SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

ON THE ŁOJASIEWICZ EXPONENT FOR ANALYTIC CURVES

JACEK CHĄDZYŃSKI and TADEUSZ KRASIŃSKI

Faculty of Mathematics, University of Łódź
S. Banacha 22, 90-238 Łódź, Poland
E-mail: jachadzy@imul.uni.lodz.pl, krasinsk@krysia.uni.lodz.pl

Dedicated to Professor Stanisław Łojasiewicz
on the occasion of his 70th birthday

Abstract. An effective formula for the Łojasiewicz exponent for analytic curves in a neighbourhood of $0 \in \mathbb{C}^m$ is given.

1. The Łojasiewicz exponent for sets. In this section we shall assume that Ω is a neighbourhood of $0 \in \mathbb{C}^m$ ($m \geq 2$), X, Y — analytic sets in Ω and $X \cap Y = \{0\}$.

Let

$$N(X,Y) = \{\nu \in \mathbb{R}_+ : \exists A > 0,\ \exists B > 0,\ \forall z \in \Omega,\ |z| < B \Rightarrow \varrho(z,X) + \varrho(z,Y) \geq A|z|^\nu\},$$

here $|\cdot|$ is the polycylindric norm and $\varrho(\cdot, Z)$ is the distance function to a set Z. One can prove (see [Ł$_1$], IV.7) that under the above assumption $N(X,Y)$ is not empty.

By the *Łojasiewicz exponent of* X, Y *at* 0 we mean $\inf N(X,Y)$ and denote it by $\mathcal{L}_0(X,Y)$.

One can prove

PROPOSITION 1 ([Ł$_2$], s. 18). *If X, Y satisfy the above assumptions and* 0 *is an accumulation point of X, then*

$$N(X,Y) = \{\nu \in \mathbb{R}_+ : \exists A > 0,\ \exists B > 0,\ \forall x \in X,\ |x| < B \Rightarrow \varrho(x,Y) \geq A|x|^\nu\}.$$

This research was realized within the project No. 2P03A05010 financed in 1996–1998 by KBN.
1991 *Mathematics Subject Classification*: Primary 32S05; secondary 32B10.
Key words: Łojasiewicz exponent, analytic curve.
Received by the editors: September 27, 1996.
The paper is in final form and no version of it will be published elsewhere.

PROPOSITION 2 ([T], Thm. 3.2). *If X, Y satisfy the same assumptions as above, then* $\mathcal{L}_0(X,Y) \in N(X,Y)$.

Estimations of $\mathcal{L}_0(X,Y)$ from above are known. P. Tworzewski and E. Cygan in [T] and [CT] gave such estimations in terms of the intersection multiplicity of X and Y in both cases: 0 is or not an isolated point of $X \cap Y$.

Let us note an easy property of $\mathcal{L}_0(X,Y)$. Let $X = X_1 \cup \ldots \cup X_r$, $Y = Y_1 \cup \ldots \cup Y_s$, where $X_1, \ldots, X_r$, $Y_1, \ldots, Y_s$ are analytic sets in Ω passing through $0 \in \mathbb{C}^m$.

PROPOSITION 3. *Under the above assumptions*

$$\mathcal{L}_0(X,Y) = \max_{k,\,l} \mathcal{L}_0(X_k, Y_l).$$

2. The Łojasiewicz exponent for mappings. Let $\Omega \subset \mathbb{C}^n$ $(n \ge 2)$ be a neighbourhood of the origin, $F = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ be a holomorphic mapping having an isolated zero at $0 \in \mathbb{C}^n$. Let S be an analytic set in Ω such that 0 is an accumulation point of S. Put

$$N(F|S) = \{\nu \in \mathbb{R}_+ : \exists A > 0,\ \exists B > 0,\ \forall z \in S,\ |z| < B \Rightarrow A|z|^\nu \le |F(z)|\}.$$

When $S = \Omega$ we define $N(F) = N(F|\Omega)$.

By the *Łojasiewicz exponent of* $F|S$ *at* 0 we mean $\mathcal{L}_0(F|S) = \inf N(F|S)$. Analogously, $\mathcal{L}_0(F) = \inf N(F)$.

