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# TOPOLOGICAL TYPES OF FEWNOMIALS

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*Dedicated to Professor Stanisław Łojasiewicz*

**1. Introduction and statement of the results.** Let $f, g : \mathbf{R}^n \to \mathbf{R}^p$ be two continuous mappings. They are said to be *topologically equivalent* if there are homeomorphisms $\rho : \mathbf{R}^n \to \mathbf{R}^n$ and $\lambda : \mathbf{R}^p \to \mathbf{R}^p$ such that $g = \lambda \circ f \circ \rho$. A *topological type* of mappings from $\mathbf{R}^n$ to $\mathbf{R}^p$ is an equivalence class for the topological equivalence.

T. Fukuda [9] proved the following result: given two positive integers $n, k$, the number of topological types of polynomials $\mathbf{R}^n \to \mathbf{R}$ of degree $\leq k$ is finite. Khovanskii [11] and others have shown that many finiteness results on polynomials of bounded degree can also be obtained for polynomials with a bounded number of monomials (and no bound on the degree). Here we present a version of the result of Fukuda for these "fewnomials".

THEOREM 1. *Let $n$ and $k$ be two positive integers. Then the number of topological types of polynomials $\mathbf{R}^n \to \mathbf{R}$ with at most $k$ monomials is finite.*

This theorem is in the same vein as the result of van den Dries about the topological types of sets of zeros of fewnomials (two sets have the same topological type when they are homeomorphic). This result says that, given $n$ and $k$ two positive integers, the number of topological types of sets $f^{-1}(0) \subset \mathbf{R}^n$, where $f : \mathbf{R}^n \to \mathbf{R}$ is a polynomial with at most $k$ monomials, is finite. The result of van den Dries is a consequence of a generalization of the theorem of semialgebraic triviality of Hardt to the context of o-minimal structures. This generalization is in turn a consequence of the triangulability of definable sets in o-minimal structures. For these results, and for an excellent presentation of o-minimal structures, see the surveys [5] and [8], and look for the forthcoming [6]. We shall follow exactly the same lines for the proof of Theorem 1. The proof of Fukuda's theorem in [4] follows this pattern in the semialgebraic case.

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We adopt the definitions and notations of [8]. We consider an o-minimal structure $\mathcal{S}$ on a real closed field $(R,+,\cdot)$. Recall that $\mathcal{S}$ is a sequence $(\mathcal{S}_n)_{n\in\mathbf{N}}$ such that for each $n\in\mathbf{N}$:

S1. $\mathcal{S}_n$ is a boolean algebra of subsets of $R^n$, with $R^n\in\mathcal{S}_n$.

S2. $\mathcal{S}_n$ contains the diagonals $\{(x_1,\dots,x_n)\in R^n\ ;\ x_i=x_j\}$ for $1\le i<j\le n$.

S3. If $A\in\mathcal{S}_n$, then $A\times R$ and $R\times A$ belong to $\mathcal{S}_{n+1}$.

S4. If $A\in\mathcal{S}_{n+1}$, then $\pi(A)\in\mathcal{S}_n$, where $\pi:R^{n+1}\to R^n$ is the projection on the space of the first $n$ coordinates.

S5. $\mathcal{S}_3$ contains the graphs of addition and multiplication.

A sequence $\mathcal{S}$ satisfying S1–S5 is called a *structure* on $(R,+,\cdot)$. It is o-minimal if

S6. $\mathcal{S}_1$ consists exactly of the finite unions of intervals and points.

The elements of $\mathcal{S}_n$ are called *the definable subsets of* $R^n$. A mapping from a subset of $R^n$ to a subset of $R^p$ is definable if its graph is in $\mathcal{S}_{n+p}$.

The smallest structure on a real closed field $R$ has the semialgebraic sets as definable sets, and it is, of course, o-minimal. Another classical o-minimal structure is $\mathbf{R}_{\mathrm{an}}$, for which the definable subsets of $\mathbf{R}^n$ are those which are subanalytic in the projective space $\mathbf{RP}^n$. The geometric theory of o-minimal structures is thus a continuation of the pioneer work of S. Łojasiewicz in semialgebraic, semianalytic and subanalytic geometries. The feasability of such a generalization relies on the results of Knight, Pillay and Steinhorn [15], [12]. Concerning fewnomials, we shall be mainly interested in the o-minimal structure $\mathbf{R}_{\exp}$, the smallest structure containing the graph of the exponential. The fact that it is o-minimal is a fundamental result of Wilkie [19].

First, we prove that a definable function is triangulable. Note that we consider o-minimal structures not only on the field of real numbers $\mathbf{R}$, but also on other real closed fields $R$. This will be useful.

THEOREM 2. *Let $X$ be a closed and bounded definable subset of $R^n$ and $f:X\to R$ a definable continuous function. Then there exist a finite simplicial complex $K$ in $R^{n+1}$ and a definable homeomorphism $\rho:|K|_R\to X$ such that $f\circ\rho$ is linear on each simplex of $K$.*

*Moreover, given finitely many definable subsets $B_1,\dots,B_k$ of $X$, we may choose the triangulation $\rho:|K|_R\to X$ so that each $B_i$ is a union of images of open simplices of $K$.*

A result related to triangulability was obtained by T. L. Loi [13]: any definable function has a stratification satisfying the $a_f$ regularity condition. The proof of Theorem 2 is essentially the proof of M. Shiota for the triangulability of semialgebraic functions (see [16]). We will give it in Section 2. The main new point is a lemma which is a kind of "existence of good directions" (Lemma 5). The other parts of the proof are classical. Shiota has also given a theorem of triangulation of functions in the context of his $\mathcal{X}$-sets [17]. But his proof is not elementary, in the sense that it does not work for o-minimal structures on general real closed fields. This makes it useless for our purpose.

