SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

SIMPLE GERMS OF CORANK ONE AFFINE DISTRIBUTIONS

MICHAIL ZHITOMIRSKII
Department of Mathematics, Technion
32000 Haifa, Israel
E-mail: mzhi@techunix.technion.ac.il

WITOLD RESPONDEK
INSA de Rouen, LMI, UPRES-A CNRS 6085
Place Emile Blondel, 76131 Mont Saint Aignan, France
E-mail: wresp@lmi.insa-rouen.fr

1. Introduction

1.1. *Affine distributions.* All objects are assumed to be smooth (of class C^∞). An affine distribution on R^n of rank m (or corank $n-m$) is a family $A = \{A_p\}_{p\in R^n}$ of m-dimensional affine subspaces $A_p \subset T_pR^n$. If A_p is a subspace, i.e., A_p contains the zero tangent vector, then p is called an *equilibrium point* of A. Two germs A and $\tilde{A}$ of corank one affine distributions, at points p and $\tilde{p}$ respectively, are equivalent if there exists a local diffeomorphism Φ sending p to $\tilde{p}$ such that $\Phi_*(A_x) = \tilde{A}_{\Phi(x)}$ for each x close to p.

1.2. *Simple germs.* Our purpose is to list simple (of zero modality) germs of affine distributions. The definition of simplicity is the same as in any local classification problem (see [AVG, 85]). Namely, a germ A at a point p is called simple if there exist a finite number l such that A is l-determined (which means that A is equivalent to any germ $\tilde{A}$ at p such that $j^l_p\tilde{A} = j^l_pA$) and a finite tuple of germs at the origin such that any germ at p with the l-jet sufficiently close to j^l_pA is equivalent to one of the germs of this tuple. To define j^l_pA one can describe A by $m+1$ vector fields $v_0,\dots,v_m$ such that $A_x = v_0(x) + \mathrm{span}(v_1(x),\dots,v_m(x))$ for each x close to p. Then j^l_pA is equal (resp. close)

Research of the first author supported by the fund for promotion of research at the Technion.

The second author is on leave from the Institute of Mathematics, Polish Academy of Sciences, his research was partially supported by KBN grant 2P03A 004 09.

1991 *Mathematics Subject Classification*: Primary 58A30; Secondary 93B10.

Received by the editors: March 28, 1997; in the revised form: July 29, 1997.

The paper is in final form and no version of it will be published elsewhere.

to $j^l_p\tilde{A}$ if the germs A and $\tilde{A}$ can be described by tuples of vector fields with equal (resp. close) l-jets at p.

1.3. *Why of corank one?* In this paper we study corank one affine distributions only since simple dimensional arguments show that if $m \le n-2$ then there are no simple germs, see [T, 89] and [J, 90].

1.4. *Affine distributions and control systems.* Any result of this paper can be reformulated in terms of control theory: the problem of local classification of affine distributions coincides with the problem of feedback classification of control affine systems, see [J, 90]. A control affine system has the form $\dot{x} = v_0(x) + u_1 v_1(x) + \cdots + u_m v_m(x)$, where v_i's are vector fields (v_0 is called a drift vector field) and u_i's are controls. The above given description of affine distributions via tuples of vector fields allows to pass from control affine systems to affine distributions and vice versa provided that $v_1, \ldots, v_m$ are pointwise independent. In this case two control affine systems are called feedback equivalent if they define equivalent (i.e., the same up to a choice of coordinates) affine distributions.

1.5. *Germs at a nonequilibrium point*: *reduction to a classical problem.* The starting point for classification of corank one affine distributions is the following simple observation which was made in [Zh, 92], App. C: *the problem of local classification of corank one affine distributions* (*and the problem of local feedback classification of control affine systems on R^n with $n-1$ controls*) *is exactly the classical problem of classification of nonvanishing differential* 1*-forms provided one studies germs at a nonequilibrium point.*

To explain this observation it suffices to note that if p is a nonequilibrium point of a corank one affine distribution A then near p there exists a unique differential 1-form ω such that

$$A_x = \{\xi \in T_x R^n : \omega(x)(\xi) = 1\}.$$

Notation. This relation between ω and A will be denoted by $A = (\omega, 1)$.

