EXAMPLES OF FUNCTIONS $C^k$-EXTENDABLE FOR EACH $k$ FINITE, BUT NOT $C^\infty$-EXTENDABLE

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Dedicated to Professor Stanislaw Lojasiewicz

Abstract. In Example 1, we describe a subset $X$ of the plane and a function on $X$ which has a $C^k$-extension to the whole $\mathbb{R}^2$ for each $k$ finite, but has no $C^\infty$-extension to $\mathbb{R}^2$. In Example 2, we construct a similar example of a subanalytic subset of $\mathbb{R}^5$; much more sophisticated than the first one. The dimensions given here are smallest possible.

1. Introduction. Let $X$ be any subset of $\mathbb{R}^n$. Consider the following $\mathbb{R}$-algebras of functions on $X$

$$C^k(X) = \{ f : X \to \mathbb{R} \mid f = \tilde{f} \text{ on } X \text{ for some } C^k \text{ function } \tilde{f} : \mathbb{R}^n \to \mathbb{R} \},$$

where $k \in \mathbb{N} \cup \{\infty\}$, and

$$C^{(\infty)}(X) = \lim_{k \in \mathbb{N}} C^k(X) = \bigcap_{k \in \mathbb{N}} C^k(X).$$

It is clear that $C^\infty(X) \subset C^{(\infty)}(X) \subset C^k(X)$, with $k \in \mathbb{N}$. An interesting question of differential analysis is the following:

When $C^{(\infty)}(X) = C^\infty(X)$?

Of course, one can assume that $X$ is closed in $\mathbb{R}^n$. The answer to the above question is affirmative in the following cases:

1) When $n = 1$ (see [9]); it is not so when $n = 2$ (see Example 1 below).

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2) When \( X = \overline{\text{int} X} \), because then \( C^k(X) \) is naturally isomorphic to the space \( \mathcal{E}^k(X) \) of \( C^k \)-Whitney fields on \( X \) (\( k \in \mathbb{N} \cup \infty \)), and so
\[
C^{(\infty)}(X) = \lim_{k \to \infty} C^k(X) = \lim_{k \to \infty} \mathcal{E}^k(X) = \mathcal{E}^\infty(X) = C^\infty(X),
\]
(see [8; Chap. I, §4]). Observe that the isomorphisms \( C^k(X) = \mathcal{E}^k(X) \), and thus \( C^{(\infty)}(X) = C^\infty(X) \), occurs for more general sets than those satisfying the condition \( X = \overline{\text{int} X} \); e.g. for the Cantor set in \( \mathbb{R} \).

3) When \( X \) is a semianalytic or, more generally, Nash subanalytic subset of \( \mathbb{R}^n \) (see [4]). The equality \( C^{(\infty)}(X) = C^\infty(X) \) also holds if \( X \) is a subanalytic subset of \( \mathbb{R}^n \) of dimension not more than two or of pure codimension one (see [11, 4]). Bierstone and Milman ([1, 2]) give necessary and sufficient conditions for a subanalytic subset \( X \) of \( \mathbb{R}^n \) to satisfy the equality \( C^{(\infty)}(X) = C^\infty(X) \). In particular, it follows from their results and [4] that the construction from [10] provides examples of subanalytic subsets \( X \) of \( \mathbb{R}^5 \) of dimension three such that \( C^\infty(X) \not\subseteq C^{(\infty)}(X) \). In Example 2 below, we verify this explicitly, constructing a function \( f \in C^{(\infty)}(X) \setminus C^\infty(X) \).

2. Example 1. Let \( X \) denote the union of the following arcs
\[
\lambda_i = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \epsilon, y = x^{i+\frac{1}{2}}\} \quad (i = 1, 2, \ldots),
\]
and of the arc \( \lambda_0 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq \epsilon, y = 0\} \), where \( \epsilon \) is a small positive real number.

We define a function \( f : X \to \mathbb{R} \) by the following formulae
\[
f(x, y) = iy - 1 = ix^{i+\frac{1}{2}} \quad \text{for} \quad (x, y) \in \lambda_i \quad (i = 1, 2, \ldots)
\]
and \( f(x, y) = 0 \) for \( (x, y) \in \lambda_0 \).

The function \( f \) is \( C^k \)-extendable to \( \mathbb{R}^2 \) for each \( k \in \mathbb{N} \). To see this, notice that this function on \( \lambda_i \) is defined by the \( C^\infty \)-function \( f(x, y) = iy - 1 \), and by the \( C^k \)-function \( f(x, y) = ix^{i+\frac{1}{2}} \) on each \( \lambda_i \) with \( i \geq k \). Now, it is enough to glue all these \( C^k \)-functions together, by using, for example, Whitney’s extension theorem.

On the other hand, \( f \) has no \( C^\infty \)-extension to \( \mathbb{R}^2 \). The point is that \( \lambda_i \) is \( C^i \) but not \( C^{i+1} \). This implies that if \( h \in C^{i+1}(\lambda_i) \), then each \( C^{i+1} \)-extension \( \tilde{h} : \mathbb{R}^2 \to \mathbb{R} \) of \( h \) has a uniquely determined derivative \((\partial\tilde{h}/\partial y)(0, 1/i) = i \). It follows that if \( \tilde{h} \) were a \( C^\infty \)-extension of \( f \), then \((\partial\tilde{h}/\partial y)(0, 1/i) = i \), which is a contradiction.