In Section 3 we introduce real spectra for o-minimal structures. This tool can be used to translate elementary results into results on definable families. This technique was developed in the semialgebraic context in [3]. This is only a small part of the theory of the real spectrum, and actually it is a reformulation of classical model-theoretic notions and results.

We apply this technique of translation in Section 4 to Theorem 2, and we obtain the definable triviality of a definable family of functions over a finite definable partition of the parameter space (Theorem 14). Finally, in Section 5, we come back to the finiteness of topological types of fewnomials, using the definability of $x^y$ $(x > 0)$ in $\mathbf{R}_{\exp}$. We give a similar result for polynomials with bounded additive complexity.

**2. Triangulation of definable functions.** This section is devoted to the proof of Theorem 2. Actually we will need a variant of this theorem.

THEOREM 3. *Let $X$ be a closed and bounded definable subset of $R^n$ and $f : X \to R$ a definable continuous function. Then there exist a finite simplicial complex $K$ in $\mathbf{Q}^{n+1}$ and definable homeomorphisms $\rho : |K|_R \to X$ and $\tau : R \to R$ such that $\tau \circ f \circ \rho$ is the restriction to $|K|_R$ of the projection $R^{n+1} \to R$ on the last factor.*

*Moreover, given finitely many definable subsets $B_1, \ldots, B_k$ of $X$, we can choose the triangulation $\rho : |K|_R \to X$ so that each $B_i$ is a union of images of open simplices of $K$.*

We shall give only the proof of Theorem 3, since the proof of the other theorem is essentially the same, but a little simpler. We may replace $X$ with the graph $A$ of $f$ in $R^n \times R$, and the proof of Theorem 3 is reduced to the proof of the following triangulability result.

PROPOSITION 4. *Let $A$ be a closed and bounded definable subset of $R^n \times R$ and let $B_i$, $i = 1, \ldots, k$, be definable subsets of $A$. Let $\pi : R^n \times R \to R$ be the projection on the last factor. Then there exist a finite simplicial complex $K$ in $\mathbf{Q}^n \times \mathbf{Q}$ and definable homeomorphisms $\tau : R \to R$ and $\varphi : |K|_R \to A$ such that $\tau \circ \pi \circ \varphi = \pi|_{|K|_R}$ and each $B_i$ is a union of images by $\varphi$ of open simplices of $K$.*

Proof. We proceed by induction on $n$. The case of $n = 0$ is obvious. We can subdivide $R$ with finitely many points $x_1 < \ldots < x_p$ such that $A$ and the $B_i$ are unions of points $x_i$ and intervals $]x_j, x_{j+1}[$. Then we choose a definable homeomorphism $\tau : R \to R$ such that $\tau(x_i) \in \mathbf{Q}$ for $i = 1, \ldots, p$.

Given $n > 0$, assume that the proposition is proved for $n - 1$. Since every definable set is a finite union of locally closed definable sets, we may assume that the $B_i$ are locally closed. Then we may replace $B_i$ with its closure $\overline{B_i}$ and the difference $\overline{B_i} \setminus B_i$. Hence we may assume that all the $B_i$ are closed.

Let $F_0$ be the boundary of $A$ and $F_i$ the boundary of $B_i$, for $i = 1, \ldots, k$. Set $F = \bigcup_{i=0}^k F_i$. Then $F$ is a closed and bounded definable set of dimension at most $n$. Denote by $C$ the finite set of points $c \in R$ such that $\{x \in R^n \ ; \ (x, c) \in F\}$ is of dimension $n$. Let $G_i$, $i = 0, \ldots, k$ be the union of the closure of $F_i \setminus (R^n \times C)$ with the boundary of $F_i \cap (R^n \times C)$. Set $G = \bigcup_{i=0}^k G_i$. Each $G_i$ is a closed and bounded definable set, and for every $t$ in $R$, the dimension of $\{x \in R^n \ ; \ (x, t) \in G\}$ is strictly less than $n$.

Let $p : R^n \times R \to R^{n-1} \times R$ be the projection defined by $p(x_1, x', t) = (x', t)$. Assume that $p$ has the following property:

$$(\Phi) \qquad \forall (x', t) \in R^{n-1} \times R, \quad \big(p^{-1}(x', t) \cap G\big) \text{ is finite.}$$

We apply the cylindrical cell decomposition to the projection $p$ and to the definable sets $G_i$. We get a finite partition of $p(A)$ into definably connected definable subsets $X_\lambda$ of $R^{n-1} \times R$, and definable continuous functions

$$\xi_{\lambda,1} < \ldots < \xi_{\lambda,m_\lambda} : X_\lambda \longrightarrow R,$$

such that every graph of $\xi_{\lambda,\mu}$ is contained in some $G_i$ and every $G_i$ is a union of graphs of $\xi_{\lambda,\mu}$ (here the graph of $\xi_{\lambda,\mu}$ should be regarded as the set of $(\xi_{\lambda,\mu}(x), x)$ for $x \in X_\lambda$). We may moreover assume that the partition $X_\lambda$ is compatible with the subsets $R^{n-1} \times \{c\}$ for every $c \in C$. Applying the inductive assumption, we may assume that there is a simplicial complex $L$ in $\mathbf{Q}^{n-1} \times \mathbf{Q}$ such that $|L|_R = p(A)$ and all $X_\lambda$ are open simplices $\sigma_\lambda$ of $L$. Since all $G_i$ are closed and bounded and according to the assumption that property $(\Phi)$ holds for $p$, every function $\xi_{\lambda,\mu}$ may be continuously extended to the closed simplex $\overline{\sigma_\lambda}$. We denote the extension by $\overline{\xi_{\lambda,\mu}}$. Note that $A$ and the $B_i$ are disjoint unions of "thin" cells (i.e. graphs of $\xi_{\lambda,\mu}$) and "thick" cells (i.e. slices of the cylinders $p^{-1}(\sigma_\lambda)$ cut by the graphs of two successive functions $\xi_{\lambda,\mu}$ and $\xi_{\lambda,\mu+1}$. The closure $\overline{C}$ of a cell is a union of cells.