In the sequel for a holomorphic function $g : \Omega \to \mathbb{C}$ we put $V(g) := \{z \in \Omega : g(z) = 0\}$.

One can prove

THEOREM 1 ([CK]). *If $\Omega \subset \mathbb{C}^n$ $(n \ge 2)$ is a neighbourhood of the origin, $F = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ is a holomorphic mapping having an isolated zero at $0 \in \mathbb{C}^n$ and $f := f_1 \cdot \ldots \cdot f_m$, then*

$$\mathcal{L}_0(F) = \mathcal{L}_0(F|V(f)).$$

We shall now prove a theorem on the Łojasiewicz exponent, needed in the sequel.

Let $n = 2$ and Ω be a neighbourhood of $0 \in \mathbb{C}^2$, $F = (f_1, \ldots, f_m) : \Omega \to \mathbb{C}^m$ be a holomorphic mapping having an isolated zero at $0 \in \mathbb{C}^2$.

THEOREM 2. *If f_1 is a homogeneous form of degree r with r different tangent lines and $r \le \operatorname{ord} f_i < \infty$, then*

$$\mathcal{L}_0(F) = \mathcal{L}_0(F|V(f_1)).$$

Proof. Let $f_1 = L_1 \cdot \ldots \cdot L_r$ be a factorization of f_1 into linear factors. Let $\mu(g,h)$ denote the multiplicity of a mapping $(g,h) : \Omega \to \mathbb{C}^2$ at $0 \in \mathbb{C}^2$. Since

$$\mathcal{L}_0(F|V(f_1)) = \max_{i=1}^{r} \mathcal{L}_0(F|V(L_i)) = \max_{i=1}^{r} \min_{j=2}^{m} \mu(L_i, f_j),$$

then, without loss of generality, we may assume that

$$\mathcal{L}_0(F|V(f_1)) = \mu(L_1, f_m). \tag{1}$$

Hence for each $i \in \{1, \ldots, r\}$ there exists $j \in \{1, \ldots, m\}$ such that

$$\mu(L_i, f_j) \le \mu(L_1, f_m). \tag{2}$$

By Theorem 1 we have

$$\mathcal{L}_0(F) = \mathcal{L}_0(F|V(f)). \tag{3}$$

Let $\mathcal{O}^2$ be the ring of germs of holomorphic functions at $0 \in \mathbb{C}^2$, $h : \Omega \to \mathbb{C}$ — a holomorphic function and $\hat{h} \in \mathcal{O}^2$ — the germ generated by h. Assume that $\hat{h}$ is an arbitrary irreducible germ dividing $\hat{f}$. It is easy to check that

$$\mathcal{L}_0(F|V(f)) = \max_h \mathcal{L}_0(F|V(h)). \tag{4}$$

It follows from (1), (3) and (4) that it suffices to show that

$$\mathcal{L}_0(F|V(h)) \le \mu(L_1, f_m). \tag{5}$$

Assume to the contrary that (5) does not hold for some h. In the sequel $\operatorname{ord} h$ means the order of h at $0 \in \mathbb{C}^2$. Since

$$\mathcal{L}_0(F|V(h)) = (1/\operatorname{ord} h) \min_{k=1}^{m} \mu(f_k, h),$$

then for every $k \in \{1, \ldots, m\}$ we have

$$\mu(L_1, f_m) < \mu(f_k, h)/\operatorname{ord} h. \tag{6}$$

If the curve $V(h)$ has no common tangent line with the curve $V(f_1)$ at 0, then

$$\mu(f_1, h)/\operatorname{ord} h = r \le \mu(L_1, f_m),$$

which contradicts (6).