1.6. *Simple germs at a nonequilibrium point.* It was proved in [Zh, 92], Sect. 12 that any simple germ of a differential 1-form on R^{2k+1} (resp. R^{2k}) is equivalent to one and only one of three models (fixed germs at the origin): the Darboux model D (resp. D') and two Martinet models $M_\pm$ (resp. $M'_\pm$) ([1]), where

$$\begin{aligned} D &= dz + x_1 dy_1 + \cdots + x_k dy_k, \\ M_\pm &= \pm z dz + (1 + x_1) dy_1 + x_2 dy_2 + \cdots + x_k dy_k, \\ D' &= (1 + x_1) dy_1 + x_2 dy_2 + \cdots + x_k dy_k, \\ M'_\pm &= (1 \pm x_1^2) dy_1 + x_2 dy_2 + \cdots + x_k dy_k. \end{aligned}$$

Therefore the following statement holds:

THEOREM 1 ([Zh, 92], App. C). *All simple germs of corank one affine distributions at a nonequilibrium point are exhausted*, *up to equivalence*, *by the models* $(D, 1)$ *and* $(M_\pm, 1)$

([1]) This result was announced in [Zh, 85] (with the proof for the 2-dimensional case).

if $n = 2k + 1$ *and by the models* $(D', 1)$ *and* $(M'_{\pm}, 1)$ *if* $n = 2k$. [2]

Using results of the works [M, 70] and [Zh, 92], Sect. 11, one can easily determine whether a germ $(\omega, 1)$ is simple or not. Namely, given a germ $A = (\omega, 1)$ at a point $p \in R^{2k+1}$ (resp. R^{2k}) we take a nondegenerate volume form Ω and define the function $F = (\omega \wedge (d\omega)^k)/\Omega$ (resp. $F = (d\omega)^k/\Omega$). Then A is equivalent to the Darboux model if and only if $F(p) \neq 0$ and to one of the Martinet models if and only if $F(p) = 0$ and the n-form $\Omega_1 = dF \wedge (d\omega)^k$ (resp. $\Omega_1 = dF \wedge \omega \wedge (d\omega)^{k-1}$) does not vanish at p. For any n the sign $+$ (resp. $-$) in the Martinet models corresponds to the case where the n-forms $\Omega(p)$ and $\Omega_1(p)$ define the same (resp. different) orientations.

1.7. *Germs at an equilibrium point.* Given a corank one distribution A and any point p (equilibrium or not) there exist a nonvanishing differential 1-form ω and a function f such that for each x close to p we have

$$A_x = \{\xi \in T_x R^n : \omega(x)(\xi) = f(x)\}.$$

Notation. We will write this relation in the form $A = (\omega, f)$.

Note that (ω, f) and $(\tilde{\omega}, \tilde{f})$ is the same affine distribution if and only if there exists a function T such that $T(p) \neq 0$ and $\tilde{\omega} = T\omega, \tilde{f} = Tf$. The point p is an equilibrium point of A if and only if $f(p) = 0$. Therefore *classification of germs of corank one affine distributions at an equilibrium point p is the local classification of pairs* (ω, f), $\omega(p) \neq 0$, $f(p) = 0$, *with respect to the following equivalence*: (ω, f) *is equivalent to* $(\tilde{\omega}, \tilde{f})$ *if there are a local diffeomorphism* Φ *and a nonvanishing function* T *such that* $\Phi^*\omega = T\tilde{\omega}, f \circ \Phi = T\tilde{f}$.

1.8. 2-*dimensional and* 3-*dimensional cases.* A classification of simple germs of corank one affine distributions on R^n, $n = 2, 3$, at an equilibrium point is contained in [JR, 90] for the 2-dimensional case and in [RZh, 95] for the 3-dimensional case (in both papers — in terms of control theory). Using the notation in Section 1.7 we can reformulate this classification as follows: any simple germ of a corank one distribution on R^2 (resp. R^3) at an equilibrium point is equivalent to the germ at the origin of the affine distribution (dx, y) (resp. to the germ at the origin of one of the three affine distributions $(dz + xdy, y)$, $(dz + x^2dy, y)$, $(dz - x^2dy, y)$).