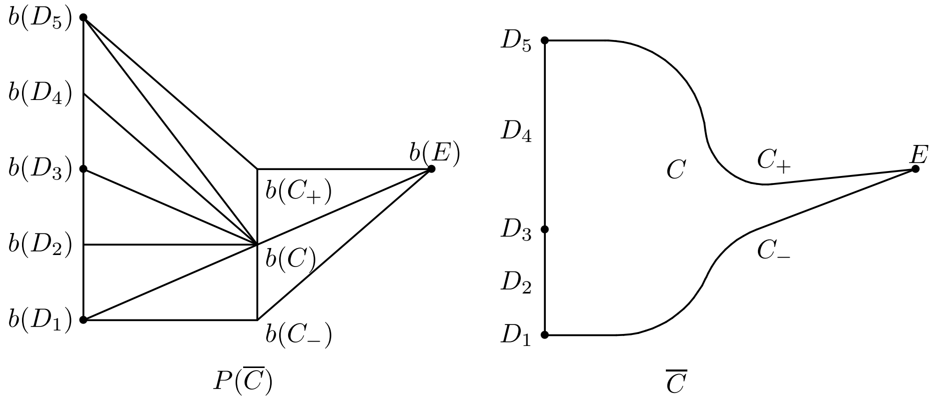


Figure 1. The construction of $K$

From now on, the construction of the triangulation of $A$ is more or less classical. We will follow closely [10]. However we detail the construction for the convenience of the reader and since one of the formulas of [10] has to be modified.

First we construct a simplicial complex $K$ in $\mathbf{Q}^n \times \mathbf{Q}$ such that the projection $p : \mathbf{Q} \times \mathbf{Q}^{n-1} \times \mathbf{Q} \to \mathbf{Q}^{n-1} \times \mathbf{Q}$ induces a simplicial morphism from $K$ to the barycentric subdivision $L'$.

We denote by $b(\sigma)$ the barycenter of the simplex $\sigma$ of $L$. If $C$ is a thin cell which is the graph of $\xi_\mu : \sigma \to R$, we set $b(C) = (\mu, b(\sigma))$. If $C$ is a thick cell delimited by the graphs of $\xi_\mu, \xi_{\mu+1} : \sigma \to R$, we set $b(C) = (\mu + \frac{1}{2}, b(\sigma))$. Note that in both cases $b(C)$ has coordinates in $\mathbf{Q}$. The closed simplices of $K$ are the $[b(C_0), b(C_1), \ldots, b(C_p)]$ for every sequence $(C_0, C_1, \ldots, C_p)$ of cells in $A$ such that $C_i \subset \overline{C_{i+1}}$ for $i = 0, \ldots, p-1$. We denote by $P(\overline{C})$ the polytope which is the union of all simplices $[b(C_0), b(C_1), \ldots, b(C_p)]$ of $K$ such that $C_p$ is contained in the closure $\overline{C}$.

Let $C$ be a cell such that $p(C) = \sigma$. If $C$ is a thin cell, then $P(\overline{C})$ is the graph of a function $h_{\overline{C}} : \overline{\sigma} \to R$ linear on each simplex of the subdivision of $\overline{\sigma}$. If $C$ is a thick cell bounded from above (resp. from below) by the thin cell $C_+$ (resp. $C_-$), we denote by $W(\overline{C})$ the polytope $P(\overline{C}) \cap p^{-1}(\overline{\sigma} \setminus \sigma)$. Note that $P(\overline{C})$ is the cone with vertex $b(C)$ and base the union of $P(\overline{C_-})$ (the floor), $P(\overline{C_+})$ (the ceiling) and $W(\overline{C})$ (the walls).

Now it suffices to construct a definable homeomorphism $\varphi : |K|_R \to A$ such that $p \circ \varphi = p$ and $\varphi(P(\overline{C})) = \overline{C}$ for every cell $C$. Note that, if $C$ is a thin cell which is the graph of $\xi : \sigma \to R$, we must have $\varphi(h_{\overline{C}}(x), x) = (\overline{\xi}(x), x)$ for every $x \in \overline{\sigma}$. Let $C$ be a thick cell bounded from above (resp. from below) by the thin cell $C_+$ (resp. $C_-$) which is the graph of $\xi_+ : \sigma \to R$ (resp. $\xi_- : \sigma \to R$). Let $\theta_{\overline{C}} : \overline{\sigma} \to R$ be the function which maps $z \in \overline{\sigma}$ to the "height" of $P(\overline{C})$ at the vertical of $z$, i.e. $\theta_{\overline{C}}(z) = h_{C_+}(z) - h_{C_-}(z)$. We proceed by induction on the dimension of $\sigma$ and assume that $\varphi$ has already been constructed on $W(\overline{C})$. It is also defined on $P(\overline{C_-}) \cup P(\overline{C_+})$ by the preceding remark. To define $\varphi$ inside $P(\overline{C})$, we use its conic structure. Every point $x$ inside $P(\overline{C})$ can be represented as $x = (1-r)b(C) + ry$, where $0 \leq r < 1$ and $y \in P(\overline{C_-}) \cup P(\overline{C_+}) \cup W(\overline{C})$. This representation is unique if $x \neq b(C)$.

