

## ON THE EULER CHARACTERISTIC OF FIBRES OF REAL POLYNOMIAL MAPS

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**Abstract.** Let  $Y$  be a real algebraic subset of  $\mathbf{R}^m$  and  $F : Y \rightarrow \mathbf{R}^n$  be a polynomial map. We show that there exist real polynomial functions  $g_1, \dots, g_s$  on  $\mathbf{R}^n$  such that the Euler characteristic of fibres of  $F$  is the sum of signs of  $g_i$ .

The purpose of this paper is to give a new, self-contained and elementary proof of the following result.

**THEOREM.** *Let  $Y$  be a real algebraic subset of  $\mathbf{R}^m$  and  $F : Y \rightarrow \mathbf{R}^n$  be a polynomial map. Then there exist real polynomials  $g_1(y), \dots, g_s(y)$  on  $\mathbf{R}^n$  such that the Euler characteristic of fibres of  $F$  is the sum of signs of  $g_i$ , that is*

$$\chi(F^{-1}(y)) = \operatorname{sgn} g_1(y) + \dots + \operatorname{sgn} g_s(y).$$

Our proof is based on a classical and elementary result expressing the number of real roots of a real polynomial of one variable as the signature of an associated quadratic form known already to Hermite [He1, He2] and Sylvester [Sy1], see also [B], [BW], [BCR, p. 97]. In the proof we use a modern generalized version of this result presented in [PRS] (note that we need only a one variable case of [PRS], that is precisely [BR, Proposition p. 18]). Our original proof of the theorem [PS] used different means such as the theory of local topological degree of polynomial mappings, Gröbner bases and the Eisenbud-Levine Theorem and was not so explicit as the one presented below.

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The paper is organized as follows. In Section 1 we prepare the algebraic part of the proof. In particular we recall Hermite and Sylvester’s theorem. In Section 2 we present basic properties of Euler characteristic of semialgebraic sets and a useful formalism of Euler integral of constructible functions. The proof of the theorem for  $F$  proper is presented in Section 3. This case is particularly simple and the proof is obtained by an effective elimination procedure. In Section 4 we complete the proof in general (non-proper) case.

The main theorem of the paper was inspired by the first preprint version of [CK]. For its applications see [MP] and [PS].

**1. Preliminaries I.** Let  $W$  be an irreducible real algebraic subset of  $\mathbf{R}^n$ , let  $\mathcal{A}$  denote the ring of real polynomials on  $W$ , and  $\mathcal{K}$ —the field of fractions of  $\mathcal{A}$ .

Let  $X$  be a real indeterminate. Take  $h^1, \dots, h^k \in \mathcal{K}[X]$  and denote by  $I$  the ideal in  $\mathcal{K}[X]$  generated by  $h^1, \dots, h^k$ . We assume that at least one of  $h^i$  is nonzero. The ring  $\mathcal{K}[X]$  is a principal ideals domain, and  $h = \text{g.c.d.}(h^1, \dots, h^k)$  is a generator of  $I$ . One may find polynomials  $H_1, \dots, H_k, P_1, \dots, P_k \in \mathcal{K}[X]$  such that

$$h = H_1 h^1 + \dots + H_k h^k, \quad h^i = P_i h.$$

Set  $d = \text{deg } h$ . Then  $Q = \mathcal{K}[X]/I$  is a  $\mathcal{K}$ -algebra and  $d = \dim_{\mathcal{K}} Q$ . The monomials  $1, X, \dots, X^{d-1}$  form a basis in  $Q$ . In particular, if  $d = 0$ , then  $I = \mathcal{K}[X]$  and  $Q = \{0\}$ .

For any  $f \in \mathcal{K}[X]$  there are unique  $q, r \in \mathcal{K}[X]$  such that  $f = qh + r$  and  $\text{deg } r < d$ . Thus  $r = a_0 + a_1 X + \dots + a_{d-1} X^{d-1}$ . Put  $r_i(f) = a_i \in \mathcal{K}$ . This way  $r_i(f)$  is the  $i$ -th coordinate of  $f$  in  $Q$ .