3. Example 2. In this section we will give an example of a subanalytic subset \( X \) of \( \mathbb{R}^n \) and of a function \( f \in C^{(\infty)}(X) \setminus C^\infty(X) \). As for the definitions and basic properties of subanalytic sets, we refer the reader to [5], [6], [7] or [3].

Before describing the example observe that if \( \varphi : G \to H \) is an analytic mapping, where \( G \subset \mathbb{R}^m \) and \( H \subset \mathbb{R}^n \) are open subsets, then, for each point \( y \in G \), \( \varphi \) induces a homomorphism of the algebras of germs of analytic functions
\[
\varphi^*_y : \mathcal{O}_{H, \varphi(y)} \to \mathcal{O}_{G, y}, \quad \varphi^*_y(g) = g \circ \varphi_y.
\]

We will also need its completion
\[
\tilde{\varphi}^*_y : \hat{\mathcal{O}}_{H, \varphi(y)} \to \hat{\mathcal{O}}_{G, y}
\]
which can be identified with the homomorphism
\[ \hat{\varphi}_y : \mathbb{R}[[x_1, \ldots, x_n]] \rightarrow \mathbb{R}[[y_1, \ldots, y_m]], \]
defined by the formula \( \hat{\varphi}_y(Q) = Q \circ (T_y \varphi) - \varphi(y) \).

Then \( \ker \varphi^*_y \) is the ideal of analytic relations among \( \varphi_1, \ldots, \varphi_n \) at \( y \), and \( \ker \varphi^*_y \) is the ideal of formal relations at \( y \).

**Theorem (see [10]).** Let \( I = (-1/2, 1/2) \) and \( J = I \times 0 \times 0 \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3 \). Let \( A = \{(a_\nu, 0, 0) \mid \nu = 1, 2, \ldots \} \) be any countable subset of \( J \). Then there exists an analytic mapping \( \varphi = (\varphi_1, \ldots, \varphi_5) : I^3 \rightarrow \mathbb{R}^5 \) such that

1. \( \ker \varphi^*_y = 0 \), whenever \( y \notin A \);
2. \( \ker \varphi^*_y \neq 0 \), whenever \( y \in J \setminus \overline{A} \);
3. \( \ker \varphi^*_y = 0 \neq \ker \varphi^*_y \), whenever \( y \in J \cap (\overline{A} \setminus A) \).

We are going to recall the construction of \( \varphi = \varphi(u, w, t) = (\varphi_1, \ldots, \varphi_5) \).

We put \( \varphi_1(u, w, t) = u, \varphi_2(u, w, t) = t, \varphi_3(u, w, t) = tw \). Take two sequences \( \{r(n)\} \) \( (n = 1, 2, \ldots) \) and \( \{\rho(n)\} \) \( (n = 1, 2, \ldots) \) such that \( r(n) \in \mathbb{Z}, 0 < r(n) \leq r(n + 1), \limsup r(n)/n = +\infty, \rho(n) \in \mathbb{R}, 0 < \rho(n) \leq n^{-r(n)} \), for each \( n \), and \( \rho(n + 1) < \rho(n) \).

Put
\[ p_n(u) = [(u - a_1) \ldots (u - a_n)]^r(n), \quad n = 1, 2, \ldots. \]

We define \( \varphi_4 \) by the formula
\[ \varphi_4(u, w, t) = t \sum_{n=1}^{\infty} p_n(u)w^n. \]

To define \( \varphi_5 \) we need the following sequence of rational functions
\[ f_n = p_n^{-1}(u)\left[ t^{n-1}y - \sum_{\nu=1}^{n-1} \rho(\nu)p_\nu(u)t^{n-\nu}x^\nu \right] (n = 1, 2, \ldots). \]

Then
\[ f_n(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = t^n \cdot \sum_{\nu=n}^{\infty} p_n^{-1}(u)p_\nu(u)w^\nu, \]
and we define \( \varphi_5 \) by the formula
\[ \varphi_5(u, w, t) = \sum_{n=1}^{\infty} \rho(n)f_n(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \sum_{n=1}^{\infty} \rho(n)t^n \cdot \sum_{\nu=n}^{\infty} p_n^{-1}(u)p_\nu(u)w^\nu. \]

The formula
\[ F(u, t, x, y, z) = z - \sum_{n=1}^{\infty} \rho(n)f_n(u, t, x, y) \]
defines an analytic function on \((I \setminus \overline{Z}) \times \mathbb{R}^4\), where \( Z = \{a_\nu \mid \nu = 1, 2, \ldots\} \), and \( F(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) = 0 \) on \((I \setminus \overline{Z}) \times I^2 \).

Now we will choose \( A \) in a special way: assume that \( 0 < a_{n+1} < a_n < 1/4 \) and \( \lim a_n = 0 \).