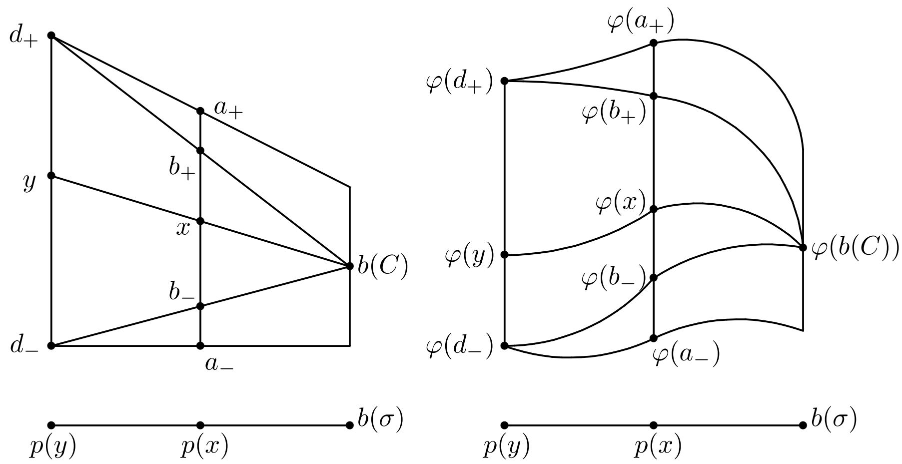


Figure 2. The construction of $\varphi$

First we consider the case where $y \in P(C_+) \cup P(C_-)$. This corresponds to the case where $x$ is in the segment $[a_-, b_-]$ or in the segment $[b_+, a_+]$ (see Figure 2). Then we take for $\varphi(x)$ the image of $x$ by the affine map sending the segment $[a_-, a_+]$ onto the segment $[\varphi(a_-), \varphi(a_+)]$. If $x = (x_1, p(x))$, the first coordinate of $\varphi(x)$ is

$$\frac{x_1 - h_{\overline{C_-}}(p(x))}{\theta_{\overline{C}}(p(x))}\,\overline{\xi_+}(p(x)) + \frac{h_{\overline{C_+}}(p(x)) - x_1}{\theta_{\overline{C}}(p(x))}\,\overline{\xi_-}(p(x)).$$

Next we consider the case where $y \in W(\overline{C})$. This corresponds to the case where $x$ is in the segment $[b_-, b_+]$ (see Figure 2). The images $\varphi(b_-)$ and $\varphi(b_+)$ have already been defined. Then we take for $\varphi(x)$ the point which divides the segment $[\varphi(b_-), \varphi(b_+)]$ in the same way as $\varphi(y)$ divides the segment $[\varphi(d_-), \varphi(d_+)]$. The first coordinate of $\varphi(x)$ is

$$\frac{rs\theta_{\overline{C}}(p(y)) + \frac{1-r}{2}\,\theta_{\overline{C}}(b(\sigma))}{\theta_{\overline{C}}(p(x))}\,\overline{\xi_+}(p(x)) + \frac{r(1-s)\theta_{\overline{C}}(p(y)) + \frac{1-r}{2}\,\theta_{\overline{C}}(b(\sigma))}{\theta_{\overline{C}}(p(x))}\,\overline{\xi_-}(p(x)),$$

where $s \in [0,1]$ is such that $\varphi(y) = s(\xi_+(p(y), p(y)) + (1-s)(\xi_-(p(y), p(y))$. Note that $s$ is not well defined if and only if $\theta_{\overline{C}}(p(y)) = 0$. In this case the formula gives $\frac{1}{2}\big(\overline{\xi_+}(p(x)) + \overline{\xi_-}(p(x))\big)$.

The geometric description should convince the reader that $\varphi$ is a homeomorphism.

To complete the proof, we have to show that we may assume that $p$ verifies $(\Phi)$. For this we use the following lemma.

LEMMA 5. *Let $W \subset R^n \times R^p$ $(n \geq 2)$ be a definable set. For $s \in R^p$, define $W_s = \{y \in R^n \; ; \; (y, s) \in W\}$. Assume that for all $s$ in $R^p$, $\dim(W_s) < n$. Then there exists a polynomial mapping $v' : R \to R^{n-1}$ of degree not greater than $p$ such that, for all $s$ in $R^p$, the set of $x_1 \in R$ such that $(x_1, v'(x_1)) \in W_s$ is finite.*

Proof. We proceed by induction on $n$. We begin with $n = 2$. Let $V$ be a definable subset of dimension 1 of $R^2$. Then $V$ is a disjoint union of finitely many points, vertical open intervals and graphs of definable continuous functions $\xi_i : I_i \to R$, where $I_i$ is an open interval. Consider such a function $\xi_i$. Let $(f_a)_{a \in R^{p+1}}$ be the family of polynomials in one variable of degree not greater than $p$, parametrized by the $(p + 1)$-tuple of the coefficients. Then the open definable set

$$\Big\{x_1 \in I_i \; ; \; \exists a \in R^{p+1} \; \exists \epsilon \in R \; \Big(\epsilon > 0 \text{ and } \forall y \in I_i \; \big(|y - x_1| < \epsilon \Rightarrow f_a(y) = \xi_i(y)\big)\Big)\Big\}$$

is a finite union of disjoint open intervals contained in $I_i$, and for every such interval $U$ there is a unique polynomial $f_a$ such that $f_a|_U = \xi_i|_U$. Hence there is a finite number of polynomials $f_a$ such that the set of $x_1 \in R$ such that $(x_1, f_a(x_1)) \in V$ is infinite. From this we deduce that, for all $s \in R^p$, there are finitely many $a \in R^{p+1}$ such that the set $B_{a,s} = \{x_1 \in R \; ; \; (x_1, f_a(x_1)) \in W_s\}$ is infinite. Therefore the set of $a \in R^{p+1}$ such that there is $s \in R^p$ such that $B_{a,s}$ is infinite, is of dimension at most $p$. Hence there exists a polynomial $f_a$ of degree not greater than $p$ such that, for all $s \in R^p$, the set $B_{a,s}$ is finite.