2. New results. The new result of this paper is a complete classification of simple germs at an equilibrium point of corank one affine distributions on R^n, $n \geq 4$.

2.1. *Classification of simple germs.* In the following theorem (which is valid for $n \geq 3$) one meets the Darboux and Martinet models D and $M_{\pm}$ in the odd-dimensional case (see Section 1.6), and the differential 1-form

$$Q = dy_1 + x_2dy_2 + \cdots + x_kdy_k$$

on R^{2k}. Note that by one of Darboux theorems the field of kernels of Q is equivalent to generic germ of corank one distribution on R^{2k}. We use the notation of Section 1.7.

[2] In the 2-dimensional case ($k = 1$) this result is contained, in terms of control theory, in the work [JR, 90], where Darboux and Martinet models and results of the work [Zh, 85] are not used.

THEOREM 2. *Any simple germ at an equilibrium point of a corank one affine distribution on R^{2k+1}, $k \geq 1$ (resp. R^{2k}, $k \geq 2$), is equivalent to the germ at the origin of one and only one of the three distributions (D, x_1), $(M_\pm, x_1)$ (resp. two distributions (Q, x_1), $(Q, x_2 + x_1^2)$).*

2.2. *Singularity classes.* Now we give a way to determine whether a given germ $A = (\omega, f)$ at an equilibrium point p is simple or not and, if it is, to which of the models it is equivalent. There is no loss of generality to assume that p is the origin. Fix a nondegenerate volume form Ω. Given a germ $A = (\omega, f)$ we define the function

$$F = (\omega \wedge (d\omega)^k)/\Omega,$$

if $n = 2k + 1$, and the function

$$G = (\omega \wedge (d\omega)^{k-1} \wedge df)/\Omega,$$

if $n = 2k$. We introduce the following sets of germs:

the set $\mathrm{Orb}(D, x_1)$ consisting of germs of affine distributions (ω, f) at $0 \in R^{2k+1}$ such that $f(0) = 0$, $F(0) \neq 0$, $(\omega \wedge df)(0) \neq 0$;

the set $\mathrm{Orb}(M_+, x_1)$ (resp. $\mathrm{Orb}(M_-, x_1)$) consisting of germs of affine distributions (ω, f) at $0 \in R^{2k+1}$ such that $f(0) = 0$, $F(0) = 0$, the n-form $\Omega_1 = \omega \wedge (d\omega)^{k-1} \wedge dF \wedge df$ does not vanish at 0 and the n-forms $\Omega(0)$ and $\Omega_1(0)$ define the same (resp. different) orientations;

the set $\mathrm{Orb}(Q, x_1)$ consisting of germs of affine distributions (ω, f) at $0 \in R^{2k}$ such that $f(0) = 0$ and $G(0) \neq 0$;

the set $\mathrm{Orb}(Q, x_2 + x_1^2)$ consisting of germs of affine distributions (ω, f) at $0 \in R^{2k}$ such that $f(0) = 0$, $G(0) = 0$, $(\omega \wedge (d\omega)^{k-1} \wedge dG)(0) \neq 0$ and $(\omega \wedge df)(0) \neq 0$.

One can easily check that these sets do not depend on the volume form Ω and they are well-defined disjoint sets of germs of corank one distributions: any of the above given sets is distinguished by a condition which is invariant with respect to the change of a pair (ω, f) by a pair $(T\omega, Tf)$, where T is a nonvanishing function.

THEOREM 3. *Denote by α any of 5 models in Theorem 2. A germ (ω, f) of a corank one affine distribution at equilibrium point $0 \in R^n$ is equivalent to the model α if and only if it belongs to the above described set $\mathrm{Orb}(\alpha)$.*

In other words, $\mathrm{Orb}(\alpha)$ is the orbit of the model α. Taking into account results of Section 1.6, we obtain that the set of all simple germs at the origin of corank one affine distributions on R^{2k+1} (resp. R^{2k}, $k \geq 2$) consists of one open orbit, three codimension one orbits and two (resp. one) codimension two orbits. The set of non-simple germs has codimension two.

