SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

A SINGULAR RIEMANN-ROCH FOR HIRZEBRUCH CHARACTERISTICS

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1. Introduction. The Hirzebruch-Riemann-Roch theorem (HRR) for vector bundles on a non-singular complex variety was generalized by Grothendieck (GRR) and further extended to singular varieties by Baum, Fulton and MacPherson (BFM-RR). The HRR says, symbolically speaking, that $\chi = T$, where χ denotes the Euler-Poincaré characteristic of the bundle and T denotes its Todd characteristic. Hirzebruch generalized these two characteristics to χ_y and T_y, introducing a parameter y, and he showed that $\chi_y = T_y$. In this paper we give a "BFM-RR version" of this generalized HRR.

2. Riemann-Roch theorems. In this section we briefly recall the above-mentioned three Riemann-Roch theorems in a historical order.

Let X be a non-singular complex projective variety and E a holomorphic vector bundle over X. Let $\chi(X, E) = \sum_{i=0}^{\infty}(-1)^i \dim_{\mathbf{C}} H^i(X; \Omega(E))$ be the Euler-Poincaré characteristic, where $\Omega(E)$ is the coherent sheaf of germs of sections of E. J.-P. Serre conjectured and F. Hirzebruch solved that this Euler number can be expressed in terms of Chern classes of E and the tangent bundle T_X. Namely, the following theorem is a celebrated theorem, usually called the Hirzebruch-Riemann-Roch theorem:

THEOREM 1 (HRR).

$$\chi(X, E) = T(X, E).$$

Here $T(X, E) := \int_X \operatorname{td}(T_X) \operatorname{ch}(E) \cap [X]$ is called the T-characteristic ([Hi-1]), where $\operatorname{ch}(E)$ is the total Chern character of E and $\operatorname{td}(T_X)$ is the total Todd class of the tangent

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Partially supported by Grant-in-Aid for Scientific Research (No. 06640162 and No. 07640129), the Japanese Ministry of Education, Science and Culture.

1991 *Mathematics Subject Classification*: 14B05, 14C40, 14F99, 18F99, 57R20.

Received by the editors: October 20, 1996.

The paper is in final form and no version of it will be published elsewhere.

bundle T_X of X. For the sake of later use, we recall that for a complex vector bundle V $\mathrm{ch}(V)$ and $\mathrm{td}(V)$ are defined as follows:

$$\mathrm{ch}(V) = \sum_{i=1}^{\operatorname{rank} V} e^{\alpha_i}$$

and

$$\mathrm{td}(V) = \prod_{i=1}^{\operatorname{rank} V} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

where α_i's are the Chern roots of V.

As a reference for a more detailed historical aspect of HRR, see Hirzebruch's article [Hi-2].

Grothendieck (cf. [BS]) generalized HRR for non-singular quasi-projective algebraic varieties over any field and proper morphisms with Chow cohomology ring theory instead of ordinary cohomology theory. For the complex case we can still take the ordinary cohomology theory (or the homology theory by the Poincaré duality). Here we stick ourselves to complex projective algebraic varieties for the sake of simplicity. For a variety X, let $\mathbf{K}_0(X)$ denote the Grothendieck group of algebraic coherent sheaves on X and for a morphism $f : X \to Y$ the pushforward $f_! : \mathbf{K}_0(X) \to \mathbf{K}_0(Y)$ is defined by

$$f_!(\mathcal{F}) := \sum_{i=0}^{\infty} (-1)^i \mathbf{R}^i f_* \mathcal{F},$$

where $\mathbf{R}^i f_* \mathcal{F}$ is (the class of) the higher direct image sheaf of $\mathcal{F}$. Then $\mathbf{K}_0$ is a covariant functor with the above pushforward (see [Gr] and [Man]). Then Grothendieck showed the existence of a natural transformation from the covariant functor $\mathbf{K}_0$ to the $\mathbf{Q}$-homology covariant functor $H_*(\ ;\mathbf{Q})$ (see [BS]), which is called the Grothendieck-Riemann-Roch theorem:

Theorem 2 (GRR). *Let the transformation $\tau : \mathbf{K}_0(\ ) \to H_*(\ ;\mathbf{Q})$ be defined by $\tau(\mathcal{F}) = \mathrm{td}(X)\,\mathrm{ch}(\mathcal{F}) \cap [X]$ for any smooth variety X. Then τ is actually natural, i.e., for any morphism $f : X \to Y$ the following diagram commutes*:

$$\begin{array}{ccc} \mathbf{K}_0(X) & \xrightarrow{\tau} & H_*(X;\mathbf{Q}) \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ \mathbf{K}_0(Y) & \xrightarrow[\tau]{} & H_*(Y;\mathbf{Q}) \end{array}$$

i.e.,

$$\mathrm{td}(T_Y)\,\mathrm{ch}(f_!\mathcal{F}) \cap [Y] = f_*(\mathrm{td}(T_X)\,\mathrm{ch}(\mathcal{F}) \cap [X]).$$

Clearly HRR is induced from GRR by considering a map from X to a point.