Any  $f \in Q$  defines a  $\mathcal{K}$ -linear endomorphism  $A_f : Q \rightarrow Q$  by multiplication  $A_f(p) = fp$ . Let  $\text{Tr}(f) \in \mathcal{K}$  be the trace of  $A_f$ . Clearly

$$\text{Tr}(f) = r_0(f) + r_1(fX) + \dots + r_{d-1}(fX^{d-1}).$$

The trace map  $\text{Tr} : Q \rightarrow \mathcal{K}$  is  $\mathcal{K}$ -linear.

Fix  $g \in \mathcal{K}[X]$ . Define a symmetric bilinear form  $\Theta^g : Q \times Q \rightarrow \mathcal{K}$  by  $\Theta^g(a, b) = \text{Tr}(gab)$ . For  $0 \leq i, j \leq d-1$  put  $T^{ij} = r_0(gX^{i+j}) + \dots + r_{d-1}(gX^{i+j+d-1}) \in \mathcal{K}$ , so that the matrix of  $\Theta^g$  is  $[T^{ij}]$ .

Recall that  $\mathcal{K}$  is the field of fractions of the polynomial ring  $\mathcal{A}$  of an irreducible real algebraic set  $W \subset \mathbf{R}^n$ . There exists a proper algebraic subset  $\Sigma \subset W$  such that the numerators and the denominators of all non-trivial elements of  $\mathcal{K}$ , which have appeared above, do not vanish on  $W \setminus \Sigma$ . Given  $f \in \mathcal{K}[X]$ ,  $f = a_d X^d + \dots + a_0$ ,  $a_i \in \mathcal{K}$ . For  $w \in W$  we denote by  $f_w$  the evaluation of  $f$  at  $w$ , that is,  $f_w = a_d(w)X^d + \dots + a_0(w)$ . Such  $f_w$  is well-defined provided all  $a_i(w)$  exist. Fix  $w \in W \setminus \Sigma$ . Then  $\text{deg } h_w = d$  and  $h_w$  is the greatest common divisor of  $h_w^1, \dots, h_w^k$ . Let  $I_w$  denote the ideal generated by  $h_w^1, \dots, h_w^k$ . Hence  $I_w = (h_w)$ ,  $Q_w = \mathbf{R}[X]/I_w$  is an  $\mathbf{R}$ -algebra,  $\dim_{\mathbf{R}} Q_w = d$ , and  $1, X, \dots, X^{d-1}$  form a basis in  $Q_w$ .

Given  $g \in \mathcal{K}[X]$ . In the same way as above we define the trace map  $\text{Tr}_w : Q_w \rightarrow \mathbf{R}$ , and the symmetric bilinear form  $\Theta_w^g : Q_w \times Q_w \rightarrow \mathbf{R}$ , given by  $\Theta_w^g(a, b) = \text{Tr}(g_w ab)$ . Then the evaluation  $[T_w^{ij}] = [T^{ij}(w)]$  is the matrix of  $\Theta_w^g$ .

Let  $V_w = \{x \in \mathbf{R} \mid h^1(w, x) = \dots = h^k(w, x) = 0\} = \{x \in \mathbf{R} \mid h_w(x) = 0\}$ . If

$w \in W \setminus \Sigma$  then  $I_w \neq \{0\}$  and  $V_w$  is finite. Set

$$(1) \quad A_w = \sum_{x \in V_w} \operatorname{sgn} g(w, x).$$

If  $g = 1$  then  $A_w$  is just the number of real roots of  $h_w(x)$ . The following result follows from [He1, He2] and [Syl], see [BR, Proposition p. 18] and [PRS, Theorem 2.1].

PROPOSITION 1. For  $w \in W \setminus \Sigma$

$$A_w = \operatorname{signature} \Theta_w^g = \operatorname{signature}[T_w^{ij}].$$

Now we apply the lemma of Descartes in order to describe the way  $A_w$  depends on  $w$ .