3. Proofs.

3.1. *Homotopy method.* Beginning from this section all objects are assumed to be germs at the origin. To prove Theorem 3 we use the homotopy method. In the following proposition all families depend smoothly on a parameter $t \in [0, 1]$. By $L_X\omega$ we denote the Lie derivative of a differential 1-form ω along a vector field X: $L_X\omega = X \rfloor d\omega + d(X \rfloor \omega)$.

PROPOSITION 1. *Consider a differential* 1*-form* ω, *functions* f *and* δ, *and let* $f_t = f + t\delta$. *Assume that there exist a family of vector fields* X_t, $X_t(0) = 0$, *and a family of functions* h_t, *such that* $L_{X_t}\omega = h_t\omega$ *and* $X_t(f_t) = h_t f_t - \delta$ *for each* $t \in [0,1]$. *Then the affine distribution* (ω, f) *is equivalent to the affine distribution* $(\omega, f+\delta)$.

Proof. Define a family of local diffeomorphisms ϕ_t by the equation $d\phi_t/dt = X_t(\phi_t)$, $\phi_0 = \mathrm{id}$, and a family of nonvanishing functions T_t by the equation $dT_t/dt = (h_t \circ \phi_t)T_t$, $T_0 \equiv 1$. Then

$$\frac{d}{dt}(\phi_t^*\omega) = \phi_t^*(L_{X_t}\omega) = (h_t \circ \phi_t)\phi_t^*\omega, \quad \frac{d}{dt}(f_t(\phi_t)) = (X_t(f_t) + \delta) \circ \phi_t = (h_t f_t) \circ \phi_t.$$

It follows that $\phi_t^*\omega = T_t\omega$ and $f_t \circ \phi_t = T_t f_0$. In particular, as $t = 1$ we obtain $\phi_1^*\omega = T_1\omega$, $(f+\delta)\circ\phi_1 = T_1 f$. ∎

3.2. *Proof of Theorem* 3. For each model α in Theorem 2 it is easy to check that $\alpha \in \mathrm{Orb}(\alpha)$. Thus it is enough to prove that if $(\omega, f) \in \mathrm{Orb}(\alpha)$ then (ω, f) is equivalent to α.

a) Let $(\omega, f) \in \mathrm{Orb}(D, x_1)$. Then ω is a contact form and we can assume that $\omega = D$. The condition $(D \wedge df)(0) \neq 0$ implies that there is no loss of generality to assume $(\partial f/\partial x_1)(0) \neq 0$ and, moreover, $(\partial f/\partial x_1)(0) > 0$ (we change, if necessary, the signs of the coordinates x_1 and y_1). Now we will reduce f to x_1 using Proposition 1. Let $\delta = f - x_1$, $f_t = x_1 + t\delta$. The set of solutions (X, h) of the equation $L_X D = hD$, $X(0) = 0$ is well known, see [Ar, 78]:

$$h = \frac{\partial u}{\partial z},\ X = \sum A_i \frac{\partial}{\partial x_i} + \sum B_i \frac{\partial}{\partial y_i} + C\frac{\partial}{\partial z},$$
$$A_i = x_i h - \frac{\partial u}{\partial y_i},\ B_i = \frac{\partial u}{\partial x_i},\ C = u - x_1\frac{\partial}{\partial x_1} - \ldots - x_k\frac{\partial}{\partial x_k},$$

where u is an arbitrary function in n variables such that $(\partial u/\partial x_i)(0) = (\partial u/\partial y_i)(0) = 0$. The equation $X_t(f_t) = h_t f_t - \delta$ with X_t and h_t of the above form for each t reduces to the equation

$$Z_t(u_t) + a_t u_t + \delta = 0,$$

where Z_t is a family of vector fields and a_t is a family of functions. This equation has a solution u_t with vanishing at 0 derivatives (for all t) since $\delta(0) = 0$ and $Z_t(0) \neq 0$ for all $t \in [0,1]$. The latter follows from the relations

$$Z_t(y_1) = -\frac{\partial f_t}{\partial x_1},\ \frac{\partial f}{\partial x_1}(0) > 0$$

which imply $Z_t(y_1)(0) < 0$, $t \in [0,1]$.