So, assume that the line $L_i = 0$ is tangent to $V(h)$ at 0. Then, there exists $j \in \{1, \ldots, m\}$ such that (2) holds. If $L_i = 0$ is not tangent to $V(f_j)$ at 0, then

$$\mu(h, f_j)/\operatorname{ord} h = \operatorname{ord} f_j = \mu(L_i, f_j) \le \mu(L_1, f_m),$$

which contradicts (6). If $L_i = 0$ is tangent to $V(f_j)$ at 0, we put $s := \mu(L_i, f_j)$. Then we have $r \le r_j := \operatorname{ord} f_j < s$. Since the considerations are local, then shrinking Ω, if necessary, we may assume that $f_j = \sum_{\nu=r_j}^{\infty} P_\nu$, where P_ν is a homogeneous polynomial of degree ν. Let $f_j^* := \sum_{\nu=s}^{\infty} P_\nu$. Take arbitrary $\nu \in \{r_j, \ldots, s-1\}$. Then from the assumption that f_1 has r different tangent lines we have

$$\mu(P_\nu, h) \ge \mu(L_i, h) + (\nu - 1) \operatorname{ord} h \ge \mu(L_i, h) + (r-1) \operatorname{ord} h = \mu(f_1, h).$$

Hence

$$\mu(f_j - f_j^*, h) \ge \mu(f_1, h). \tag{7}$$

On the other hand, from (2) and (6) for $k = 1$ we have

$$\mu(f_j^*, h) = s \operatorname{ord} h = \mu(L_i, f_j) \operatorname{ord} h \le \mu(L_1, f_m) \operatorname{ord} h < \mu(f_1, h).$$

Hence and from (7)

$$\mu(f_j, h) = \mu(f_j^*, h) \le \mu(L_1, f_m) \operatorname{ord} h,$$

which contradicts (6).

This ends the proof. ∎

3. Main results. In this section we shall give an effective formula for the Łojasiewicz exponent for analytic curves (Theorems 3 and 4).

Let, in the sequel, Ω be a neighbourhood of $0 \in \mathbb{C}^m$ $(m \geq 2)$, X, Y — analytic curves in Ω (i.e. analytic sets of pure dimension 1) and $X \cap Y = \{0\}$. Since the considerations are local, we may assume that $X = X_1 \cup \ldots \cup X_r$, $Y = Y_1 \cup \ldots \cup Y_s$, where X_i, Y_j are analytic curves in Ω generating irreducible germs at 0. Hence and from Proposition 3 it follows that the problem of finding the Łojasiewicz exponent for X, Y reduces to the case when X and Y generate irreducible germs at 0.

Let now Z be an analytic curve in Ω generating an irreducible germ at 0. Then Z has only one tangent at 0. Without loss of generality, changing the coordinates linearly in $\mathbb{C}^n$, if necessary, we may assume that this tangent does not lie in the hyperplane $H_1 := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : z_1 = 0\}$. Shrinking Ω, we may equivalently express this situation in terms of a holomorphic description of Z. Namely, by the second version of the Puiseux theorem ([Ł$_1$], II.6.2) we get easily

PROPOSITION 4. *A curve Z generates an irreducible germ at* 0 *and has the tangent not lying in H_1 if and only if in a neighbourhood $\Omega' \subset \Omega$, Z can be represented in the form*

$$Z \cap \Omega' = \{(t^r, \lambda_2(t), \ldots, \lambda_m(t)) : t \in W\},$$

where r is a positive integer, W — a neighbourhood of 0 *in* $\mathbb{C}$, λ_j *— holomorphic functions in W such that* $\operatorname{ord} \lambda_j \geq r$ *for* $j = 2, \ldots, m$.

If the above mapping $W \ni t \mapsto (t^r, \lambda_2(t), \ldots, \lambda_m(t)) \in Z \cap \Omega'$ is a homeomorphism we shall call this mapping a *parametrization of* $Z \cap \Omega'$.

Now, we shall give a formula for $\mathcal{L}_0(X, Y)$ in terms of holomorphic descriptions of X and Y. The assumptions, under which the formula will be obtained, are not restrictive. It follows from both Proposition 4 and its precedent considerations.

First, we fix some standard notations. Let $\lambda = (\lambda_2, \ldots, \lambda_m)$, $\varphi = (\varphi_2, \ldots, \varphi_m)$, $\psi = (\psi_2 \ldots, \psi_m)$ be holomorphic mappings in a neighbourhood of $0 \in \mathbb{C}$. Then we define $\operatorname{ord} \lambda := \min_{i=2}^m \operatorname{ord} \lambda_i$ and $\varphi - \psi := (\varphi_2 - \psi_2, \ldots, \varphi_m - \psi_m)$.

