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ON SECTORIAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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À S. Łojasiewicz, en témoignage d'une longue amitié

This note answers a question posed to me by B. Jakubczyk.

THEOREM. *Given m holomorphic functions $f_1, \dots, f_m$ at 0 of $2n+1$ variables $(x, y_1, \dots, y_n, z_1, \dots, z_n)$, suppose that the formal power series $\overline{y}_i = \sum_{j\geq 2} \overline{y}_{i,j} x^j$ satisfy $f_j(x, \overline{y}_i, \overline{y}'_i) = 0$. Then, given a direction θ at $0 \in \mathbb{C}$, there exist $\varepsilon > 0$ and $y = (y_1, \dots, y_n)$, holomorphic in $(\arg x - \theta) < \varepsilon$, $0 < |x| < \varepsilon$, having $\overline{y}$ as asymptotic expansion at 0, and satisfying $f_j(x, y_i(x), y'_i(x)) = 0$.*

Note that here m and n are arbitrary.

Proof. We can replace the equations $f_j = 0$ by $\frac{d}{dx} f_j = 0$ (since $f_j(0) = 0$). Adding the y'_i as new unknown functions and changing a little bit the notation, we are reduced to the quasi-linear case:

$$\sum a_{ij}(x, y) y'_j = b_i(x, y) \qquad (1 \leq i \leq m;\ 1 \leq j \leq n),$$

a_{ij} and b_i holomorphic at $0 \in \mathbb{C}^{n+1}$.

Let $\mathfrak{p}$ be the kernel of the map $\mathbb{C}\{x, y_1, \dots, y_n\} \xrightarrow{\varphi} \mathbb{C}[[x]]$ given by $\varphi(f) = f(x, \overline{y}(x))$. Obviously, $\mathfrak{p}$ is a prime ideal.

Case 1. Suppose $\mathfrak{p} = 0$, i.e., φ is injective. Let r be the rank of the matrix $\big(a_{ij}(x, y)\big)$ in the germs at $0 \in \mathbb{C}^{n+1}$, and suppose for instance $\det(a_{ij}(x, y)) \neq 0$, $1 \leq i \leq r$, $1 \leq j \leq r$. I claim that we can forget the $m - r$ last equations; to prove this, it is sufficient to prove the following result: if we have $c_i(x, y)$ holomorphic at 0, satisfying $\sum_i c_i a_{ij} = 0$ $(1 \leq j \leq n)$, then $\sum c_i b_i = 0$. But this is true, since we have a formal solution of the system, and φ is injective.

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To solve the r first equations, we choose first $y_{r+1}, \dots, y_n$ arbitrarily, by the classical theorem of Borel-Ritt. Then we are reduced to the "determined" case, treated by Ramis-Sibuya [R-S].

Actually they state their results in the case $x^{r+1}y' = f(x, y)$; but the reduction to this case is easy: one has just to substitute $y = a + x^\ell \widetilde{y}$, a polynomial in x, $\ell \gg 0$; cf. for instance [M].

General case. One reduces it to the preceding case by *desingularization.* Let $Y \subset (\mathbb{C}^{n+1}, 0)$ be the germ defined by $\mathfrak{p}$; according to Hironaka, there is a proper map $Z \xrightarrow{\pi} (\mathbb{C}^{n+1}, 0)$, with $Z \setminus \pi^{-1}(Y) \simeq (\mathbb{C}^{n+1} \setminus Y, 0)$, such that the strict transform $\widetilde{Y}$ of Y is non-singular; and (Z, π) is built-up by a sequence of blowing-up of smooth center. One verifies that the formal curve $x \mapsto (x, \overline{y}(x))$ can be lifted at each step of the desingularization, and is finally lifted to a formal curve $x \mapsto \widetilde{y}(x)$ with values in $\widetilde{Y}$.

Let p be the projection $\widetilde{Y} \to \mathbb{C}$ composed of $\widetilde{Y} \to \mathbb{C}^{n+1}$, and of the projection $(x, y) \mapsto x : \mathbb{C}^{n+1} \to \mathbb{C}$; the composition $\mathbb{C} \xrightarrow{\tilde{y}} \widetilde{Y} \xrightarrow{p} \mathbb{C}$ is the identity; therefore p is smooth at $\widetilde{y}(0)$, and p (or "x") can be chosen as one of the local coordinates at $\widetilde{y}(0) \in \widetilde{Y}$.

Now, we add to our equations a system of generators of $\mathfrak{p} : f_k(x, y_1, \dots, y_n) = 0$. If we denote by ω_i the form $\sum a_{ij}(x, y)dy_j - b_i(x, y)dx$, and by $\overline{\omega}_i$ its restriction to Y (i.e. the class of ω_i modulo $\mathfrak{p}$ and $d\mathfrak{p}$), our system is equivalent to $\overline{\omega}_i = 0$ $(1 \leq i \leq m)$. As in Case 1, one sees that one can restrict oneself to the case where the $\overline{\omega}_i$ are independent. Now, by desingularization, we are reduced to the case where Y is non-singular, and we end as in Case 1.

I also mention, without giving details, that one can eliminate the use of desingularization by a more careful study of the situation.

References

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