface $C = \bigcup \Sigma_i$ with at most double points is defined as the sum of the genus of its normalization and the first Betti number of a graph associated to it. Here the vertices of the graph are irreducible components. For double points, we put an edge joining vertices corresponding to irreducible components containing the double point. This gives a graph associated to C. A possibly singular Riemann surface C of genus g and with k marked points and at most double points is called a stable curve, if each irreducible component Σ_i of C satisfies the following:

Either the genus of Σ_i is at least 2, the number of marked points and double points on Σ is at least 1 if its genus is 1 and at least 3 if its genus is 0.

This condition is equivalent to the condition that the automorphism group of C is finite. Here a homeomorphism $\phi : C \to C$ is an automorphism if its restriction to each irreducible component is biholomorphic and ϕ preserves marked points. The moduli space of stable curves of genus g and with k marked points is a compact orbifold.

When we consider the limit of a sequence of J-holomorphic curves with varying complex structure on the domain, we have to take the limit of complex structures on the domain into account. However this is not enough. For instance, regular fibers of an elliptic surface are holomorphic curves of genus 1. Singular fibers are limits of such curves and classified by Kodaira. In the list of singular fibers, there are "unstable curves". Kontsevich considered a stability condition for J-holomorphic curves.

Let $C = \sum \Sigma_i$ be a singular Riemann surface with at most double points. A J-holomorphic map $f : C \to M$ is called a stable map, if each irreducible component satisfies one of the following:

1. The restriction of f to Σ_i is not a constant map.
2. The genus of Σ_i is at least 2.
3. The number of marked points and double points on Σ is at least 1 if its genus is 1 and at least 3 if its genus is 0.

A biholomorphic automorphism ϕ of $(C, x_1, \ldots, x_k)$ is called an automorphism of $f : C \to M$, if $f = f \circ \phi$ holds. The condition of stability for f is, again, equivalent to the condition that its automorphism group is finite. We can introduce a topology on the moduli space of stable curves and show that it is compact and Hausdorff. A similar construction can be done for connecting orbits in Floer theory.

This space may be pathological and we may not define its fundamental class. We describe this space by "local defining equations" and "perturbing the equations" so that the zero locus becomes nice. The first step is based on the Kuranishi method [17] and the gluing argument for J-holomorphic curves in the appendix of [24]. The second step is subtle since usual perturbation may not work. This is because of the existence of automorphisms. We introduced the notion of multi-valued perturbation and making the zero locus nice.

Recall that a map $f : (\Sigma, j) \to (M, J)$ is (j, J)-holomorphic or simply J-holomorphic if it satisfies

$$df + J \circ df \circ j = 0.$$

Consider the left hand side of this equation as a section $\overline{\partial}$ of a Banach bundle $\mathcal{E}$ with fiber at f being $\Gamma(\Sigma, T^*_{0,1}\Sigma \otimes_{\mathbf{C}} f^*T^{1,0}M)$. For fixed j and J, $\overline{\partial}$ is a Fredholm section. Locally we can choose a finite dimensional subbundle F_f in $\mathcal{E}$ such that $\overline{\partial}$ is transversal to F_f. Then the inverse image of F_f is a finite dimensional manifold V_f. In fact, we may have non-trivial automorphisms of f. In this case, we can choose the subbundle so that it is invariant under the automorphism group Γ_f, i.e., an equivariant vector bundle over V_f. A stability condition for f is finiteness of Γ_f. On V_f, the section $\overline{\partial}$ takes values in F_f and gives a section s_f of $F_f \to V_f$. For a stable map $f : C \to M$, we modify the gluing argument in [24] in the situation with "obstruction bundles" and give a local description of the moduli space in a similar way. The moduli space is locally isomorphic to the image of the zero locus of s_f in the quotient space $U_f = V_f/\Gamma_f$. (U_f is an orbifold.) If s_f is perturbed to a section transversal to the zero section in equivariant way, the zero locus in U_f becomes an orbifold. However such a perturbation is not always possible.

So we consider a multi-valued section, each branch of which is transversal to the zero section. We can perturb s_f, not necessarily in an equivariant way, so that it is transversal to the zero section (Thom's transversality theorem). Then we take all the images of this section under Γ_f. Consider the set of points where one of branches of the multi-section vanishes. For a point in this set, we associate the weight by the ratio of the number of branches vanishing at the point to the number of all the branches. This construction is local and we have to make such multi-valued perturbations everywhere in a compatible way. This is discussed in terms of Kuranishi structure defined below.

A Kuranishi structure of dimension n on a compact metrizable space Y is a collection $(U_p, E_p, s_p, \psi_p, \phi_{pq}, \hat{\phi}_{pq})$ for each $p \in Y$ such that

1. $U_p = V_p/\Gamma_p$ is an orbifold and E_p is an orbibundle on it.
2. s_p is a single valued section of E_p.
3. ψ_p is a homeomorphism from $s_p^{-1}(0)$ to a neighborhood of p in Y.
4. Let $q \in \psi_p(s_p^{-1}(0))$. Then there exists an embedding $\phi_{pq} : U_q \to U_p$ in the category of orbifolds, which is covered by an embedding of orbibundles $\hat{\phi}_{pq} : E_q \to E_p$.
5. $s_p \circ \phi_{pq} = \hat{\phi}_{pq} \circ s_q$, $\psi_p \circ \phi_{pq} = \psi_q$.
6. If $r \in \psi_q(s_q^{-1}(0))$, then $\phi_{pq} \circ \phi_{qr} = \phi_{pr}$, $\hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \hat{\phi}_{pr}$.
7. dim U_p − rank $E_p = n$ for all $p \in Y$.

We can find multi-valued sections for E_p in a compatible way so that each branch is transversal to the zero section (see [12]). The last condition in the definition of Kuranishi structure is satisfied in our case, because the linearization operator of the J-holomorphic curve equation is a Fredholm operator and its index is invariant under compact perturbation. Note that for a higher dimensional complex manifold Z and a holomorphic vector bundle E, the difference of dimensions of $H^0(Z;E)$ and $H^1(Z;E)$ is not necessarily invariant under deformation. Only the alternating sum of the dimensions of $H^k(Z;E)$ is given by the Riemann-Roch-Hirzebruch formula or the Atiyah-Singer index formula and topological invariants.

We can also define an orientation of a Kuranishi structure and show that there is a canonical orientation for the Kuranishi structure on the moduli space of J-holomorphic curves. This enables us to define the "fundamental cycle" with rational coefficients. This is the way to overcome the negative multiple bubble problem and we can define Gromov-Witten invariants.

The construction of Floer homology also follows by a similar argument and we get the following:

THEOREM. *Let (M,ω) be a closed symplectic manifold and ϕ an exact symplectomorphsm of (M,ω) with only non-degenerate fixed points. Then the number of fixed points of ϕ is at least the sum of Betti numbers of M, i.e.,* $\sum_k \dim H^k(M;\mathbf{Q})$.

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