Remark 1. The target of the transformation of the original GRR is the cohomology $H^*(\ ;\mathbf{Q})$ with the Gysin homomorphism, instead of the homology $H_*(\ ;\mathbf{Q})$. But, by the definition of the Gysin homomorphism the original GRR can be put in as above.

Baum, Fulton and MacPherson [BFM] have extended GRR to singular varieties, by introducing the so-called *localized Chern character* $\mathrm{ch}_X^M(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ with X embedded into a non-singular quasi-projective variety M, as a substitute of

$\mathrm{ch}(F) \cap [X]$ in the above GRR. Let X be a compact complex subspace of a complex manifold M and let $E_\bullet$ be a complex of topological vector bundles on M which is exact off X. Then the localized Chern character $\mathrm{ch}_X^M(E_\bullet)$ is defined as follows (for more details see [BFM]). Let $d(E_\bullet) \in \mathbf{K}_0(M, M - X)$ be the difference-bundle of the complex $E_\bullet$, $\mathrm{ch} : \mathbf{K}_0(M, M - X) \to H_*(M, M - X; \mathbf{Q})$ the Chern character and $\mathbf{L} : \mathbf{K}_0(M, M - X) \to H_*(X; \mathbf{Q})$ the Lefschetz duality isomorphism. Then $\mathrm{ch}_X^M(E_\bullet)$ is defined to be the value of $d(E_\bullet)$ by the composite $\mathbf{L} \circ \mathrm{ch}$, i.e., $\mathrm{ch}_X^M(E_\bullet) := \mathbf{L} \circ \mathrm{ch}(d(E_\bullet))$. For a coherent sheaf $\mathcal{F}$ on X, the localized Chern character $\mathrm{ch}_X^M(\mathcal{F})$ is defined by: $\mathrm{ch}_X^M(\mathcal{F}) := \mathrm{ch}_X^M(E_\bullet)$, where $E_\bullet$ is a resolution of the coherent sheaf $i_{M*}\mathcal{F}$ with $i_M : X \to M$ being the embedding of X into smooth M. Note that if X is smooth $\mathrm{ch}_X^X(\mathcal{F}) = \mathrm{ch}(F) \cap [X]$. In [BFM] Baum, Fulton and MacPherson showed the following theorem:

Theorem 3 (BFM-RR).
(i) $\tau(\mathcal{F}) := \mathrm{td}(i_M^* T_M) \cap \mathrm{ch}_X^M(\mathcal{F})$ *is independent of the embedding* $i_M : X \to M$.
(ii) *Let the transformation* $\tau : \mathbf{K}_0(\ ) \to H_*(\ ; \mathbf{Q})$ *be defined by* $\tau(\mathcal{F}) = \mathrm{td}(i_M^* T_M) \cap \mathrm{ch}_X^M(\mathcal{F})$ *for any variety* X*. Then* τ *is actually natural, i.e., for any morphism* $f : X \to Y$ *the following diagram commutes*:

$$\begin{tikzcd} \mathbf{K}_0(X) \arrow[r, "\tau"] \arrow[d, "f_!"'] & H_*(X; \mathbf{Q}) \arrow[d, "f_*"] \\ \mathbf{K}_0(Y) \arrow[r, "\tau"'] & H_*(Y; \mathbf{Q}) \end{tikzcd}$$

i.e., for any embeddings $i_M : X \to M$ *and* $i_N : Y \to N$

$$\mathrm{td}(i_N^* T_N) \cap \mathrm{ch}_Y^N(f_!\mathcal{F}) = f_*(\mathrm{td}(i_M^* T_M) \cap \mathrm{ch}_X^M(\mathcal{F})).$$

When X and Y are smooth, it is clear that BFM-RR becomes GRR.

Summing up we have the following relationships between these three Riemann-Roch's:

$$\text{HRR} \xleftarrow{\text{“mapping to a point”}} \text{GRR} \xleftarrow{\text{“smooth”}} \text{BFM-RR}$$

Here the first arrow means that if we consider GRR for a map from a variety to a point, then we get HRR, and the second arrow means that if we restrict BFM-RR to the category of smooth varieties, then we get GRR.

3. Hirzebruch characteristics. In Hirzebruch's book [Hi-1, §12.1 and §15.5] he has generalized the characteristics $\chi(X, E)$ and $T(X, E)$ to the so-called χ_y-characteristic $\chi_y(X, E)$ and T_y-characteristic $T_y(X, E)$ as follows, using a parameter y (see also [HBJ, Chapter 5]).