LEMMA 2. Let  $f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_0$  be a real polynomial and assume that all roots of  $f(x)$  are real. Let  $p_+$  (resp.  $p_-$ ) denote the number of positive (resp. negative) roots of  $f$  counted with multiplicities. Let  $\Lambda$  denote the set of all pairs  $(r, s)$  with  $0 \leq r < s \leq n$  such that  $a_r \neq 0, a_s \neq 0$ , and  $a_i = 0$  for  $r < i < s$ . Define  $\Lambda' = \{(r, s) \in \Lambda \mid r + s \text{ is odd}\}$ . Then

$$p_+ - p_- = - \sum_{(r,s) \in \Lambda'} \operatorname{sgn}(a_r a_s).$$

Proof. We say that the pair of real numbers  $(a, b)$  changes sign if  $ab < 0$ . If this is the case then  $(1 - \operatorname{sgn} ab)/2 = 1$ , if  $ab > 0$  then  $(1 - \operatorname{sgn} ab)/2 = 0$ .

By Descartes' lemma (see [MS, Theorem 6, p. 232], or [BR, Proposition 1.1.10, p. 14]),  $p_+$  equals the number of sign changes in the sequence of non-zero coefficients of  $f(x)$ , that is

$$p_+ = \sum_{(r,s) \in \Lambda} (1 - \operatorname{sgn}(a_r a_s))/2.$$

According to the same fact,  $p_-$  equals the number of sign changes in the sequence of non-zero coefficients of  $f(-x)$ , i.e.

$$p_- = \sum_{(r,s) \in \Lambda} (1 - (-1)^{r+s} \operatorname{sgn}(a_r a_s))/2.$$

Hence  $p_+ - p_- = - \sum_{(r,s) \in \Lambda'} \operatorname{sgn} a_r a_s$  as required. ■

COROLLARY 3. There exist polynomials  $\varphi_1, \dots, \varphi_t \in \mathcal{A}$  and a proper algebraic subset  $\Sigma \subset W$  such that for every  $w \in W \setminus \Sigma$

$$A_w = \sum_{i=1}^t \operatorname{sgn} \varphi_i(w).$$

Proof. Let  $T(\lambda) = T_d \lambda^d + \dots + T_0$ , where  $T_i \in \mathcal{K}$ , be the characteristic polynomial of  $[T^{ij}]$ . Define  $\Lambda'$  in the same way as above. In particular  $T_r T_s \in \mathcal{K} \setminus \{0\}$  for each  $(r, s) \in \Lambda'$ . Then the evaluation  $T_w(\lambda) = T_d(w) \lambda^d + \dots + T_0(w)$  is the characteristic polynomial of  $[T_w^{ij}]$ . We enlarge  $\Sigma$  so that  $T_r(w) T_s(w) \neq 0$  for  $w \in W \setminus \Sigma$  and  $(r, s) \in \Lambda'$ . Then by Lemma 2

$$(2) \quad \operatorname{signature} \Theta_w^g = \operatorname{signature}[T_w^{ij}] = - \sum_{(r,s) \in \Lambda'} \operatorname{sgn} T_r(w) T_s(w)$$

as required. ■

**2. Preliminaries II.** In this section we recall the construction and some basic properties of the Euler integral of a constructible function, see e.g. [MP] for more information.

An integer-valued function  $\varphi : \mathbf{R}^n \rightarrow \mathbf{Z}$  is called *constructible* if it admits a presentation as a finite sum

$$(3) \quad \varphi = \sum m_i \mathbf{1}_{X_i},$$

where for each  $i$ ,  $X_i$  is a closed semialgebraic subset of  $\mathbf{R}^n$ ,  $\mathbf{1}_{X_i}$  is the characteristic function of  $X_i$ , and  $m_i$ 's are integers. The presentation (3) is not unique. If the support of  $\varphi$  is compact, then we may choose all  $X_i$  compact. Then the *Euler integral* of  $\varphi$  is defined as

$$\int \varphi = \sum m_i \chi(X_i).$$

For  $R > 0$  let  $\Psi_R^n$  denote the characteristic function of the ball  $\mathbf{B}_R$  centred at the origin and of radius  $R$ . For any constructible  $\varphi$ , the product  $\varphi\Psi_R^n$  has compact support, and we define the Euler integral of  $\varphi$  as

$$\int \varphi = \int \varphi\Psi_R^n, \quad \text{for } R \gg 0.$$

This definition makes sense because of the following

LEMMA 4. *Let  $X \subset \mathbf{R}^n$  be semialgebraic. Then if  $R > 0$  is large enough,  $X \cap \mathbf{B}_R$  is a deformation retract of  $X$ .*