b) Let $(\omega, f) \in \mathrm{Orb}(M_+, x_1) \cup \mathrm{Orb}(M_-, x_1)$. Then $(\omega \wedge (d\omega)^{k-1} \wedge dF)(0) \neq 0$, therefore the field of kernels of ω is equivalent to the field of kernels of the 1-form M_+, see [M, 70]. This means that there is no loss of generality to assume that $\omega = M_+$. The condition $(M_+, f) \in \mathrm{Orb}(M_+, x_1)$ (resp. $(M_+, f) \in \mathrm{Orb}(M_-, x_1)$) is equivalent to the condition $(\partial f/\partial x_1)(0) > 0$ (resp. $(\partial f/\partial x_1)(0) < 0$). Note that the affine distributions $(M_+, -x_1)$ and (M_-, x_1) are equivalent (the form $(-M_+)$ can be obtained from M_- by the change of signs of the coordinates $y_1, \ldots, y_k$). Therefore to prove that (M_+, f) is equivalent to

(M_+, x_1) (resp. to (M_-, x_1)) we have to reduce f to x_1 (resp. to $-x_1$). In what follows we consider only the case $(M_+, f) \in \mathrm{Orb}(M_+, x_1)$. In the case $(M_+, f) \in \mathrm{Orb}(M_-, x_1)$ the arguments are similar.

Let $\delta = f - x_1,\ f_t = x_1 + t\delta$. By Proposition 1 in order to reduce f to x_1 it suffices to prove the solvability of the system $L_{X_t} M_+ = h_t M_+,\ X_t(f_t) = h_t f_t - \delta$ with respect to a family (X_t, h_t) such that $X_t(0) = 0$. The equation $L_{X_t} M_+ = h_t M_+,\ X_t(0) = 0$ can be easily solved (see [RZh, 95]). The set of all solutions (X_t, h_t) contains solutions of the form

$$h_t = 2v_t + z\frac{\partial v_t}{\partial z},\ X_t = \sum A_{i,t}\frac{\partial}{\partial x_i} + \sum B_{i,t}\frac{\partial}{\partial y_i} + C_t\frac{\partial}{\partial z},$$

$$A_{1,t} = -z^2\frac{\partial v_t}{\partial y_1} + (1 + x_1)h_t,\ A_{i,t} = -z^2\frac{\partial v_t}{\partial y_i} + x_i h_t\ (i = 2, \ldots, k),$$

$$B_{i,t} = z^2\frac{\partial v_t}{\partial x_i},\ C = zv_t - (1 + x_1)z\frac{\partial v}{\partial x_1} - x_2 z\frac{\partial v_t}{\partial x_2} - \cdots - x_k z\frac{\partial v_t}{\partial x_k},$$

where v_t is an arbitrary family of function in n variables such that $v_t(0) = 0$. The equation $X_t(f_t) = h_t f_t - \delta$ with such X_t and h_t can be written as a system for h_t and v_t of the form

$$W_t h_t + z\frac{\partial f_t}{\partial z}v_t + zR_t(v_t) + \delta = 0, \quad h_t = 2v_t + z\frac{\partial v_t}{\partial z},$$

where R_t is a family of vector fields such that $R_t(z) \equiv 0$ and W_t is a family of functions such that $W_t(0) = (\partial f_t/\partial x_1)(0)$. Now we use our condition $(\partial f/\partial x_1)(0) > 0$. It follows that $(\partial f_t/\partial x_1)(0) > 0,\ t \in [0, 1]$, and eliminating h_t from the first equation we reduce the system to an equation for v_t of the form

$$(2 + g_t)v_t + z\frac{\partial v_t}{\partial z} + zE_t(v_t) = \delta_t,$$

where E_t is a family of vector fields such that $E_t(z) \equiv 0,\ \ g_t$ and δ_t are families of functions, $g_t(0) = \delta_t(0) = 0$. The solvability of this equation follows from its solvability in formal series with respect to z and hyperbolicity of the vector field $z(\partial/\partial z) + zE_t$ on the hypersurface $\{z = 0\}$ (see [Zh, 92], Ch. 2 and [RZh, 95]). It remains to note that $v_t(0) = 0$ since $\delta_t(0) = 0$.