Definition 1.

$$\begin{aligned} \chi_y(X, E) : &= \sum_{p=0}^{\infty} \Big(\sum_{q=0}^{\infty} (-1)^q \dim_{\mathbf{C}} H^q(X, \Omega(E) \otimes \Lambda^P T_X^\vee) \Big) y^p \\ &= \sum_{p=0}^{\infty} \chi(X, E \otimes \Lambda^P T_X^\vee) y^p \end{aligned}$$

where $T_X^\vee$ is the dual of the tangent bundle T_X, i.e., the cotangent bundle of X.

$$T_y(X,E) := \int_X \widetilde{\mathrm{td}_{(y)}}(T_X)\,\mathrm{ch}_{(1+y)}(E)\cap[X],$$

$$\widetilde{\mathrm{td}_{(y)}}(T_X) := \prod_{i=1}^{\dim X}\Big(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i(1+y)}}-\alpha_i y\Big),$$

$$\mathrm{ch}_{(1+y)}(E) := \sum_{j=1}^{\operatorname{rank} E} e^{\beta_j(1+y)},$$

where α_i's are the Chern roots of T_X and β_j's are the Chern roots of E.

With the above definition Hirzebruch [Hi-1, §21.3] showed the following generalized Hirzebruch-Riemann-Roch (abbr. g-HRR):

Theorem 4 (g-HRR).

$$\chi_y(X,E) = T_y(X,E).$$

Note that when $y=0$ we get the original HRR.

Remark 2. The modified Todd class $\widetilde{\mathrm{td}_{(y)}}(T_X)$ defined above unifies the following three important characteristic cohomology classes:

($y=-1$): $\widetilde{\mathrm{td}_{(-1)}}(T_X) = c(T_X)$, the total Chern class,
($y=0$): $\widetilde{\mathrm{td}_{(0)}}(T_X) = \mathrm{td}(T_X)$, the total Todd class,
($y=1$): $\widetilde{\mathrm{td}_{(1)}}(T_X) = L(T_X)$, the total Thom-Hirzebruch L-class.

In particular, for $E=$ the trivial line bundle, for these special values $y=-1,0,1$, the g-HRR reads as follows, which are well-known facts:

($y=-1$): topological Euler-Poincaré characteristic $e(X)=\int_X c(T_X)\cap[X]$,
($y=0$): arithmetic genus $\chi(X)=\int_X \mathrm{td}(T_X)\cap[X]$,
($y=1$): signature $\sigma(X)=\int_X L(T_X)\cap[X]$.

There is another simpler version of modified Todd class parametrized by y, which is sort of a "Todd version" of the Chern polynomial $c_t=\sum_{i=0}^\infty t^i c_i$ with $c=\sum_{i=0}^\infty c_i$ the total Chern class.

Definition 2 ("Todd polynomial"). For a variable q and a complex vector bundle V

$$\mathrm{td}_{(q)}(V) := \sum_{i=0}^\infty q^i\,\mathrm{td}_i(V) = \prod_{p=1}^{\operatorname{rank} V}\frac{q\gamma_p}{1-e^{-q\gamma_p}}$$

where γ_p's are the Chern roots of V.

There is a fundamental relationship between these two types of "parametrized" Todd classes:

Proposition 5 ([Y-2, Lemma (2.3.7)]). *For any virtual bundle V*

$$\widetilde{\mathrm{td}_{(y)}}(V) = (1+y)^{-\operatorname{rank} V}\Big(\sum_{p=0}^\infty \mathrm{ch}_{(1+y)}(\Lambda^p V^\vee)y^p\Big)\,\mathrm{td}_{(1+y)}(V),$$

or, if we use the Grothendieck λ-ring structure, $\lambda_y(V) := \sum_{p=0}^{\infty}[\Lambda^p V]y^p$, then the equality becomes

$$\widetilde{\mathrm{td}_{(y)}}(V) = (1+y)^{-\operatorname{rank} V}\,\mathrm{ch}_{(1+y)}(\lambda_y(V^\vee))\,\mathrm{td}_{(1+y)}(V).$$

Then, since $\mathrm{ch}_{(1+y)}(E)\,\mathrm{ch}_{(1+y)}(\lambda_y T_X^\vee) = \mathrm{ch}_{(1+y)}(E \otimes \lambda_y T_X^\vee)$, the generalized Hirzebruch-Riemann-Roch can be rewritten as follows:

Corollary 6.

$$\chi(X, E \otimes \lambda_y T_X^\vee) = (1+y)^{-\dim X} \int_X \mathrm{td}_{(1+y)}(T_X)\,\mathrm{ch}_{(1+y)}(E \otimes \lambda_y T_X^\vee) \cap [X].$$

We will see later that this simple observation is quite significant.