Let \( X = \varphi([-1/4, 1/4]^3) \). Take a sequence \( \{\epsilon_n\} \ (n = 1, 2, \ldots) \) such that \( \epsilon_n > 0, a_{n+1} + \epsilon_{n+1} < a_n - \epsilon_n \).
There are $C^\infty$-functions $\lambda_n : \mathbb{R} \to [0,1]$ ($n = 1, 2, \ldots$) such that $\lambda_n = 1$ in a neighbourhood of $a_n$, $\lambda_n(u) = 0$ if $|u - a_n| \geq \epsilon_n$ and $|\lambda_n^{(k)}(u)| \leq C_k \cdot \epsilon_n^{-k}$ for each $u \in \mathbb{R}$, where $C_k$ is a constant depending only on $k$ (see [8; Chap. I, Lemma 4.2]).

Consider the following sequence of $C^\infty$-functions on $\mathbb{R}^5$

$$G_m(u, t, x, y, z) = \left[z - \sum_{n=1}^{m-1} \rho(n)f_n(u, t, x, y)\right] \cdot m \cdot \lambda_m(u), \quad m = 1, 2, \ldots$$

Now we have

$$G_m(\varphi_1, \ldots, \varphi_5) = \left[\varphi_5 - \sum_{n=1}^{m-1} \rho(n)f_n(\varphi_1, \ldots, \varphi_4)\right] \cdot m \cdot \lambda_m(\varphi_1)
= \sum_{n=m}^\infty m\lambda_m(u)\rho(n)t^n\omega_n(u, w), \quad \text{where} \quad \omega_n(u, w) = \sum_{n=0}^\infty p_n^{-1}p_\nu(u)w^\nu.$$

Consider now the function

$$h = \sum_{m=1}^\infty G_m(\varphi_1, \ldots, \varphi_5) = \sum_{m=1}^\infty \sum_{n=m}^\infty m\lambda_m(u)\rho(n)t^n\omega_n(u, w).$$

It is a simple matter to check that $h$ is a $C^\infty$-function on $[-1/4,1/4]^5$. It is easily seen that there is a function $h_0 : X \to \mathbb{R}$ such that $h = h_0(\varphi_1, \ldots, \varphi_5)$.

We will show that $h_0 \in C^{(\infty)}(X) \setminus C^\infty(X)$. If there were a $C^\infty$-extension $\tilde{h}_0$ of $h_0$ to $\mathbb{R}^5$, then we would have the equality $h = G_m(\varphi_1, \ldots, \varphi_5)$ near $(a_m, 0, 0)$ for each $m$, hence, in view of (1), $(\partial h_0/\partial z)(a_m, 0, 0) = m$, which should tend to $(\partial h_0/\partial z)(0, 0, 0)$, when $m$ tends to infinity, a contradiction.

Now fix any $k \in \mathbb{N}$. We will show that there is a $C^k$-function $H_k$ on $\mathbb{R}^5$ such that $H_k = h_0$ on $X$.

Put

$$\Omega = \{(u, t, x) \in \mathbb{R}^3 \mid |u| < 1/4, \; |t| < 1/4, \; |x| < (1/4)|t|\}.$$ 

Observe that if $(u, t, x, y, z) \in X$ and $t \neq 0$, then

$$h_0(u, t, x, y, z) = \sum_{m=1}^k G_m(u, t, x, y, z) + \sum_{m=k+1}^\infty \sum_{n=m}^\infty \sum_{\nu=0}^\infty \theta_{mn\nu}(u, t, x),$$

where

$$\theta_{mn\nu}(u, t, x) = m\lambda_m(u)\rho(n)(p_n^{-1}p_\nu(u))x^\nu t^n - \nu.$$

Let $\alpha, \beta, \gamma \in \mathbb{N}$ be such that $\alpha + \beta + \gamma \leq k$. Then $\partial^{\alpha+\beta+\gamma}\theta_{mn\nu}/\partial u^\alpha\partial t^\beta\partial x^\gamma$ is equal to

$$\sum_{i=0}^\alpha \frac{m\alpha!}{i!(\alpha - i)!} \lambda_m^{(i)}(u)\rho(n)(p_n^{-1}p_\nu)\frac{(\alpha - i)!}{(\nu - \gamma)!}(n - \nu - \beta)! x^\nu t^n - \nu - \gamma.$$ 

Since $n - \gamma - \beta \geq 1$, this derivative extends continuously to $\Omega$. Estimating the absolute value of this derivative on $\Omega$, the reader can easily check that there is a $C^k$-function $\tilde{H}_k$ on $\mathbb{R}^3$ such that

$$\tilde{H}_k(u, t, x) = \sum_{m=k+1}^\infty \sum_{n=m}^\infty \sum_{\nu=0}^\infty \theta_{mn\nu}(u, t, x)$$.
on $\Omega$. Thus, the formula

$$H_k(u, t, x, y, z) = \tilde{H}_k(u, t, x) + \sum_{m=1}^{k} G_m(u, t, x, y, z)$$

defines the required extension.

References