SINGULARITIES SYMPOSIUM — ŁOJASIEWICZ 70
BANACH CENTER PUBLICATIONS, VOLUME 44
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

MANIS VALUATIONS AND PRÜFER EXTENSIONS*

MANFRED KNEBUSCH
University of Regensburg, Department of Mathematics
Universitätsstr. 31, D-93040 Regensburg
E-mail: Manfred.Knebusch@mathematik.uni-regensburg.de

A *Manis valuation* on a ring R (always commutative, with 1) is a *surjective* map $v : R \to \Gamma \cup \{\infty\}$ with Γ a totally ordered abelian group fulfilling the usual valuation axioms $v(xy) = v(x)+v(y)$, $v(x+y) \geq \mathrm{Min}(v(x), v(y))$, and (to avoid trivialities) $v(1) = 0$, $v(0) = \infty$. Then $v^{-1}(\infty)$ is a prime ideal $\mathfrak{q}$ of R, called the *support* $\mathrm{supp}\, v$ of v, and v induces a Krull valuation $\hat{v}$ on the residue class field $k(\mathfrak{q}) = \mathrm{Quot}(R/\mathfrak{q})$ of $\mathfrak{q}$. A Manis valuation v is uniquely determined—up to equivalence in a natural sense—by the pair $(A_v, \mathfrak{p}_v)$, with $A_v := \{x \in R \mid v(x) \geq 0\}$ and $\mathfrak{p}_v := \{x \in R \mid v(x) > 0\}$, cf. [M].

From a geometric viewpoint the definition of Manis valuations is often too narrow. One then has to drop the surjectivity condition and arrives at the Bourbaki valuations [Bo, VI §3]. These are the basic objects for the valuation spectra (cf. my talk at Warszawa, [Kn]). But Manis valuations are much better amenable to algebraic manipulations than Bourbaki valuations, and under happy circumstances (see below) it suffices to work with Manis valuations.

A ring extension $A \subset R$ is called *Prüfer*, if for every prime ideal $\mathfrak{p}$ of A there exists a Manis valuation v on R with $A_v = \{x \in R \mid sx \in A \text{ for some } s \in A \setminus \mathfrak{p}\}$ and $\mathfrak{p}_v = \{x \in R \mid sx \in \mathfrak{p} \text{ for some } s \in A \setminus \mathfrak{p}\}$. We then also say that A is an *R-Prüfer ring.* (This is a slight modification of the original definition of Prüfer extensions by Griffin [G_2]. The modification seems to be necessary, cf. [Gr, p. 285].) In my talk I gave a report on joint work with Digen Zhang. We intend to write a book on Manis valuations and Prüfer extensions. A preliminary version of the first chapter has been published in a newly founded German electronic journal [KZ], and my report has been on that chapter. (The journal can be reached under the Internet-address:
`http:/www.mathematik.uni-bielefeld.de/documenta/vol-01/vol-01.html`.)

1991 *Mathematics Subject Classification*: Primary 13A18; Secondary 13B02.
Received by the editors: January 10, 1997.

* Lecture given by the author at the *Symposium on Singularities, on the Occasion of the* 70^{th} *Birthday of Stanisław Łojasiewicz*, Cracow.

I now give some indications why it seems to be useful from a geometric viewpoint to study Prüfer extensions.

If F is a formally real field then it is well known that the intersection of the real valuation rings of F is a Prüfer domain H, and that H has the quotient field F. (A valuation ring is called real if its residue class field is formally real.) H is the so called *real holomorphy ring* of F, cf. [B, §2], [S], [KS, Chap. III §12]. If F is the function field $k(V)$ of an algebraic variety V over a real closed field k (e.g. $k = \mathbf{R}$), suitable overrings of H in R can tell us a lot about the algebraic and the semialgebraic geometry of $V(k)$.

These rings, of course, are again Prüfer domains. A very interesting and—to our opinion—still mysterious role is played by some of these rings which are related to the orderings of higher level of F, cf. e.g. [B_2], [B_3]. Here we meet a remarkable phenomenon. For orderings of level 1 (i.e. orderings in the classical sense) the usual procedure is to observe first that the convex subrings of ordered fields are valuation rings, and then to go on to Prüfer domains as intersections of such valuation rings, cf. e.g. [B], [S], [KS]. But for higher levels, up to now, the best method is to construct directly a Prüfer domain A in F from a "torsion preordering" of F, and then to obtain the valuation rings necessary for analyzing the preordering as localizations $A_{\mathfrak{p}}$ of A, cf. [B_2, p. 1956 f], [B_3]. Thus there is a two way traffic between valuations and Prüfer domains.