Given $n > 2$, assume the lemma is proved for $n - 1$. Let $Z$ be the definable set of $(x_1, u, s) \in R \times R^{n-2} \times R^p$ such that $\{x_n \in R \; ; \; (x_1, u, x_n, s) \in W\}$ is infinite. For all $s \in R^p$, the set $Z_s$ has dimension at most $n-2$ and therefore we can apply the inductive assumption. We obtain a polynomial mapping $u : R \to R^{n-2}$ of degree at most $p$ such that, for all $s \in R^p$, the set of $x_1 \in R$ such that $(x_1, u(x_1)) \in Z_s$ is finite. Consider the definable subset of $R^2 \times R^p$

$$M = \{(x_1, x_n, s) \in R^2 \times R^p \; ; \; (x_1, u(x_1), x_n, s) \in W\}.$$

For all $s \in R^p$, the set $M_s$ has dimension at most 1. Therefore, by the argument above, there is a polynomial $f$ of degree at most $p$ such that for all $s \in R^p$ the set of $x_1 \in R$ such that $(x_1, f(x_1)) \in M_s$ is finite. Set $v' = (u, f)$, and the proof is complete. ∎

Proof of Proposition 4 (continued). Set

$$M = \{(x_1, v', x', t) \in R \times R^{n-1} \times R^{n-1} \times R \; ; \; (x_1, x' - v', t) \in G\}$$
$$M_{(x',t)} = \{(x_1, v') \in R \times R^{n-1} \; ; \; (x_1, v', x', t) \in M\}.$$

Since, for all $t$ in $R^n$, the dimension of $\{x \in R^n \; ; \; (c, t) \in G\}$ is not greater than $n$, we have, for all $(x', t)$ in $R^{n-1} \times R$, $\dim(M_{(x',t)}) < n$. By Lemma 5, there is a polynomial mapping $v' : R \to R^{n-1}$ such that the set of $x_1$ in $R$ such that $(x_1, v'(x_1)) \in M_{(x',t)}$ is finite. Now set $v(x_1, x') = (x_1, x' + v'(x_1))$. Then $v$ is a polynomial automorphism of $R^n$, and

$$\forall (x', t) \in R^{n-1} \times R \quad \big(p^{-1}(x', t) \cap (v \times \mathrm{Id}_R)(G)\big) \text{ is finite.}$$

Therefore we may assume that $p$ satisfies $\Phi$. ∎

**3. Definable families and real spectrum.** We introduce "ideal points" which will be useful for the study of definable families. Let $\widetilde{R^p}$ be the Stone space of the boolean algebra $\mathcal{S}_p$. The points of $\widetilde{R^p}$ are the ultrafilters of $\mathcal{S}_p$, and the topology of $\widetilde{R^p}$ has a basis of closed and open subsets consisting of the $\widetilde{A} = \{\alpha \in \widetilde{R^p} \; ; \; A \in \alpha\}$, for $A \in \mathcal{S}_p$. With this topology $\widetilde{R^p}$ is compact, Hausdorff, totally disconnected. We shall call this topology the *constructible* topology. There is another, coarser, topology on $\widetilde{R^p}$ which has a basis of open subsets consisting of the $\widetilde{U}$ for $U$ open definable subset of $R^p$. We shall not consider this coarser topology in this paper. See [14] where this space is considered, together with a structural sheaf. We may consider $R^p$ as a subset of $\widetilde{R^p}$ by identifying a point $a \in R^p$ with the principal ultrafilter of elements of $\mathcal{S}_p$ containing $a$.

Let us take an example. The intervals $]r, +\infty[$, for $r \in R$, generate an ultrafilter of $\mathcal{S}_1$. We denote by $+\infty$ this point of $\widetilde{R}$. Two definable functions $f_1$ and $f_2$ defined respectively on intervals $]r_1, +\infty[$ and $]r_2, +\infty[$ are said to have the same germ at $+\infty$ if they coincide on an interval $]s, +\infty[$. The germs of definable functions at $+\infty$ form a field $\kappa(+\infty)$ (actually a *Hardy field* since every definable function is differentiable on some interval $]r, +\infty[$). This field is important because it contains the information about the asymptotic behavior of definable functions. In the case of the o-minimal structure $\mathbf{R}_{\mathrm{an}}$, it is the field of convergent Puiseux series in $1/x$, and the Łojasiewicz exponents can be seen in its valuation group $\mathbf{Q}$. There is no such nice description for the o-minimal structure $\mathbf{R}_{\exp}$, and the valuation group in this case is awfully complicated. See [7] for the importance of the Hardy field $\kappa(+\infty)$ for o-minimal structures. Following the same pattern, we now associate a field $\kappa(\alpha)$ with every $\alpha \in \widetilde{R^p}$.

If $S$ is a definable subset of $R^p$, denote by $\mathcal{D}(S)$ the ring of definable functions from $S$ to $R$. For $\alpha \in \widetilde{R^p}$, define $\kappa(\alpha)$ as the inductive limit of the $\mathcal{D}(S)$ for $S \in \alpha$. If $f$ is a definable function on $S \in \alpha$, we denote by $f(\alpha)$ its image in $\kappa(\alpha)$. Note that if $\alpha = a \in R^p$, then $\kappa(a) = R$ and $f(a) \in R$ is the value of $f$ at $a$.