Remark 3. As long as we are just concerned with expressing $\chi_y(X,E)$ in terms of characteristic classes of E and T_X and the parameter y, the following formula is also perfectly all right, for example.

$$\begin{aligned}
\chi_y(X,E) &= \sum_p \chi(X, E \otimes \Lambda^p T_X^\vee) y^p \\
&= \sum_p \int_X (\mathrm{td}(T_X)\,\mathrm{ch}(E \otimes \Lambda^p T_X^\vee) \cap [X]) y^p \\
&= \int_X \mathrm{td}(T_X)\,\mathrm{ch}(E \otimes \Lambda^p T_X^\vee) \cap [X] \\
&= \int_X \mathrm{td}(T_X)\,\mathrm{ch}(\lambda_y T_X^\vee)\,\mathrm{ch}(E) \cap [X].
\end{aligned}$$

Here we set

$$\begin{aligned}
\widehat{\mathrm{td}_{(y)}}(T_X) := \mathrm{td}(T_X)\,\mathrm{ch}(\lambda_y T_X^\vee) &= \prod_{i=1}^{\dim X} \frac{\alpha_i}{1-e^{-\alpha_i}} \prod_{i=1}^{\dim X} (1 + y e^{-\alpha_i}) \\
&= \prod_{i=1}^{\dim X} \frac{\alpha_i(1+ye^{-\alpha_i})}{1-e^{-\alpha_i}} \\
&= \prod_{i=1}^{\dim X} \Big(\frac{\alpha_i(1+y)}{1-e^{-\alpha_i}} - \alpha_i y \Big).
\end{aligned}$$

However, unlike the modified Todd class $\widetilde{\mathrm{td}_{(y)}}(T_X)$, this class $\widehat{\mathrm{td}_{(y)}}(T_X)$ does not give rise to the Chern class when $y=-1$ nor the Thom-Hirzebruch class when $y=1$, although it of course gives rise to the Todd class when $y=0$. Indeed, when $y=-1$ we have

$$\widehat{\mathrm{td}_{(-1)}}(T_X) = \prod_{i=1}^{\dim X} \alpha_i = c_n(X), \text{ the top Chern class,}$$

and when $y=1$ we have

$$\begin{aligned}
\widehat{\mathrm{td}_{(1)}}(T_X) &= \prod_{i=1}^{\dim X} \alpha_i \frac{1+e^{\alpha_i}}{1-e^{-\alpha_i}} = \prod_{i=1}^{\dim X} \frac{\alpha_i}{\tanh(\frac{1}{2}\alpha_i)} \\
&\neq \prod_{i=1}^{\dim X} \frac{\alpha_i}{\tanh(\alpha_i)} = L(T_X).
\end{aligned}$$

So the significance of HRR is that it unifies the formulae about the three important genera and also that the modified Todd class $\widetilde{\mathrm{td}_{(y)}}(T_X)$ unifies the three corresponding characteristic classes.

4. A generalized Baum-Fulton-MacPherson's Riemann-Roch. In this section we shall get a "BFM-RR version" of the generalized Hirzebruch-Riemann-Roch in a similar manner to that of the construction of BFM-RR.

It seems quite natural to speculate that when one tries to get a "BFM-RR version" of g-HRR, simply thinking, if one replaces $\mathrm{td}(i_M^* T_M) \cap \mathrm{ch}_X^M(\mathcal{F})$ in BFM-RR by the following ingredient

$$\widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$$

then it might work perfectly, just as in the case of BFM-RR. Here ${\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$ should be similar to $\mathrm{ch}_X^M(\mathcal{F})$. As a candidate for ${\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$, we recall a definition and results from our previous paper [Y-1].

In [Y-1] we give modified (or "twisted") versions of $\mathrm{ch}_X^M(E_\bullet)$ and $\mathrm{ch}_X^M(\mathcal{F})$ as follows: ${\mathrm{ch}_{(q)}}_X^M(E_\bullet)$ is defined by simply replacing the Chern character ch by the "Chern character polynomial" $\mathrm{ch}_{(q)} = \sum_{i=0}^{\infty} q^i \,\mathrm{ch}_i$ in the construction of $\mathrm{ch}_X^M(E_\bullet)$.

Definition 3 ("twisted" localized Chern character).

$${\mathrm{ch}_{(q)}}_X^M(\mathcal{F}) := q^{\dim X - \dim M} {\mathrm{ch}_{(q)}}_X^M(E_\bullet).$$

In particular, when X is smooth, then ${\mathrm{ch}_{(q)}}_X^X(\mathcal{F}) = \mathrm{ch}_{(q)}(\mathcal{F}) \cap [X]$.

Theorem 7 ([Y-1, Proposition (3.9)]). *The homology class*

$$\tau_{(q)}(\mathcal{F}) := \mathrm{td}_{(q)}(i_M^* T_M) \cap {\mathrm{ch}_{(q)}}_X^M(\mathcal{F})$$

is independent of the embedding $i_M : X \to M$.