Let Ω be a neighbourhood of $0 \in \mathbb{C}^m$ $(m \geq 2)$ and X, Y — analytic curves in Ω.

THEOREM 3. *If $X = \{(t^p, \varphi(t)) : t \in U\}$, $Y = \{(t^q, \psi(t)) : t \in V\}$, where p, q are positive integers, U, V — neighbourhoods of* 0 *in* $\mathbb{C}$, φ, ψ *— holomorphic mappings satisfying* $\operatorname{ord} \varphi \geq p$, $\operatorname{ord} \psi \geq q$ *and* $X \cap Y = \{0\}$, *then*

$$\mathcal{L}_0(X, Y) = (1/pq) \max_{i=1}^{q} \operatorname{ord} \big(\varphi(t^q) - \psi(\eta^i t^p)\big) \tag{8}$$
$$= (1/pq) \max_{i=1}^{p} \operatorname{ord} \big(\psi(t^p) - \varphi(\varepsilon^i t^q)\big),$$

where η, ε mean primitive roots of unity of degree q and p, respectively.

Proof. By the symmetry of X and Y it suffices to prove the first formula in (8). Denote by ν the right hand side of the first equality in (8). For simplicity, we may assume that

$$\nu = (1/pq) \operatorname{ord}(\varphi(t^q) - \psi(t^p)). \tag{9}$$

Put $d := pq$. From the assumptions and (9) we get that there exist constants C_1, D_1, D_2, $r > 0$ such that for $|t| < r$

$$t^q \in U, \qquad t^p \in U \cap V, \tag{10}$$

$$C_1|t|^d \le |(t^d, \varphi(t^q))| \le D_1|t|^d, \tag{11}$$

$$|\varphi(t^q) - \psi(t^p)| \le D_2|t|^{\nu d}. \tag{12}$$

Let $P(\delta) := \{z \in \mathbb{C}^m : |z| < \delta\}$. Take additionally $\delta > 0$ such that $P(2\delta) \subset \Omega$ and $2\delta < r^d$.

Since $0 \in \mathbb{C}^m$ is an accumulation point of X, then by Proposition 1 it suffices for $x \in X \cap P(\delta)$ to estimate $\varrho(x, Y)$ from above and from below by $|x|^\nu$.

Let $U^* := \{t \in \mathbb{C} : t^q \in U\}$ and $V^* := \{t \in \mathbb{C} : t^p \in V\}$.

First, we estimate $\varrho(x, Y)$ from above for $x \in X \cap P(\delta)$. Let $x = (t^d, \varphi(t^q))$. From the definition of infimum and (10), (11), (12) we have

$$\begin{aligned}\varrho(x, Y) &= \inf_{\tau \in V^*} |(t^d - \tau^d, \varphi(t^q) - \psi(\tau^p))| \\ &\le |(0, \varphi(t^q) - \psi(t^p))| \le D_2|t|^{\nu d} \le D|x|^\nu,\end{aligned} \tag{13}$$

where $D := D_2/C_1^\nu$.

Consider the mapping $F : U^* \times V^* \ni (t, \tau) \mapsto (t^d - \tau^d, \varphi(t^q) - \psi(\tau^p)) \in \mathbb{C}^m$. The mapping has an isolated zero at $0 \in \mathbb{C}^2$. From the definition of the Łojasiewicz exponent, diminishing r if necessary, we have that there exists $C_2 > 0$ such that for $|(t, \tau)| < r$

$$|F(t, \tau)| \ge C_2|(t, \tau)|^{\mathcal{L}_0(F)}. \tag{14}$$

Let us calculate $\mathcal{L}_0(F)$. It is easy to check that F satisfies the assumption of Theorem 2. Then $\mathcal{L}_0(F) = \mathcal{L}_0(F|\Gamma_1)$, where $\Gamma_1 := \{(t, \tau) \in U^* \times V^* : t^d - \tau^d = 0\}$. Hence and from the simple fact that

$$\mathcal{L}_0(F|\Gamma_1) = \max_{i=1}^{d} \mathcal{L}_0(F|\Gamma_{1i}),$$

where $\Gamma_{1i} := \{(t, \tau) \in U^* \times V^* : \tau = \theta^i t\}$ and θ is a primitive root of unity of degree d, we get

$$\mathcal{L}_0(F) = \max_{i=1}^{d} \operatorname{ord}\bigl(\varphi(t^q) - \psi((\theta^i t)^p)\bigr). \tag{15}$$