Proof. Let  $\rho : X \rightarrow \mathbf{R}$  denote the distance to the origin. By topological triviality of semialgebraic mappings [BCR, Thm. 9.3.1], there is a finite subset  $\{y_1, \dots, y_p\} \subset \mathbf{R}$  such that

$$\rho : X \setminus \rho^{-1}(\{y_1, \dots, y_p\}) \rightarrow \mathbf{R} \setminus \{y_1, \dots, y_p\}$$

is a locally trivial fibration. In particular,  $R > \max\{|y_1|, \dots, |y_p|\}$  satisfies the statement of the lemma. ■

COROLLARY 5. *If  $X \subset \mathbf{R}^n$  is closed semialgebraic then  $\int \mathbf{1}_X = \chi(X)$ .*

Proof. If  $R$  is large enough then  $X \cap \mathbf{B}_R$  is a deformation retract of  $X$ . Hence  $\int \mathbf{1}_X = \int \mathbf{1}_X \Psi_R^n = \chi(X \cap \mathbf{B}_R) = \chi(X)$ , as required. ■

LEMMA 6. *Let  $f(x) = a_d x^d + \dots + a_0$ ,  $a_d \neq 0$ , be a polynomial. Then  $\varphi(x) = \text{sgn } f(x)$  is a constructible function. For  $2 \leq k \leq d$  define  $V_k = \{x \in \mathbf{R} \mid f(x) = \dots = f^{(k-1)}(x) = 0\}$  and put  $Z_k = \sum_{x \in V_k} \text{sgn } f^{(k)}(x)$ . Then*

$$\int \varphi = \begin{cases} \text{sgn } a_d - (Z_2 + Z_4 + \dots + Z_d), & \text{for } d \text{ even;} \\ -(Z_2 + Z_4 + \dots + Z_{d-1}), & \text{for } d \text{ odd.} \end{cases}$$

Proof. Clearly  $\text{sgn } f(x)$  is constructible. Let  $x_1 < \dots < x_p$  be the real roots of  $f$ . Take  $R > 0$  such that  $-R < x_1$  and  $x_p < R$ . Put  $x_0 = -R$  and  $x_{p+1} = R$ . For  $0 \leq i \leq p + 1$ , let  $k(i) = \min\{j \geq 0 \mid f^{(j)}(x_i) \neq 0\}$ , and  $S_i = \text{sgn } f^{(k(i))}(x_i)$ .

The sign of  $f$  on  $(x_i, x_{i+1})$  is constant and equals  $S_i = (-1)^{k(i+1)} S_{i+1}$ . We have

$$\varphi\psi_R^1 = \sum_{i=0}^p \frac{1}{2} (S_i + (-1)^{k(i+1)} S_{i+1}) \mathbf{1}_{[x_i, x_{i+1}]} - \sum_{i=1}^p ((-1)^{k(i)} + 1) S_i \mathbf{1}_{\{x_i\}}.$$

Hence

$$\int \varphi = \frac{1}{2}(S_0 + S_{p+1}) - \sum_{i=1}^p \frac{1}{2}((-1)^{k(i)} + 1)S_i = \frac{1}{2}(S_0 + S_{p+1}) - (Z_2 + Z_4 + \dots).$$

If  $d$  is even then  $\frac{1}{2}(S_0 + S_{p+1}) = \text{sgn } a_d$ . If  $d$  is odd then  $\frac{1}{2}(S_0 + S_{p+1}) = 0$ . This ends the proof of Lemma 6. ■