Thus we get the following

Proposition 8. *The homology class $\widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$ depends on the embeddings of X into smooth varieties M.*

Proof. Using the formula in Proposition 5, we can get the following equality:

$$\widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F}) = (1+y)^{-\dim M}\,\mathrm{ch}_{(1+y)}\big(\lambda_y(i_M^* T_M^\vee)\big)\,\mathrm{td}_{(1+y)}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F}).$$

Here, thanks to Theorem 7 above, $\mathrm{td}_{(1+y)}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$ is independent of the embeddings of X into smooth varieties M, but $(1+y)^{-\dim M}\,\mathrm{ch}_{(1+y)}(\lambda_y i_M^* T_M^\vee)$ does depend on the embeddings. Thus the homology class $\widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$ does depend on the embeddings of X into smooth varieties. ∎

In spite of this unpleasant dependence, however, we still want to use this homology class, which is quite natural and similar to the definition of Baum-Fulton-MacPherson's Todd class. For the sake of simplicity we introduce the following notation:

Definition 4.

$$\tau_{(q)}^{\mathrm{BFM}}(\mathcal{F}) := \mathrm{td}_{(q)}(i_M^* T_M) \cap {\mathrm{ch}_{(q)}}_X^M(\mathcal{F})$$

is called the *parametrized Baum-Fulton-MacPherson's Todd class* of the coherent sheaf $\mathcal{F}$.

THEOREM 9. *Let* $\mathbf{K}_0(X)[y] := \mathbf{K}_0(X)\otimes\mathbf{Z}[\mathbf{y}]$ *and let the pushforward* $f_! : \mathbf{K}_0(X)[y] \to \mathbf{K}_0(Y)[y]$ *be defined by extending* $f_! : \mathbf{K}_0(X) \to \mathbf{K}_0(Y)$ *linearly with respect to the polynomial ring* $\mathbf{Z}[\mathbf{y}]$. *Define* $\tau^{\mathrm{BFM}}_{(1+y)} : \mathbf{K}_0(X)[y] \to H_*(X : \mathbf{Q})[y]$ *by* $\tau^{\mathrm{BFM}}_{(1+y)}(\sum f_i[\mathcal{F}_i]) := \sum f_i \tau^{\mathrm{BFM}}_{(1+y)}(\mathcal{F}_i)$, *where* $f_i \in \mathbf{Z}[y]$ *and* $\mathcal{F}_i \in \mathbf{K}_0(X)$. *Then for a morphism* $f : X \to Y$ *the following diagram commutes*:

$$\begin{array}{ccc}
\mathbf{K}_0(X)[y] & \xrightarrow{(1+y)^{-\dim X}\,\tau^{\mathrm{BFM}}_{(1+y)}} & H_*(X;\mathbf{Q})[y,y^{-1}] \\
{\scriptstyle f_!}\big\downarrow & & \big\downarrow{\scriptstyle f_*} \\
\mathbf{K}_0(Y)[y] & \xrightarrow[(1+y)^{-\dim Y}\,\tau^{\mathrm{BFM}}_{(1+y)}]{} & H_*(Y;\mathbf{Q})[y,y^{-1}]
\end{array}$$

When $y = 0$, this gives us the original BFM-RR. This theorem shall be provisionally called a generalized Baum-Fulton-MacPherson's Riemann-Roch (g-BFM-RR).

Remark 4. Consider the diagram in Theorem 9 in the case when X is smooth, $f : X \to pt$ is a mapping to a point pt and $E \otimes \lambda_y T_X^\vee \in \mathbf{K}_0(X)[y]$. Then we have

$$f_!(E \otimes \lambda_y T_X^\vee) = \chi(X, E \otimes \lambda_y T_X^\vee)$$

and

$$\begin{aligned}
&f_*(1+y)^{-\dim X}\tau^{\mathrm{BFM}}_{(1+y)}(E \otimes \lambda_y T_X^\vee) \\
&\qquad = (1+y)^{-\dim X}\int_X \mathrm{td}_{(1+y)}(T_X)\,\mathrm{ch}_{(1+y)}(E \otimes \lambda_y T_X^\vee) \cap [X].
\end{aligned}$$

The above theorem is a slight modification of the following theorem.

THEOREM 10 ([Y-1, Theorem (3.10)]). *Let* $\mathbf{K}_0(X)[q,q^{-1}] := \mathbf{K}_0(X)\otimes\mathbf{Z}[q,q^{-1}]$ *and for a morphism* $f : X \to Y$ *the "twisted" pushforward* $f_!^{(q)} : \mathbf{K}_0(X)[q,q^{-1}] \to \mathbf{K}_0(Y)[q,q^{-1}]$ *be defined by*

$$f_!^{(q)} := q^{\dim X - \dim Y} f_!,$$

where $f_! : \mathbf{K}_0(X) \to \mathbf{K}_0(Y)$ *is the original Grothendieck pushforward.* (*Obviously* $\mathbf{K}_0(\ )[q,q^{-1}]$ *is a covariant functor with this "twisted" pushforward.*) *Let the homomorphism* $\tau^{\mathrm{BFM}}_{(q)} : \mathbf{K}_0(X)[q,q^{-1}] \to H_*(X;\mathbf{Q})[q,q^{-1}]$ *be defined by* $\tau^{\mathrm{BFM}}_{(q)}(\sum f_i[\mathcal{F}_i]) := \sum f_i\tau^{\mathrm{BFM}}_{(q)}(\mathcal{F}_i)$, *where* $f_i \in \mathbf{Z}[q,q^{-1}]$ *and* $\mathcal{F}_i \in \mathbf{K}_0(X)$. *Then* $\tau^{\mathrm{BFM}}_{(q)}$ *is a natural transformation, i.e., for any morphism* $f : X \to Y$