c) Let $(\omega, f) \in \mathrm{Orb}(Q, x_1) \cup \mathrm{Orb}(Q, x_2 + x_1^2)$. Then $\omega \wedge (d\omega)^{k-1}(0) \neq 0$ and by Darboux theorem the field of kernels of ω is equivalent to the field of kernels of the 1-form Q. Therefore there is no loss of generality to assume that $\omega = Q$. If $(Q, f) \in \mathrm{Orb}(Q, x_1)$ then $(Q \wedge (dQ)^{k-1} \wedge df)(0) \neq 0$, or, equivalently, $(\partial f/\partial x_1)(0) \neq 0$. The latter condition implies that (Q, f) is equivalent to (Q, x_1).

Assume now that $(Q, f) \in \mathrm{Orb}(Q, x_2 + x_1^2)$. Then the function f has the following properties:

$$\frac{\partial f}{\partial x_1}(0) = 0,\ \frac{\partial^2 f}{\partial x_1^2}(0) \neq 0, \quad (Q \wedge df)(0) \neq 0.$$

It follows that (Q, f) is equivalent to $(Q, \pm x_1^2 + \nu)$, where ν is a function of the variables $y_1, x_2, y_2, \ldots, x_k, y_k$ such that $(Q \wedge d\nu)(0) \neq 0$. By a) the affine distribution (Q, ν) on $R^{2k-1}(y_1, x_2, y_2, \ldots, x_k, y_k)$ is equivalent to (Q, x_2). Then $(Q, \pm x_1^2 + \nu)$ is equivalent to $(Q, x_2 \pm x_1^2 T)$, where T is a nonvanishing function. Changing the coordinate x_1 we can reduce T to 1. It remains to note that $(Q, x_2 - x_1^2)$ is equivalent to $(Q, x_2 + x_1^2)$ since the change of the signs of the coordinates $y_1, x_2, \dots, x_k$ brings Q to $-Q$. ■

3.3. *Proof of Theorem* 2. Throughout the proof we use Theorem 3 and the description of the orbits in Sections 1.6 and 2.2. At first we consider the odd-dimensional case, $n = 2k+1$. Denote by S_1 the union of the orbits of the models $(D, 1)$ and $(M_\pm, 1)$ and by S the union of the orbits of the models $(D, 1), (M_\pm, 1), (D, x_1)$ and $(M_\pm, x_1)$. Given a 1-form ω we associate to it a function $F = \omega \wedge (d\omega)^k/\Omega$, where Ω is a nondegenerate volume form. Introduce the following sets of germs of corank one affine distributions:

$$B_1 = \{(\omega, f): \ F(0) = 0, \ (f(d\omega)^k \wedge dF + k\omega \wedge (d\omega)^{k-1} \wedge df \wedge dF)(0) = 0\};$$
$$B_2 = \{(\omega, f): f(0) = 0, \ (\omega \wedge df)(0) = 0\}.$$

LEMMA 1. *Let $f(0) \neq 0$. Then $(\omega, f) \in S_1$ if and only if $(\omega, f) \notin B_1$.*

To prove the lemma we write the affine distribution (ω, f) in the form $(\omega/f, 1)$ and use the description of the models in Section 1.6.

Using the description of the models in Section 2.2 we obtain the following corollary: $(\omega, f) \in S$ if and only if $(\omega, f) \notin B_1 \cup B_2$. It follows that the set of 2-jets of germs of S is open and we obtain that any germ of S is simple. Another corollary of Lemma 1 (and results of Section 1.6) is the absence of simple germs in the set $B_1 \cap \{(\omega, f): \ f(0) \neq 0\}$ and thus in the set B_1.

To prove Theorem 2 in the odd-dimensional case it remains to show that there are no simple germs in the set B_2. It suffices to prove that a germ $(\omega, f) \in B_2$ such that $df(0) \neq 0$ is not simple. Let $\mu = \mu(\omega, f)$ be the pullback of ω to the hypersurface $\{f = 0\}$. Note that $\mu(0) = 0$ and that if (ω, f) is equivalent to $(\tilde\omega, \tilde f)$ then the Pfaffian equations generated by $\mu(\omega, f)$ and $\mu(\tilde\omega, \tilde f)$ are equivalent. It follows that the problem of classification of generic germs of the set B_2 contains the problem of classification of generic Pfaffian equations generated by 1-forms which vanish at 0. In the latter problem there are no simple germs, see [L, 75] and [Zh, 92], Sect. 21, 29.