Less is done up to now for F the function field $k(V)$ of an algebraic variety V over a p-adically closed field k (e.g. $k = \mathbf{Q}_p$). But work of Kochen and Roquette (cf. §6 and §7 in the book [PR] by Prestel and Roquette) gives ample evidence, that also here Prüfer domains play a prominent role. In particular, every formally p-adic field F contains a "*p-adic holomorphy ring*", called the *Kochen ring*, in complete analogy to the formally real case [PR, §6]. Actually the Kochen ring has been discovered and studied much earlier than the real holomorphy ring ([Ko], [R]).

If R is a commutative ring (with 1) and k is a subring of R then we can still define a real holomorphy ring $H(R/k)$ consisting of those elements a of R which on the real spectrum of R (cf. [BCR], [B_1], [KS]) can be bounded by elements of k. (If R is a formally real field F and k the prime ring of F this coincides with the real holomorphy ring H from above.) These rings $H(R/k)$ have proved to be very useful in real semialgebraic geometry. In particular, N. Schwartz and M. Prechtel have used them for completing a real closed space and, more generally, to turn a morphism between real closed spaces into a proper one in a universal way ([Sch, Chap V, §7], [Pt]).

The algebra of these holomorphy rings turns out to be particularly good natured if we assume that $1+\Sigma R^2 \subset R^*$, i.e. that all elements $1+a_1^2+\ldots+a_n^2$ ($n \in \mathbf{N}$, $a_i \in R$) are units in R. This is a natural condition in real algebra. The rings used by Schwartz and Prechtel, consisting of abstract semialgebraic functions, fulfil the condition automatically. More generally, if A is any commutative ring (always with 1) then the localization $S^{-1}A$ with respect to the multiplicative set $S = 1 + \Sigma A^2$ is a ring R fulfilling the condition, and R has the same real spectrum as A. Thus for many problems in real geometry we may replace A by R.

Recently V. Powers has proved that, if $1 + \Sigma R^2 \subset R^*$, the real holomorphy ring

$H(R/k)$ with respect to any subring k is an R-Prüfer ring. More generally V. Powers proved that, if $1 + \Sigma R^{2d} \subset R^*$ for some even number $2d$, every subring A of R containing the elements $\frac{1}{1+q}$ with $q \in \Sigma R^{2d}$ is R-Prüfer ([P, Th. 1.7], cf. also [BP]).

An R-Prüfer ring is related to Manis valuations on R in much the same way as a Prüfer domain is related to valuations of its quotient field. Why should not we try to repeat the success story of Prüfer domains and real valuations on the level of relative Prüfer rings and Manis valuations? Already Marshall in his important paper [Mar] has followed such a program. He has worked there with "Manis places" in a ring R with $1 + \Sigma R^2 \subset R^*$, and has related them to the points of the real spectrum $\operatorname{Sper} R$.

We mention that Marshall's notion of Manis places is slightly misleading. By his definition these places do not correspond to Manis valuations but to a broader class of valuations which we call "special valuations". (These are the valuations which in the valuation spectrum $\operatorname{Spev} R$ are maximal with respect to primary specializations.) But then V. Powers (and independently one of us, D. Z.) observed that, in the case $1 + \Sigma R^2 \subset R^*$, the places of Marshall in fact do correspond to the Manis valuations of R [P]. In [KZ] we prove that every special valuation of R is Manis under a much weaker condition on R, cf. [KZ, Th. 1.1].

The program to study Manis valuations and relative Prüfer rings in rings of real functions has gained new impetus and urgency from the fact that the theory of orderings of higher level has recently been pushed from fields to rings leading to *real spectra of higher level*. These spectra in turn have already proved to be useful for ordinary real semialgebraic geometry. We mention an opus magnum by Ralph Berr [Be], where spectra of higher level are used in a fascinating way to classify the singularities of real semialgebraic functions.

p-adic semialgebraic geometry seems to be accessible as well. L. Bröcker and H.-J. Schinke [BS] have brought the theory of p-adic spectra to a rather satisfactory level by studying the "L-spectrum" $L\text{-spec}\,A$ of a commutative ring A with respect to a given non-archimedian local field L (e.g. $L = \mathbf{Q}_p$). There seems to be no major obstacle in sight which prevents us from defining and studying rings of semialgebraic functions on a constructible (or even proconstructible) subset X of $L\text{-spec}\,A$. Here "semialgebraic" means definability in a model theoretic sense plus a suitable continuity condition. Relative Prüfer subrings of such rings should be quite interesting.