PROPOSITION 6. *$\kappa(\alpha)$ is a real closed field.*

Proof. We know that $\kappa(\alpha)$ is a ring. If $f(\alpha)$ is a nonzero element of $\kappa(\alpha)$, there is $S \in \alpha$ such that either $f > 0$ on $S$ or $f < 0$ on $S$. In both cases $1/f$ belongs to $\mathcal{D}(S)$. In the first case $\sqrt{f} \in \mathcal{D}(S)$. Hence $\kappa(\alpha)$ is an ordered field in which every positive element is a square. The fact that every polynomial of odd degree

$$X^{2k+1} + f_{2k}(\alpha)X^{2k} + \cdots + f_0(\alpha) \in \kappa(\alpha)[X]$$

has a root in $\kappa(\alpha)$ is a consequence of the following lemma, applied to

$$F = \{(x,t) \in R \times S\ ;\ x^{2k+1} + f_{2k}(t)x^{2k} + \cdots + f_0(t) = 0\},$$

where $S \in \alpha$ is such that all $f_i$ are defined on $S$. ∎

LEMMA 7 (Definable choice [8]). *Let $F$ be a definable subset of $R \times R^p$ and assume that there exists $S \in \mathcal{S}_p$ such that for all $t \in S$, there exists $x \in R$ such that $(x,t) \in F$. Then there exists $f \in \mathcal{D}(S)$ such that $(f(t), t) \in F$ for all $t \in S$.*

The ideal points $\alpha \in \widetilde{R^p}$ play the role of the generic points of algebraic geometry. For the study of definable families of sets of functions, we shall use "generic fibers". Consider a definable family of subsets of $R^n$ parametrized by $R^p$. This is simply a definable subset $F$ of $R^n \times R^p$.

DEFINITION 8. If $\alpha \in \widetilde{R^p}$, the *fiber $F_\alpha$ at $\alpha$ of the definable family $F \subset R^n \times R^p$* is the set of $(f_1(\alpha), \ldots, f_n(\alpha))$ in $\kappa(\alpha)^n$ such that there exists $S \in \alpha$ on which all $f_i$ are defined and such that $(f_1(t), \ldots, f_n(t), t) \in F$ for all $t \in S$.

If $A$ is a definable subset of $R^n$, *the extension $A_{\kappa(\alpha)}$ of $A$ to $\kappa(\alpha)$* is the fiber at $\alpha$ of the constant family $A \times R^p$.

Of course, for an ordinary point $t \in R^p$ we have $F_t = \{x \in R^n \; ; \; (x,t) \in F\}$.

As an immediate consequence of the definable choice, we obtain

PROPOSITION 9. *Let $F, G \subset R^n \times R^p$ be two definable families. If $F_\alpha = G_\alpha$, then there is $S \in \alpha$ such that $F \cap (R^n \times S) = G \cap (R^n \times S)$.*

Now we formulate the main result about fibers at $\alpha$. Actually, what is hidden here is the model-theoretic construction of the *definable ultrapower*.

THEOREM 10. *Let $\mathcal{S}_n(\kappa(\alpha))$ be the set of fibers $F_\alpha$ of definable families $F \subset R^n \times R^p$. The sequence of $\mathcal{S}_n(\kappa(\alpha))$ for $n \in \mathbf{N}$ defines an o-minimal structure $\mathcal{S}(\kappa(\alpha))$ on the field $\kappa(\alpha)$.*

Proof. It is obvious that $F \mapsto F_\alpha$ preserves the boolean operations. Hence we get property S1 of o-minimal structures. Properties S2, S3 and S5 are also almost immediate. Note that $(F \times_{R^p} G)_\alpha = F_\alpha \times G_\alpha$, where $F \subset R^n \times R^p$ and $G \subset R^m \times R^p$ are definable families and $F \times_{R^p} G$ their fiber product above $R^p$. Property S4 is a consequence of the definable choice: if $F \subset R^{n+1} \times R^p$ is a definable family, $\pi : R^{n+1} \times R^p \to R^n \times R^p$ the projection defined by $\pi(x_1, \ldots, x_{n+1}, t) = (x_1, \ldots, x_n, t)$ and $\pi_\alpha : \kappa(\alpha)^{n+1} \to \kappa(\alpha)^n$ the projection on the space of the first $n$ coordinates, then $(\pi(F))_\alpha = \pi_\alpha(F_\alpha)$. Finally, property S6 comes from the cylindrical cell decomposition: if $F \subset R \times R^p$ is a definable family, then there are $S \in \alpha$ and definable functions $f_1 < \ldots < f_\ell : S \to R$ such that $F \cap R \times S$ is a union of graphs of $f_i$ and slices of the cylinder $R \times S$ bounded by the graphs of $f_i$. ∎

If $F \subset R^n \times R^p$ and $G \subset R^k \times R^p$ are two definable families, a *definable family of mappings from $F$ to $G$* is a definable mapping $f : F \to G$ such that the composite mapping $F \to G \to R^p$ is equal to the projection $F \to R^p$. In other words, there exists a definable mapping $\overline{f} : F \to R^k$ such that $f(x,t) = (\overline{f}(x,t), t)$ for all $(x,t) \in F$. Set

$$\Gamma = \{(x,y,t) \in R^n \times R^k \times R^p \; ; \; (x,t) \in F, \; y = \overline{f}(x,t)\}.$$

It is easily verified that $\Gamma_\alpha$ is the graph of a definable mapping $f_\alpha : F_\alpha \to G_\alpha$ which is by definition the *fiber of the family $f$ at $\alpha$*.

If $f : A \to B$ is a definable mapping (for the o-minimal structure on $R$), we define its extension $f_{\kappa(\alpha)} : A_{\kappa(\alpha)} \to B_{\kappa(\alpha)}$ as the fiber of the constant family $f \times R^p : A \times R^p \to B \times R^p$.