$$\tau^{\mathrm{BFM}}_{(q)} f_!^{(q)} = f_*\tau^{\mathrm{BFM}}_{(q)}.$$

Remark 5. There is a simple relationship between $\tau^{\mathrm{BFM}}_{(q)}$ and τ: For a coherent sheaf $\mathcal{F}$ on X we have

$$\tau^{\mathrm{BFM}}_{(q)}(\mathcal{F}) = \sum_{i\geq 0} q^{\dim X - i}\tau_i(\mathcal{F}),$$

where $\tau_i(\mathcal{F})$ is the $2i$-dimensional component of the total homology class $\tau(\mathcal{F})$. Theorem 10 was originally obtained from a motivation of trying to get a "twisted version" $\tau^{\mathrm{BFM}}_{(q)} : \mathbf{K}_0(X)[q,q^{-1}] \to H_*(X;\mathbf{Q})[q,q^{-1}]$ of $\tau : \mathbf{K}_0(X) \to H_*(X;\mathbf{Q})$, satisfying that $\tau^{\mathrm{BFM}}_{(q)}$ is the unique natural transformation so that if X is smooth then $\tau^{\mathrm{BFM}}_{(q)}(\mathcal{O}_X) = \mathrm{td}_{(q)}(T_X) \cap [X]$.

Now, using Theorem 9 and introducing the following definitions, we can capture and understand the total homology class $\widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$ (depending on the embedding of X into a smooth variety M) much clearer as follows.

Definition 5. Let $i_M : X \to M$ be an embedding of X into a smooth variety M. The isomorphism

$$\Lambda_M : \mathbf{K}_0(X)[y] \to \mathbf{K}_0(X)[y]$$

defined by extending the following linearly with respect to $\mathbf{Z}[y]$

$$\Lambda_M(\mathcal{F}) := \mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee)$$

is called the *twisting isomorphism via the embedding* i_M.

Definition 6. For a morphism $f : X \to Y$ and embeddings $i_M : X \to M$ and $i_N : Y \to N$ of X and Y into smooth varieties, the homomorphism

$$(f_!^{[y]})_M^N : \mathbf{K}_0(X)[y] \to \mathbf{K}_0(Y)[y]$$

defined by extending the following linearly with respect to $\mathbf{Z}[y]$

$$(f_!^{[y]})_M^N(\mathcal{F}) = f_!\bigl(\mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee - f^* i_N^* T_N^\vee)\bigr)$$

is called the *twisted homomorphism* with respect to the embeddings i_M and i_N.

Lemma 1. *The following diagram commutes*:

$$\begin{array}{ccc} \mathbf{K}_0(X)[y] & \xrightarrow{\Lambda_M} & \mathbf{K}_0(X)[y] \\ {\scriptstyle f_!^{[y]}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbf{K}_0(Y)[y] & \xrightarrow[\Lambda_N]{} & \mathbf{K}_0(Y)[y] \end{array}$$

Proof. It suffices to show that for a coherent sheaf $\mathcal{F}$

$$\Lambda_N (f_!^{[y]})_M^N(\mathcal{F}) = f_* \Lambda_M(\mathcal{F}).$$

Then by the projection formula and the multiplicativity of λ_y we have that

$$\begin{aligned} \Lambda_N (f_!^{[y]})_M^N(\mathcal{F}) &= f_!\bigl(\mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee - f^* i_N^* T_N^\vee)\bigr) \otimes \lambda_y(i_N^* T_N^\vee) \\ &= f_!\bigl(\mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee - f^* i_N^* T_N^\vee) \otimes \lambda_y(f^* i_N^* T_N^\vee)\bigr) \\ &= f_!\bigl(\mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee - f^* i_N^* T_N^\vee + f^* i_N^* T_N^\vee)\bigr) \\ &= f_!(\mathcal{F} \otimes \lambda_y(i_M^* T_M^\vee)) \\ &= f_* \Lambda_M(\mathcal{F}). \quad \blacksquare \end{aligned}$$

Theorem 11 (g-BFM-RR's associated to embeddings). *For an embedding* $i_M : X \to M$ *we define the homomorphism*

$$\tau_M^{[y]} : \mathbf{K}_0(X)[y] \to H_*(X; \mathbf{Q})[y]$$

by (*for a coherent sheaf* $\mathcal{F}$)

$$\tau_M^{[y]}(\mathcal{F}) = (1+y)^{\dim M - \dim X} \widetilde{\mathrm{td}_{(y)}}(i_M^* T_M) \cap {\mathrm{ch}_{(1+y)}}_X^M(\mathcal{F})$$

and extend it linearly with respect to the polynomial ring $\mathbf{Z}[y]$. *Then for a morphism* $f : X \to Y$ *and embeddings* $i_M : X \to M$ *and* $i_N : Y \to N$ *of* X *and* Y *into smooth varieties the following diagram commutes*:

$$\begin{array}{ccc} \mathbf{K}_0(X)[y] & \xrightarrow{\tau_M^{[y]}} & H_*(X;\mathbf{Q})[y] \\ {\scriptstyle f_!^{[y]}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathbf{K}_0(Y)[y] & \xrightarrow[\tau_N^{[y]}]{} & H_*(Y;\mathbf{Q})[y] \end{array}$$