PROPOSITION 7. *Let  $W$  be an irreducible real algebraic subset of  $\mathbf{R}^n$  and  $\mathcal{A}$  denote the ring of real polynomials on  $W$ . Let  $f \in \mathcal{A}[X]$ . Then there exist a finite family of polynomials  $\psi_i \in \mathcal{A}$  and a proper algebraic subset  $\Sigma \subset W$  such that for every  $w \in W \setminus \Sigma$*

$$\int_{\mathbf{R}} \text{sgn } f(w, \cdot) = \sum \text{sgn } \psi_i(w).$$

Proof. Let  $f = a_r x^r + \dots + a_0$ , where  $a_i \in \mathcal{A}$ . We may suppose  $a_r \neq 0$  in  $\mathcal{A}$ . Then for  $w \in W \setminus a_r^{-1}(0)$ ,  $\int_{\mathbf{R}} \text{sgn } f(w, \cdot)$  is calculated in Lemma 6.

For each  $1 \leq k \leq r$ , we apply the construction of Section 1 to the ideal  $I = (h^1, \dots, h^k)$  generated by  $h^1 = f, \dots, h^k = \partial^{k-1} f / \partial X^{k-1}$  and  $g = \partial^k f / \partial X^k$ . Then, by Proposition 1 and Corollary 3, each of  $Z_k$  of Lemma 6 is a finite sum of signs of polynomials in  $w$  as required. This shows the proposition. ■

PROPOSITION 8. *Let  $W$  be a real algebraic set and  $\mathcal{A}$  denote the ring of polynomials on  $W$ . Let  $f \in \mathcal{A}[X]$ . Then there exists a finite family of polynomials  $\gamma_i \in \mathcal{A}$  such that for every  $w \in W$*

$$(4) \quad \int_{\mathbf{R}} \text{sgn } f(w, \cdot) = \sum_i \text{sgn } \gamma_i(w).$$

Proof. The proof is by induction on  $\dim W$  and the number of irreducible components of  $W$ .

If  $W$  is irreducible then, by Proposition 7, we may find a finite family of polynomials  $\gamma'_k \in \mathcal{A}$  satisfying (4) in the complement of the proper algebraic subset  $\Sigma$  of  $W$ . By the inductive assumption there exists a finite family of polynomials  $\gamma''_j$  on  $\Sigma$  satisfying (4) on  $\Sigma$ . We consider  $\gamma''_j$  as the restriction of polynomials on  $W$ . Let  $P \in \mathcal{A}$  be any non-negative polynomial such that  $P^{-1}(0) = \Sigma$ . Then

$$\text{sgn } P\gamma'_k = \begin{cases} \text{sgn } \gamma'_k, & \text{on } W \setminus \Sigma; \\ 0, & \text{on } \Sigma. \end{cases}$$

Similarly

$$\text{sgn } \gamma''_j - \text{sgn } P\gamma''_j = \begin{cases} 0, & \text{on } W \setminus \Sigma; \\ \text{sgn } \gamma''_j, & \text{on } \Sigma, \end{cases}$$

and hence the family  $\{P\gamma'_k, \gamma''_j, -P\gamma''_j\}$  satisfies the statement.

Suppose  $W' \subset W$  is an irreducible component of  $W$  and let  $W''$  be the union of the other components. Let  $\gamma'_k$  (resp.  $\gamma''_j$ ) be the family satisfying (4) on  $W'$  (resp.  $W''$ ). Let  $P \in \mathcal{A}$  be any non-negative polynomial such that  $P^{-1}(0) = W'$ . Then, by the same argument as above, the family  $\{P\gamma'_k, \gamma''_j, -P\gamma''_j\}$  satisfies the statement. This ends the proof. ■

**3. Proof of Theorem (proper case).** Suppose  $F : Y \rightarrow \mathbf{R}^n$  is proper. Replacing  $Y$  by the graph of  $F$  we may assume that  $Y \subset \mathbf{R}^n \times \mathbf{R}^m$  and  $F$  is induced by the projection on the first factor.