We easily check that $\{\theta^{ip} : 1 \le i \le d\} = \{\eta^i : 1 \le i \le q\}$. Hence

$$\max_{i=1}^{d} \operatorname{ord}\bigl(\varphi(t^q) - \psi((\theta^i t)^p)\bigr) = \max_{i=1}^{q} \operatorname{ord}\bigl(\varphi(t^q) - \psi(\eta^i t^p)\bigr). \tag{16}$$

From (15), (16) and the definition of ν we get

$$\mathcal{L}_0(F) = \max_{i=1}^{q} \operatorname{ord}\bigl(\varphi(t^q) - \psi(\eta^i t^p)\bigr) = d\nu.$$

Hence and from (14) for $|(t, \tau)| < r$ we get

$$|F(t, \tau)| \ge C_2|t|^{d\nu}. \tag{17}$$

Now, we estimate $\varrho(x, Y)$ from below for $x \in X \cap P(\delta)$. Since $P(2\delta) \subset \Omega$, then there exists $y_0 \in Y \cap P(2\delta)$ such that $\varrho(x, Y) = \varrho(x, y_0)$. Let $x = (t^d, \varphi(t^q))$, $y_0 = (\tau_0^d, \psi(\tau_0^p))$.

Since for $x \in P(\delta)$, $|t| < \delta^{1/d} < r$ and for $y_0 \in P(2\delta)$, $|\tau_0| < (2\delta)^{1/d} < r$, then from (17) and (11) we get

$$\varrho(x,Y) = \varrho(x,y_0) = |F(t,\tau_0)| \geq C_2|t|^{d\nu} \geq C|x|^\nu, \tag{18}$$

where $C := C_2/D_1^\nu$.

Summing up, from (13) and (18) for $x \in X \cap P(\delta)$ we obtain

$$C|x|^\nu \leq \varrho(x,Y) \leq D|x|^\nu,$$

which gives that $\mathcal{L}_0(X,Y) = \nu$.

This ends the proof. ∎

We shall now give a second formula for $\mathcal{L}_0(X,Y)$ in terms of the first version of the Puiseux Theorem ([Ł$_1$], II.6.1) in the two-dimensional case.

First we give a simple lemma. Let Ω be a neighbourhood of $0 \in \mathbb{C}^2$, $h : \Omega \to \mathbb{C}$ a distinguished pseudopolynomial in y of degree r and $Z := V(h)$. Assume additionally that $\hat{h} \in \mathcal{O}^2$ is irreducible and that $W \ni t \mapsto (t^r, \lambda(t)) \in \Omega$ is a parametrization of Z.

Lemma 1. *If there exist a positive integer D, a disc $\Delta = \{t \in \mathbb{C} : |t| < \delta\}$ and functions $\gamma_1, \ldots, \gamma_r$ — holomorphic in Δ, such that $\{t \in \mathbb{C} : |t| < \delta^{D/r}\} \subset W$ and $h(t^D, y) = \prod_{i=1}^r (y - \gamma_i(t))$, then*

(a) $r|D$,

(b) *after an appropriate renumbering of γ_i we have $\gamma_i(t) = \lambda(\varepsilon^i t^{D/r})$ in Δ where ε is a primitive root of unity of degree r.*