P r o o f. First we observe the following key lemma (cf. the proof of [BFM, Proposition (3.3)]):

LEMMA 2. *For a coherent sheaf* $\mathcal{F}$ *and a vector bundle* E *the following equality holds*:

$$\mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F}\otimes E) = \mathrm{ch}_{(1+y)}(E) \cap \mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F}).$$

The proof of the theorem goes as follows:

$$\begin{aligned} &\tau_M^{[y]}(\mathcal{F}) \\ &= (1+y)^{\dim M - \dim X}\widetilde{\mathrm{td}_{(y)}}(i_M^*T_M) \cap \mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F}) \\ &= (1+y)^{\dim M - \dim X}((1+y)^{-\dim M}\,\mathrm{td}_{(1+y)}(i_M^*T_M)\,\mathrm{ch}_{(1+y)}(\lambda_y i_M^* T_M^\vee) \cap \mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F})) \\ &= (1+y)^{-\dim X}\,\mathrm{td}_{(1+y)}(i_M^*T_M) \cap \mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F}\otimes\lambda_y i_M^*T_M^\vee) \\ &= (1+y)^{-\dim X}\tau_{(1+y)}^{\mathrm{BFM}}(\Lambda_M(\mathcal{F})). \end{aligned}$$

Hence the assertion follows from the two commutative diagrams in Theorem 9 and Lemma 1. Since the homology class $\tau_M^{[y]}(\mathcal{F}) = (1+y)^{\dim M-\dim X}\widetilde{\mathrm{td}_{(y)}}(i_M^*T_M) \cap \mathrm{ch}_{(1+y)}{}_X^M(\mathcal{F})$ does not involve y^{-1} at all, as the target of the homomorphism $\tau_M^{[y]}$ we can take $H_*(X;\mathbf{Q})[y]$ instead of $H_*(X;\mathbf{Q})[y,y^{-1}]$. ∎

The above theorem means that our g-BFM-RR gives rise to infinitely many commutative diagrams as in the theorem according to the embeddings of the target and source varieties via the commutative diagrams in Lemma 1. In particular, when X and Y are smooth, g-BFM-RR gives rise to the following special case, which is the generalized Grothendieck-Riemann-Roch (abbr. g-GRR) given in [Y-2] and the proof of which is now much more compact and clearer than that in [Y-2].

THEOREM 12 ([Y-2, Theorem (2.1) and Theorem (2.3)]).

(i) *For a morphism* $f : X \to Y$, *the pushforward* $f_!^{[y]} : \mathbf{K}_0(X)[y] \to \mathbf{K}_0(Y)[y]$ *is defined as follows*:

$$f_!^{[y]}(\mathcal{F}) := \sum_{p=0}^{\infty} f_!(\mathcal{F}\otimes\Lambda^p T_f^\vee)y^p = f_!(\mathcal{F}\otimes\lambda_y T_f^\vee)$$

where $f_! : \mathbf{K}_0(X) \to \mathbf{K}_0(Y)$ *is the original Grothendieck pushforward and* $T_f^\vee := T_X^\vee - f^*T_Y^\vee$ *is a virtual "relative cotangent bundle" as an element of* $\mathbf{K}_0(X)$, *and this pushforward is extended linearly with respect to the polynomial ring* $\mathbf{Z}[y]$. *Then* $\mathbf{K}_0(\ )[y]$ *is a covariant functor with the above pushforward, i.e., for morphisms* $f : X \to Y$ *and* $g : Y \to Z$, $g_!^{[y]} \circ f_!^{[y]} = (g\circ f)_!^{[y]}$.

(ii) *If we define the homomorphism* $\tau^{[y]} : \mathbf{K}_0(X)[y] \to H_*(X;\mathbf{Q})[y]$ *by*

$$\tau^{[y]}(\mathcal{F}) := \widetilde{\mathrm{td}_{(y)}}(T_X)\,\mathrm{ch}_{(1+y)}(\mathcal{F}) \cap [X],$$

which is extended linearly with respect to the polynomial ring $\mathbf{Z}[y]$, *then* $\tau^{[y]} : \mathbf{K}_0(\ )[y] \to H_*(\ ;\mathbf{Q})[y]$ *is a natural transformation, i.e., for any morphism* $f : X \to Y$ *the following diagram commutes*:

$$\begin{tikzcd} \mathbf{K}_0(X)[y] \arrow[r, "\tau^{[y]}"] \arrow[d, "f_!^{[y]}"'] & H_*(X;\mathbf{Q})[y] \arrow[d, "f_*"] \\ \mathbf{K}_0(Y)[y] \arrow[r, "\tau^{[y]}"'] & H_*(Y;\mathbf{Q})[y] \end{tikzcd}$$

i.e.,

$$\widetilde{\mathrm{td}_{(y)}}(T_Y)\,\mathrm{ch}_{(1+y)}(f_!^{[y]}\mathcal{F}) \cap [Y] = f_*\Big(\widetilde{\mathrm{td}_{(y)}}(T_X)\,\mathrm{ch}_{(1+y)}(\mathcal{F}) \cap [X]\Big).$$