The proof is by induction on  $m$  (we do not assume  $n$  to be fixed). Suppose  $m = 1$  and  $Y$  is the zero set of a non-negative polynomial function  $f(y, x)$ . We apply Proposition 8 to  $W = \mathbf{R}^n$  and  $\mathcal{A} = \mathbf{R}[y]$ . Note that, since  $F$  is proper,  $F^{-1}(y)$  has to be finite and  $\chi(F^{-1}(y)) = 1 - \int_{\mathbf{R}} \operatorname{sgn} f(y, \cdot)$ . Hence the result follows from Proposition 8.

*Inductive step.* Let  $p : Y \rightarrow \mathbf{R}^n \times \mathbf{R}^1$  denote the projection. By the inductive hypothesis we may assume that there exists a finite family of polynomials  $f_i(y, x_1)$  such that for each  $(y, x_1) \in \mathbf{R}^n \times \mathbf{R}$

$$(5) \quad \chi(p^{-1}((y, x_1))) = \sum_i \operatorname{sgn} f_i.$$

Define  $\varphi(y, x_1) = \chi(p^{-1}((y, x_1)))$ . We claim that

$$(6) \quad \int_{\mathbf{R}} \varphi(y, \cdot) = \chi(F^{-1}(y)).$$

Indeed, this follows from a ‘‘Fubini-type’’ formula for the Euler integral,

$$(7) \quad \int_{\{y\} \times \mathbf{R}} \varphi(y, \cdot) = \int_{\{y\} \times \mathbf{R}} p_* \mathbf{1}_Y = \int_{F^{-1}(y)} \mathbf{1}_Y = \chi(F^{-1}(y)),$$

see, e.g., [V], [MP, A.4.2]. Then, by (5) and (6)

$$\chi(F^{-1}(y)) = \sum_i \int_{\mathbf{R}} \operatorname{sgn} f_i(y, \cdot),$$

and the theorem again follows from Proposition 8.

**4. Proof of Theorem (general case).** The proof presented in the previous sections does not work in general since the ‘‘Fubini type’’ formula (7) fails for non-proper maps. To complete the proof in the general case we use the projective compactification and the link at infinity. We shall also need the following corollary of the theorem.

**PROPOSITION 9.** *Let  $Y \subset \mathbf{R}^m$  be algebraic and  $F : Y \rightarrow \mathbf{R} \times \mathbf{R}^n$  be a proper polynomial map. Then there exists a finite family of polynomials  $\gamma_i(y)$ , where  $y \in \mathbf{R}^n$ , such that*

$$\lim_{t \rightarrow 0^+} \chi(F^{-1}(t, y)) = \sum \operatorname{sgn} \gamma_i(y).$$

*Proof.* The proposition follows easily from the theorem and the following lemma.

**LEMMA 10.** *Let  $W$  be real algebraic and  $g(t, w)$  be a polynomial on  $\mathbf{R} \times W$ . Then there exists a finite family of polynomials  $\gamma_i(w)$  on  $W$  such that*

$$(9) \quad \lim_{t \rightarrow 0^+} \operatorname{sgn} g(t, w) = \sum \operatorname{sgn} \gamma_i(w).$$

*Proof.* Let  $\psi(w) = \lim_{t \rightarrow 0^+} g(t, w)$ . We proceed by induction on  $\dim W$ . Without loss of generality we may assume that  $W$  is irreducible. We shall show that the statement of lemma holds generically on  $W$ , that is to say, there exist a proper algebraic subset  $W'$  of  $W$  and the polynomials  $\gamma_i$  such that (9) holds in the complement of  $W'$ . Then the lemma will follow from the inductive assumption.

We may also assume that  $g$  does not vanish identically, and then there exists a non-negative integer  $k$  such that  $g(t, w) = t^k h(t, w)$ , where  $h(t, w)$  is a polynomial which does not vanish identically on  $\{0\} \times W$ . Then, in the complement of  $W' = \{w \mid h(0, w) = 0\}$ ,  $\psi(w) = \text{sgn } h(0, w)$  as required. This ends the proof of the lemma and the proposition. ■

**Remark 11.** The assumption of properness of  $F$  in Proposition 9 is not necessary. We made it since we are going to use Proposition 9 in the proof of theorem in general case, and it is only the proper case of theorem which has been proven till now.

