SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

ON VALUATION SPECTRA*

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If K is an ordered field then every convex subring of K is a valuation ring of K. This easy but fundamental observation has made valuation theory a very natural and important tool in real algebraic geometry. In particular many topological phenomena of semialgebraic sets and of constructible subsets of real spectra are best explained by use of valuations.

We have seen in recent years how important it is to switch from the consideration of particular orderings of fields to a study of the set of all orderings of all residue class fields of a commutative ring A, i.e. the real spectrum $\operatorname{Sper} A$ of A.

Now why not do the same with valuations? This leads to the definition of *valuation spectra*. In principle the points of the valuation spectrum $\operatorname{Spev} A$ should be pairs $(\mathfrak{p}, v)$ consisting of a prime ideal $\mathfrak{p}$ of A, i.e. a point of $\operatorname{Spec} A$, and a Krull valuation v of the residue class field $\operatorname{Quot}(A/\mathfrak{p})$. Here one has to made a decision whether one should distinguish between different valuations of $\operatorname{Quot}(A/\mathfrak{p})$ which have the same valuation ring or not. One further has to choose a topology on $\operatorname{Spev} A$, where again at least two reasonable choices can be made. Finally one should look for sheaves of "functions" on $\operatorname{Spev} A$ and some prominent subsets of $\operatorname{Spev} A$.

In recent years various authors have defined valuation spectra and/or related spaces. (Brumfiel, de la Puente, Berkovich, Robson, Huber, Schwartz). To my opinion the question which valuation spectrum is the "right" one depends on the applications one has in mind. Certain valuation spectra are important both for real algebraic and for p-adic geometry. In want to stress here a direction followed by Roland Huber which leads to a new foundation of rigid analytic geometry.

Huber defines for A in a certain class of topological rings, which he calls "f-adic rings", a ringed space $\operatorname{Spa} A$, the *analytic spectrum* of A. The points of $\operatorname{Spa} A$ are those points $(\mathfrak{p}, v)$ of the valuation spectrum $\operatorname{Spev} A$ such that a homomorphism form A to a valued field K inducing v is continuous. Analytic spectra are the building blocks of "adic

1991 *Mathematics Subject Classification*: Primary 13A18; Secondary 32D25.

Received by the editors: January 10, 1997.

* Summary of the author's lecture at the *Seminar on Singularities and Geometry*, Warsaw.

spaces" in much the same way as Zariski spectra are the building blocks of schemes. The category of adic spaces contains equivalents of the classical rigid analytic spaces as a full subcategory.

Classical rigid analytic geometry seems to suffer from the fact that only rank one valuations are admitted. This means that—in some sense—here we have only one parameter at our disposal. Huber's adic geometry works with valuations of arbitrary rank. It thus allows a natural understanding of important constructions within the framework of adic spaces where previously one had to leave rigid analytic geometry and to resort to formal algebraic geometry. Mumford's "mysterious" construction of semiabelian group schemes in his famous paper [5] is a case in point.

For the purpose of real geometry it is appropriate to replace Spev A by the *real valuation spectrum* Sperv A which is a sort of fibre product of Spev A with the real spectrum Sper A over the Zariski spectrum Spec A. Similarly one can replace an analytic spectrum Spa A by a *real analytic spectrum* Spar A. On these spaces there live natural analoga of semialgebraic functions. One thus obtains a synthesis of semialgebraic geometry and rigid analytic geometry.

Of course, valuation spectra should also be useful in (usual) algebraic geometry. Indeed, this amounts to an extension of Zariski's approach in the thirties and forties to a truly "algebraic" geometry. Zariski used valuations more or less as "ideal points" of algebraic varieties. Then we have learned from Grothendieck that for most purposes prime ideals instead of valuations suffice. But there are a lot of instances where it seems to be more natural and more promising to work with valuations instead of prime ideals. One immediately thinks of the resolution processes for algebraic singularities. Unfortunately, as far as I know, no one has yet made serious studies here using valuation spectra explicitly.

In the last section of our survey article on valuation spectra [1] we have outlined two other applications of valuation spectra to algebraic geometry. Details can be found in [2,3]. Recently Huber has developed systematically an etale cohomology for rigid analytic varieties [4]. This cohomology gives also information about the (usual) etale cohomology of algebraic varieties, some of which up to now cannot be gained by algebraic methods.

The survey article [1] contains the basic definitions and some theorems (e.g. a curve selection lemma) on valuation spectra. The book [4] contains the main references for Huber's "abstract" rigid analytic geometry and the work of others in this area.

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