PROPOSITION 11. *For every definable mapping $\phi : F_\alpha \to G_\alpha$ there exist $S \in \alpha$ and a definable family of mappings $f : F \cap (R^n \times S) \to G \cap (R^k \times S)$ such that $f_\alpha = \phi$. Two definable families of mappings $f, g : F \to G$ have the same fiber at $\alpha$ if and only if there exists $S \in \alpha$ such that $f$ and $g$ coincide in restriction to $F \cap (R^n \times S)$.*

We skip the proof which is straightforward. Note that taking the fiber at $\alpha$ preserves the composition of families of morphisms. Hence a commutative diagram of definable mappings for $\mathcal{S}(\kappa(\alpha))$ gives a commutative diagram of definable families of mappings over some $S \in \alpha$.

We have topological precisions concerning the relations between the fiber and the family.

PROPOSITION 12 ([5], [18]). *Let $U \subset F \subset R^n \times R^p$ be two definable families of sets. If the set of $t \in R^p$ such that $U_t$ is open in $F_t$ belongs to $\alpha$, then $U_\alpha$ is open in $F_\alpha$. If $U_\alpha$ is open in $F_\alpha$, then there is $S \in \alpha$ such that $U \cap (R^n \times S)$ is open in $F \cap (R^n \times S)$.*

*Let $f : F \to G$ be a definable family of mappings. If the set of $t \in R^p$ such that $f_t$ is continuous belongs to $\alpha$, then $f_\alpha$ is continuous. If $f_\alpha$ is continuous, then there is $S \in \alpha$ such that $f|_{F\cap(R^n\times S)}$ is continuous.*

In particular a definable homeomorphism for $\mathcal{S}(\kappa(\alpha))$ gives a homeomorphism between definable families over some $S \in \alpha$.

**4. Definable trivialization of definable families of functions.** We study in this section definable families of functions from $R^n$ to $R$. It will be more convenient to consider a definable family of functions from $R^n$ to $R$ as a definable function $f : R^n \times S \to R$, where $S$ is a definable subset of $R^p$. With the conventions of the preceding section, this corresponds to the definable family of mappings $R^n \times S \to R \times S$ defined by $(x,t) \mapsto (f(x,t),t)$.

DEFINITION 13. A definable family of functions $f : R^n \times S \to R$ is *definably trivial* if there are definable mappings $\lambda : R^n \times S \to R^n$ and $\rho : R \times S \to R$ and a definable function $\phi : R^n \to R$ such that

1. the mapping $R^n \times S \to R^n \times S$ sending $(x,t)$ to $(\lambda(x,t),t)$ is a homeomorphism,
2. the mapping $R \times S \to R \times S$ sending $(y,t)$ to $(\rho(y,t),t)$ is a homeomorphism,
3. $\phi \circ \lambda = \rho \circ f$.

It is obvious that, if a definable family of functions $f : R^n \times S \to R$ is definably trivial, then all the functions $f_t$ for $t \in S$ have the same topological type. Moreover, for every $t, u \in S$ there is a commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{f_t} & R \\ \big\downarrow & & \big\downarrow \\ R^n & \xrightarrow{f_u} & R \end{array},$$

where the vertical arrows are *definable* homeomorphisms. We say that $f_t$ and $f_u$ have the same *definable topological type*.

THEOREM 14 (Definable triviality of definable families of functions). *Let $f : R^n \times S \to R$ be a definable family of functions, where $S$ is a definable subset of $R^p$. Then there exist a definable finite partition $S = \bigcup_{i=1}^k S_i$ such that, for $i = 1, \ldots, k$, the restricted family $f|_{R^n\times S_i}$ is definably trivial.*

Proof. Take $\alpha \in \widetilde{S}$. We have a fiber $f_\alpha : \kappa(\alpha)^n \to \kappa(\alpha)$ which is a definable function.

Let $h : ]-1,1[_R \to R$ be the definable homeomorphism defined by $h(x) = x/(1-x^2)$. Set

$$\Gamma = \Big\{\big(x_1,\ldots,x_n,(h_{\kappa(\alpha)})^{-1}(f(h_{\kappa(\alpha)}(x_1),\ldots,h_{\kappa(\alpha)}(x_n))\big)\ ;\ (x_1,\ldots,x_n) \in\ ]-1,1[^n_{\kappa(\alpha)}\Big\}.$$

Let $\Xi$ be the closure of $\Gamma$ in $\kappa(\alpha)^{n+1}$ and $\pi_{\kappa(\alpha)} : \kappa(\alpha)^{n+1} \to \kappa(\alpha)$ the projection on the last coordinate. Since $\Xi$ is closed and bounded, we can triangulate $\pi_{\kappa(\alpha)}|_\Xi$. Applying Proposition 4, we get a finite simplicial complex $K$ in $\mathbf{Q}^n \times \mathbf{Q}$ and definable homeomorphisms $\tau : \kappa(\alpha) \to \kappa(\alpha)$ and $\varphi : |K|_{\kappa(\alpha)} \to \Xi$ such that $\tau \circ \pi_{\kappa(\alpha)} \circ \varphi = \pi_{\kappa(\alpha)}|_{|K|_{\kappa(\alpha)}}$ and $\Gamma = \varphi(V_{\kappa(\alpha)})$, where $V$ is a union of open simplices of $K$. Moreover we may assume that $\pi_{\kappa(\alpha)}(V_{\kappa(\alpha)}) \subset\ ]-1,1[_{\kappa(\alpha)}$, $\tau(-1) = -1$ and $\tau(1) = 1$.