Let  $X$  be a real algebraic subset of  $\mathbf{R}^m$  and denote by  $\tilde{X}$  the one point compactification of  $X$  (if  $X$  is compact then  $\tilde{X} = X \amalg \{\text{point}\}$ ). Let  $L_R = X \cap S_R^{m-1}$ , where  $S_R^{m-1}$  denotes the sphere of radius  $R > 0$  centred at the origin. We call such  $L_R$ , for  $R$  sufficiently large, *the link at infinity of  $X$* , and denote by  $L_\infty(X)$ . By an argument similar to the proof of Lemma 4 it is easy to see that

$$(10) \quad \chi(X) = \chi(\tilde{X}) + \chi(L_\infty(X)) - 1.$$

Let us recall the standard construction of  $\tilde{X}$  as an algebraic set (we will need it below in a parametrized case). Suppose  $X$  is given by a finite number of equations  $f_i(x) = 0$ ,  $i = 1, \dots, s$ . Set  $h(x, x_{m+1}) = (f_1^2(x) + \dots + f_s^2(x)) + (x_{m+1} - 1)^2$ , so that  $h^{-1}(0)$  is homeomorphic to  $X$  and  $h$  is a non-negative polynomial of degree, say, bounded by  $2p$ . Put  $x' = (x, x_{m+1}) \in \mathbf{R}^{m+1}$  and let  $H(s, x') = (s\|x'\|)^{4p} h(x'/(s\|x'\|)^2)$ . Then, it is easy to see that  $H$  extends to a non-negative polynomial on  $\mathbf{R} \times \mathbf{R}^{m+1}$  such that for  $s \neq 0$ ,  $\tilde{X}_s = \{x' \in \mathbf{R}^{m+1} \mid H(s, x') = 0\}$  is homeomorphic to the single point compactification of  $X$ . Also  $L_s = \{x' \in \mathbf{R}^{m+1} \mid H(s, x') = 0, \|x'\| = 1\}$ , for  $s \neq 0$  and small enough, is homeomorphic to the link at infinity of  $X$ .

Now we are ready to prove the theorem. First we assume that  $Y \subset \mathbf{R}^n \times \mathbf{R}^m$  and  $F$  is induced by the projection onto the first factor. Proceeding exactly in the same way as above we may compactify simultaneously the fibres of  $F$ . In particular the following statement holds.

**PROPOSITION 12.** *Let  $Y \subset \mathbf{R}^n \times \mathbf{R}^m$  be algebraic and  $F : Y \rightarrow \mathbf{R}^n$  be the projection on the first factor. Then there exists a non-negative polynomial  $H : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{m+1} \rightarrow \mathbf{R}$  such that for every  $y \in \mathbf{R}^n$*

- (i)  $\tilde{Y} = \{x' \in \mathbf{R}^{m+1} \mid H(s, y, x') = 0, s = 1\}$  is homeomorphic to the single point compactification of  $F^{-1}(y)$ . Moreover, the induced projection  $\tilde{Y} \rightarrow \mathbf{R}^n$  is proper;
- (ii)  $\{x' \in \mathbf{R}^{m+1} \mid H(s, y, x') = 0, \|x'\| = 1\}$ , for  $s = s(y) \neq 0$  small enough, is homeomorphic to the link at infinity of  $F^{-1}(y)$ .

Note that identity (10) holds for every  $X = F^{-1}(y)$ . Hence to prove the theorem it suffices to consider the families of compactifications and links at infinity of fibres  $F^{-1}(y)$  parametrized by  $y \in \mathbf{R}^n$ . For the first one the statement holds by Proposition 12 (i) and the proper case. For the family of links consider  $L \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{m+1}$  given by  $H(s, y, x') = 0, \|x'\| = 1$ . Then, clearly the projection  $L \rightarrow \mathbf{R} \times \mathbf{R}^n, (s, y, x') \rightarrow (s, y)$ , is proper and we apply to it Proposition 9. Now Proposition 12 (ii) gives the statement for the family of links at infinity. This, in virtue of (10), shows the theorem.

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