Proof. Let $\Phi(t) := (t^r, \lambda(t))$ and $\Psi_i(t) := (t^D, \gamma_i(t))$. Put $\delta_i : \Delta \ni t \mapsto \Phi^{-1} \circ \Psi_i(t) \in W$. The function δ_i is continuous and $[\delta_i(t)]^r = t^D$ in Δ. Hence it is a branch of r-th root of t^D in $\Delta \setminus \{0\}$, so, it is holomorphic in Δ. Hence we easily get that $r|D$ and there exists j that $\gamma_i(t) = \lambda(\varepsilon^j t^{D/r})$ for $t \in \Delta$. Since h is an irreducible polynomial, then γ_i are different. Hence by a renumbering we get $\gamma_i(t) = \lambda(\varepsilon^i t^{D/r})$ for $t \in \Delta$. This ends the proof of the lemma. ∎

Let us return to the announced theorem. Let Ω be a neighbourhood of $0 \in \mathbb{C}^2$, X, Y — analytic curves in Ω and $X \cap Y = \{0\}$. Assume that $X = V(f)$, $Y = V(g)$, where f and g are distinguished pseudopolynomials in y of degree p and q, respectively.

Theorem 4. *If there exist a positive integer D and holomorphic functions $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ in a neighbourhood of* $0 \in \mathbb{C}$ *such that* $\operatorname{ord} \alpha_i \geq D$, $\operatorname{ord} \beta_i \geq D$ *and*

$$\begin{aligned} f(t^D, y) &= \prod_{i=1}^p (y - \alpha_i(t)), \\ g(t^D, y) &= \prod_{j=1}^q (y - \beta_j(t)), \end{aligned} \tag{19}$$

then

$$\mathcal{L}_0(X,Y) = (1/D) \max_{i=1}^p \max_{j=1}^q \operatorname{ord}(\alpha_i - \beta_j). \tag{20}$$

P r o o f. By Proposition 3 we may assume that X, Y generate irreducible germs at 0. In consequence, we may also assume that $\hat{f}$, $\hat{g}$ are irreducible in $\mathcal{O}^2$. Let now $U \ni t \mapsto (t^p, \varphi(t)) \in X$, $V \ni t \mapsto (t^q, \psi(t)) \in Y$ be parametrizations of X and Y. Let us take such a small δ that the functions $\alpha_1, \ldots, \alpha_p$, $\beta_1, \ldots, \beta_q$ are defined in $\Delta := \{t \in \mathbb{C} : |t| < \delta\}$ and $\{t \in \mathbb{C} : |t| < \delta^{D/p}\} \subset U$, $\{t \in \mathbb{C} : |t| < \delta^{D/q}\} \subset V$ hold. Then from (19), by Lemma 1, we get $p|D$, $q|D$ and, after a renumbering,

$$\alpha_i(t) = \varphi(\varepsilon^i t^{D/p}), \qquad \beta_j(t) = \psi(\eta^j t^{D/q}) \qquad \text{for } t \in \Delta,$$

where ε, η are primitive roots of unity of degree p and q, respectively. Hence we immediately obtain that $\operatorname{ord}\varphi \geq p$, $\operatorname{ord}\psi \geq q$ and

$$(1/D)\operatorname{ord}(\alpha_i - \beta_j) = (1/pq)\operatorname{ord}(\varphi(\varepsilon^i t^q) - \psi(\eta^j t^p)).$$

Since for every $i \in \{1, \ldots, p\}$ the function $\{t \in \mathbb{C} : \varepsilon^i t \in U\} \ni t \mapsto (t^p, \varphi(\varepsilon^i t)) \in X$ is a parametrization of X and $\operatorname{ord}\varphi \geq p$, $\operatorname{ord}\psi \geq q$, then from Theorem 3 we have

$$(1/D)\max_{j=1}^{q}\big(\operatorname{ord}(\alpha_i - \beta_j)\big) = (1/pq)\max_{j=1}^{q}\big(\operatorname{ord}(\varphi(\varepsilon^i t^q) - \psi(\eta^j t^p))\big) = \mathcal{L}_0(X, Y).$$

Hence we get (20). This ends the proof. ∎

R e m a r k. The assumptions in Theorem 4 are not restrictive, because for any analytic curves X, Y in Ω, $X \cap Y = \{0\}$, there is a linear change of coordinates in $\mathbb{C}^2$ such that in these new coordinates X and Y satisfy these assumptions.