Let $\rho : V_{\kappa(\alpha)} \to \kappa(\alpha)^n$ be the definable homeomorphism defined by

$$\rho^{-1}(x_1,\ldots,x_n) = \varphi^{-1}(h^{-1}_{\kappa(\alpha)}(x_1),\ldots,h^{-1}_{\kappa(\alpha)}(x_n),h^{-1}_{\kappa(\alpha)}(f_\alpha(x_1,\ldots,x_n)))$$

and $\lambda = h_{\kappa(\alpha)} \circ \tau : \,]-1,1[_{\kappa(\alpha)} \to \kappa(\alpha)$. We have a commutative diagram of definable mappings for the o-minimal structure over $\kappa(\alpha)$

$$\begin{array}{ccc} V_{\kappa(\alpha)} & \xrightarrow{\pi_{\kappa(\alpha)}} & ]-1,1[_{\kappa(\alpha)} \\ \rho\big\downarrow & & \big\downarrow\lambda \\ \kappa(\alpha)^n & \xrightarrow{f_\alpha} & \kappa(\alpha) \end{array} .$$

From this and the results of Section 3 we deduce that there is a definable subset $S^\alpha$ of $S$ such that $S^\alpha \in \alpha$ and a diagram of definable mappings commutative over $S^\alpha$

$$\begin{array}{ccc} V \times S^\alpha & \xrightarrow{\pi_R \times S^\alpha} & ]-1,1[_R \times S^\alpha \\ \big\downarrow & & \big\downarrow \\ R^n \times S^\alpha & \xrightarrow{(f,p)} & R \times S^\alpha \end{array} ,$$

where the vertical arrows are homeomorphisms and $p : R^n \times S^\alpha \to S^\alpha$ is the projection. This shows that the family $f$ is definably trivial in restriction to $R^n \times S^\alpha$. The $\widetilde{S^\alpha}$ cover $\widetilde{S}$. Since $\widetilde{S}$ is compact, we can extract a finite definable cover $\widetilde{S_1},\ldots,\widetilde{S_k}$ and we may assume that the $S_i$ form a partition of $S$. The family $f$ is definably trivial over each $S_i$. ∎

**5. Application to fewnomials and polynomials of bounded additive complexity.** To apply Theorem 14 to fewnomials, we have to include the fewnomials with at most $k$ monomials in a definable family of functions for some o-minimal structure on the reals. Of course, the convenient o-minimal structure is $\mathbf{R}_{\exp}$. In this structure we have the definable power function $(x,\lambda) \mapsto x^\lambda = \exp(\lambda \log(x))$ defined on $\{x \in R \,;\, x > 0\} \times R$. We extend this power function to two definable functions $M_\epsilon : \mathbf{R}^2 \to \mathbf{R}$ for $\epsilon = 0,1$, defined by

$$M_\epsilon(x,\lambda) = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x = 0 \text{ and } \lambda \neq 0, \\ 1 & \text{if } x = 0 \text{ and } \lambda = 0, \\ (-1)^\epsilon \, |x|^\lambda & \text{if } x < 0. \end{cases}$$

Now consider the family of all functions $\mathbf{R}^n \to \mathbf{R}$

$$(x_1,\ldots,x_n) \mapsto \sum_{i=1}^{k} \Big( a_i \prod_{j=1}^{n} M_{\epsilon_{i,j}}(x_j,\lambda_{i,j}) \Big).$$

This is a definable family of functions parametrized by the $((a_i),(\lambda_{i,j}),(\epsilon_{i,j})) \in \mathbf{R}^k \times \mathbf{R}^{kn} \times \{0,1\}^{kn}$. In this family we have all the polynomials in $n$ variables with at most $k$ monomials. Hence we obtain a result which is a little stronger than Theorem 1: the polynomials in $n$ variables with at most $k$ monomials have a finite number of definable topological types.

We can generalize this result to polynomials of bounded additive complexity. Recall from [1] that a polynomial $f \in \mathbf{R}[x_1,\ldots,x_n]$ has additive complexity not greater than $k$ if there is a sequence $(g_1,\ldots,g_k,g_{k+1} = f)$ of polynomials in $\mathbf{R}[x_1,\ldots,x_n]$ such that each $g_i$

is the sum of a constant and a monomial in $x_1, \ldots, x_n, g_1, \ldots, g_{i-1}$ with coefficient 1. The additive complexity behaves well with respect to a linear change of variables, while the number of monomials does not.

We consider the functions which send $(x_1, \ldots, x_n) \in \mathbf{R}^n$ to the $y \in R$ such that there exists $(x_{n+1}, \ldots, x_{n+k}) \in \mathbf{R}^k$ with

$$x_{n+i} = a_i + \prod_{j=1}^{n+i-1} M_{\epsilon_{i,j}}(x_j, \lambda_{i,j}) \quad \text{for } i = 1, \ldots, k,$$

$$y = a_{k+1} + \prod_{j=1}^{n+k} M_{\epsilon_{k+1,j}}(x_j, \lambda_{k+1,j}).$$

These functions form a definable family for $\mathbf{R}_{\exp}$, parametrized by the

$$\big((a_i), (\lambda_{i,j}), (\epsilon_{i,j})\big) \in \mathbf{R}^{k+1} \times \mathbf{R}^{(k+1)(2n+k)/2} \times \{0,1\}^{(k+1)(2n+k)/2}.$$

Obviously this family contains all polynomials $\mathbf{R}^n \to \mathbf{R}$ with additive complexity at most $k$. Applying Theorem 14, we obtain:

THEOREM 15. *The polynomials $\mathbf{R}^n \to \mathbf{R}$ with additive complexity at most $k$ have a finite number of definable topological types.*

The method of proof gives no answer to the problem of effectively bounding the number of topological types in terms of the additive complexity and the number of variables. For such a bound with respect to the degree, see [2].

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