Now we consider the even-dimensional case, $n = 2k \geq 4$. At first let us note that a germ (ω, f) such that $(\omega \wedge (d\omega)^{k-1})(0) = 0$ is not simple since in this case the germ of the Pfaffian equation generated by ω is not simple, see [Zh, 92], Sect. 25. Therefore it suffices to prove Theorem 2 within the set C consisting of germs of the form (Q, f). Denote by C_1 the subset of C consisting of germs (Q, f) such that $(\partial f/\partial x_1)(0) = (\partial^2 f/\partial x_1^2)(0) = 0$ and by C_2 the subset of C consisting of germs (Q, f) such that $f(0) = 0, \ (Q \wedge df)(0) = 0$. Using the description of orbits in Sections 1.6 and 2.2 it is easy to show that a germ (Q, f) belongs to the union S' of the orbits of the models $(D', 1), (M'_\pm, 1), (Q, x_1), (Q, x_2 + x_1^2)$ if and only if $(Q, f) \notin (C_1 \cup C_2)$. This observation and results of Section 1.6 imply that (i) the set of 2-jets of germs of the set S' is open, therefore any germ of S' is simple; (ii) there are no simple germs belonging to the set $C_1 \cap \{(\omega, f) : f(0) \neq 0\}$, therefore there are no simple germs in the set C_1. Arguing as in the odd-dimensional case we obtain that there are no simple germs $(Q, f) \in C_2$, therefore any simple germ belongs to S'. ■

References

[AVG, 85] V. I. Arnol′d, S. M. Guseĭn-Zade, A. N. Varchenko, *Singularities of Differentiable Maps*, vol. 1, Birkhäuser, Boston, 1985.

[Ar, 78] V. I. Arnol′d, *Mathematical Methods in Classical Mechanics*, Springer-Verlag, New York, 1978.

[J, 90] B. Jakubczyk, *Equivalence and invariants of nonlinear control systems*, in: Nonlinear Controllability and Optimal Control, H. Sussmann (ed.), Marcel Dekker, New York, 1990, 177–218.

[JR, 90] B. Jakubczyk, W. Respondek, *Feedback equivalence of planar systems and stabilizability*, in: Robust Control of Linear Systems and Nonlinear Control, Proc. of the Internat. Sympos. MTNS-89, vol. 2, M. A. Kaashhoek, J. H. van Schuppen, and A. C. M. Ran (eds.), Birkhäuser, Boston, 1990, 447–456.

[L, 75] V. Lychagin, *Local classification of nonlinear first-order partial differential equations*, Uspekhi Mat. Nauk 30 (1975), 101–171; English transl. Russian Math. Surveys 30 (1975).

[M, 70] J. Martinet, *Sur les singularites des formes differentielles*, Ann. Inst. Fourier (Grenoble) 20 (1970), 95–178.

[RZh, 95] W. Respondek, M. Zhitomirskii, *Feedback classification of nonlinear control systems on* 3*-manifolds*, Math. Control Signals Systems 8 (1995), 299–333.

[T, 89] K. Tchoń, *On normal forms of affine systems under feedback*, in: New Trends in Nonlinear Control Theory, J. Descusse, M. Fliess, A. Isidori, and D. Leborgne (eds.), Lecture Notes in Control and Information Sciences 122, Springer-Verlag, 1989, 23–32.

[Zh, 85] M. Zhitomirskii, *Finitely determined* 1*-forms* $\omega,\ \omega|_0 \neq 0$ *are exhausted by the models of Darboux and Martinet*, Funktsional Anal. i Prilozhen. 19 (1985), 71–72; English transl. Funct. Anal. Appl. 19 (1985).

[Zh, 92] M. Zhitomirskii, *Typical Singularities of Differential 1-Forms and Pfaffian Equations*, Translations of Mathematical Monographs 113, AMS, Providence, 1992.