Proof. Take $M = X$ and $Y = N$ in the proof of Theorem 11. ∎

Note that our g-GRR specializes to GRR when $y = 0$ and that g-HRR follows from g-GRR by considering a mapping of a variety to a point.

5. An open problem. Summing up compactly what we have seen so far we have the following "commutative" diagram:

$$\begin{tikzcd} \text{HRR} & & \text{GRR} \arrow[ll, "\text{“mapping to a point”}"'] & & \text{BFM-RR} \arrow[ll, "\text{“smooth”}"'] \\ \text{g-HRR} \arrow[u, "\text{“}y{=}0\text{”}"'] & & \text{g-GRR} \arrow[ll, "\text{“mapping to a point”}"] \arrow[u, "\text{“}y{=}0\text{”}"'] & & \text{g-BFM-RR} \arrow[ll, "\text{“smooth”}"] \arrow[u, "\text{“}y{=}0\text{”}"'] \end{tikzcd}$$

One of the motivations for this present work is based on the following two facts.

Fact 1. The generalized Hirzebruch-Riemann-Roch g-HRR unifies the three important and distinguished characteristics (or genera):

(-1): topological Euler-Poincaré characteristic

$$e(X) = \int_X c(X) \cap [X],$$

(0): arithmetic genus

$$\chi(X) = \int_X \mathrm{td}(T_X) \cap [X],$$

(1): signature

$$\sigma(X) = \int_X L(T_X) \cap [X].$$

Fact 2. There are three distinguished characteristic homology classes of possibly singular varieties, corresponding to the above three important invariants, which are respectively,

(-1): Chern-Schwartz-MacPherson's class $C(X)$ ([BrSc], [Mac]), and

$$e(X) = \int_X C(X),$$

(0): Baum-Fulton-MacPherson's Todd class $\tau(X)$ (i.e., BFM-RR) [BFM], and

$$\chi(X) = \int_X \tau(X),$$

(1): Goresky-MacPherson's homology L-class $L(X)$ [GM]; and

$$\sigma(X) = \int_X L(X).$$

Thus we pose the following very naive problem:

PROBLEM. Is there a theory of characteristic homology classes unifying the above three characteristic homology classes of possibly singular varieties? In particular, can we find an ideal theory g-BFM-RR in the above commutative diagram?

Remark 6. Chern-Schwartz-MacPherson classes and Baum-Fulton-MacPherson's Todd classes are both formulated as natural transformations from certain covariant functors to the homology functor. But this is not the case for the Goresky-MacPherson's homology L-class. However Cappell and Shaneson [CS-1] have recently developed a theory of homology L-class, extending Goresky-MacPherson's homology L-classes, by using the sheaf-theoretic methods to define intersection homology and also by using the foundational topological aspects of the theory of perverse sheaves. Their homology L-class turns out to be formulated as a natural transformation from a certain covariant cobordism functor to the homology functor [Y-3]. In passing, it is noteworthy to remark that the relationship between Cappell-Shaneson's homology L-class and Goresky-MacPherson's homology L-class is just like that between Chern-Schwartz-MacPherson class and Chern-Mather class.

Remark 7. Cappell and Shaneson ([CS-2], [Sh]) have also recently announced their results about formulae relating the genera of algebraic varieties under a morphism. In which they claimed to have defined the characteristic classes T_yX and IT_yX parametrized by y such that when $y = -1$, $T_{-1}X$ becomes the Chern-Schwartz-MacPherson class $C(X)$, and when $y = 0$, T_0X becomes the Baum-Fulton-MacPherson's Todd class $\tau(X)$, and when $y = 1$, $IT_1(X)$ becomes Goresky-Macpherson's L-class $L(X)$. At the moment their detailed version is not available yet (in particular their definitions of T_yX and IT_yX are not given in either of their papers ([CS-2], [Sh])) and also we do not know the naturality of T_yX and IT_yX, i.e., whether they can be redefined as natural transformations from certain covariant functors to the homology functor. Incidentally we do not know what T_1X is.

Acknowledgements. Some of this work was done during the author's stay at the Erwin Schrödinger Institute, Vienna, Austria. He would like to express his gratitude to the Institute, and in particular, Professor Peter W. Michor, for their hospitality.

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