4. Concluding remarks. Let Ω be a neighbourhood of $0 \in \mathbb{C}^m$, $X, Y \subset \Omega$ analytic curves such that $X \cap Y = \{0\}$. Denote by $C(X)$, $C(Y)$ the tangent cones at 0 to X, Y, respectively. From Theorem 3 we obtain

COROLLARY ([T], Cor. 3.4). *Under the above assumptions*

(a) $\mathcal{L}_0(X, Y) \geq 1$,

(b) $\mathcal{L}_0(X, Y) = 1$ *if and only if* $C(X) \cap C(Y) = \{0\}$.

P r o o f. Let $H_1 := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : z_1 = 0\}$. Without loss of generality, at the cost of linear change of coordinates, we may assume that $H_1 \cap C(X) = \{0\}$, $H_1 \cap C(Y) = \{0\}$. We may also assume (see Proposition 3) that X and Y generate irreducible germs at $0 \in \mathbb{C}^m$. Then X, Y satisfy the assumptions of Theorem 3 and hence (a) is obvious. Moreover, $\mathcal{L}_0(X, Y) = 1$ if and only if $\operatorname{ord}(\varphi(t^q) - \psi(t^p)) = pq$. But this holds if and only if X and Y have different tangent lines at 0. ∎

Let X, Y be as at the beginning of this section. Let $\mu(X, Y)$ mean the intersection multiplicity of X and Y at 0 and $\deg X$, $\deg Y$ degrees of X and Y at 0. P. Tworzewski [T] proved that

$$\mathcal{L}_0(X, Y) \leq \mu(X, Y) - \deg X \deg Y + 1. \tag{21}$$

Now we give an example for which the equality in (21) does not hold.

EXAMPLE. Let $X = \{(x, y, z) \in \mathbb{C}^3 : x^3 - yz = 0,\ y^2 - xz = 0,\ z^2 - x^2 y = 0\}$, $Y = \{(x, y, z) \in \mathbb{C}^3 : x^3 - \varepsilon yz = 0,\ y^2 - \varepsilon xz = 0,\ z^2 - \varepsilon x^2 y = 0\}$, where ε is a primitive root of unity of degree 3. It is easy to show ([M], Ex. 3.2) that X and Y generate irreducible germs at $0 \in \mathbb{C}^3$. Moreover, $\mathbb{C} \ni t \mapsto (t^3, t^4, t^5) \in X$, $\mathbb{C} \ni t \mapsto (t^3, t^4, \varepsilon^2 t^5) \in Y$

are their parametrizations. Obviously, $\deg X = 3$, $\deg Y = 3$ and $\mu(X,Y) = 13$ (it can be calculated directly from the definition of the multiplicity, given in [T]). Whereas, from Theorem 3 we have

$$\begin{aligned}\mathcal{L}_0(X,Y) &= (1/9)\max_{i=1}^{3}\min\big(\operatorname{ord}(t^{12}-\varepsilon^i t^{12}), \operatorname{ord}(t^{15}-\varepsilon^{2+2i}t^{15})\big)\\ &= (1/9)\max(12,12,15) = (5/3).\end{aligned}$$

Acknowledgements. We thank S. Spodzieja for helpful discussions and a technical help in preparing the paper for publication and P. Tworzewski for discussions on the intersection multiplicity.

References

[CK] J. Chądzyński, T. Krasiński, *A set on which the local Łojasiewicz exponent is attained*, Ann. Polon. Math. 67 (1997), 191–197.

[CT] E. Cygan, P. Tworzewski, *Proper intersection multiplicity and regular separation of analytic sets*, Ann. Polon. Math. 59 (1994), 293–298.

[Ł1] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel, 1991.

[Ł2] S. Łojasiewicz, *Ensembles semi-analytiques*, Inst. de Hautes Études Scientifiques, Bures-sur-Yvette, 1965.

[M] A. Miodek, *On some method of calculating a multiplicity*, Bull. Soc. Sci. Lett. Łódź 37 (1987), No. 7, 9 pp.

[T] P. Tworzewski, *Isolated intersection multiplicity and regular separation of analytic sets*, Ann. Polon. Math. 58 (1993), 213–219.