Now, there exists already a rich theory of "*Prüfer rings with zero divisors*", also started by Griffin [G_1], cf. the books [LM], [Huc], and the literature cited there. But this theory seems not to be taylored to geometric needs. A Prüfer ring with zero divisors A is the same as an R-Prüfer ring with $R = \operatorname{Quot} A$, the total quotient ring of A. While this is a reasonable notion from the viewpoint of ring theory it may be artificial from a geometric viewpoint. A typical situation in real geometry is the following: R is the ring of (continuous) semialgebraic functions on a semialgebraic set M over a real closed field k or, more generally, the set of abstract semialgebraic functions on a proconstructible subset X of a real spectrum (cf. [Sch], [Sch_1]). Although the ring R has very many zero divisors we have experience that in some sense R behaves nearly as well as a field (cf. e.g. our notion of "convenient ring extensions" in [KZ, §6]). Now, if A is a subring of R, then it is natural and interesting from a geometric viewpoint to study the R-Prüfer rings $B \supset A$, while the total quotient rings Quot A and Quot B seem to bear little geometric relevance.

Except in a paper by P. L. Rhodes from 1991 [Rh] very little seems to be done on Prüfer extensions in general, and in the original paper of Griffin the proofs of important facts [G_2, Prop. 6, Th. 7] are omitted. Moreover the paper by Rhodes has a gap in the proof of his main theorem. ([Rh, Th. 2.1], condition (5b) there is apparently not a characterization of Prüfer extensions. Any algebraic field extension is a counterexample.) Thus we have been careful in [KZ] about a foundation of this theory. I give a summary about the contents of that paper [KZ].

In §1 and §2 we gather what we need about Manis valuations. Then in §3 and §4 we develop an auxiliary theory of "weakly surjective" ring homomorphisms. These form a class of epimorphisms in the category of commutative rings close to the flat epimorphisms studied by D. Lazard and others in the sixties, cf. [L], [Sa], [A]. In §5 the up to then independent theories of Manis valuations and weakly surjective homomorphisms are brought together to study Prüfer extensions. It is remarkable that, although Prüfer extensions are defined in terms of Manis valuations, they can be characterized entirely in terms of weak surjectivity. Namely, a ring extension $A \subset R$ is Prüfer iff every subextension $A \subset B$ is weakly surjective. A third way to characterize Prüfer extensions is by multiplicative ideal theory, as we will explicate in Chapter II of our planned book.

Our first major result on Prüfer extensions (Th. 5.2) gives characterizations of these extensions which sometimes make it easy to recognize a given ring extension as Prüfer, cf. the examples in §6. We then establish various permanence properties of the class of Prüfer extensions. For example we prove for Prüfer extensions $A \subset B$ and $B \subset C$ that $A \subset C$ is again Prüfer, a result already due to Rhodes [Rh].

At the end of §5 we prove that any commutative ring A has a universal Prüfer extension $A \subset P(A)$ which we call the *Prüfer hull* of A. Every other Prüfer extension $A \hookrightarrow R$ can be embedded into $A \hookrightarrow P(A)$ in a unique way. The Prüfer rings with zero divisors are just the rings A with $P(A)$ containing the total quotient ring Quot A.

In §6 we prove theorems which give us various examples of Manis valuations and Prüfer extensions. We illustrate how naturally they come up in algebraic geometry over a field k which is not algebraically closed, and in real algebraic and semialgebraic geometry. Perhaps our best result here is Theorem 6.8, giving a far-reaching generalization of an old lemma by A. Dress (cf. [D, Satz 2′]). This lemma states for F a field, in which -1 is not a square, that the subring of F generated by the elements $1/(1 + a^2)$, $a \in F$, is Prüfer in F. Dress's innocent looking lemma seems to have inspired generations of real algebraists (cf. e.g. [La, p. 86], [KS, p. 163]) and also ring theorists, cf. [Gi].

We finally prove in §7 for various Prüfer extensions $A \subset R$ that, if $\mathfrak{a}$ is a finitely generated A-submodule of R with $R\mathfrak{a} = R$, then some power $\mathfrak{a}^d$ (with d specified) is principal. Our main result here (Theorem 7.8) is a generalization of a theorem by P. Roquette [R, Th. 1] which states this for R a field (cf. also [Gi]). Roquette used his theorem to prove by general principles that the Kochen ring of a formally p-adic field is Bézout [loc. cit]. Similar applications should be possible in p-adic semialgebraic geometry. Roquette's paper has been an inspiration for our whole work since it indicates well the ubiquity of

Prüfer domains in algebraic geometry over a non-algebraically closed field.

References

[A] T. Akiba, *Remarks on generalized rings of quotients, III*, J. Math. Kyoto Univ. 9 (1969), 205–212.

[B] E. Becker, *Valuations and real places in the theory of formally real fields*, in: Géométrie algébrique réelle et formes quadratiques, J.-L. Colliot-Théléne et al. (ed.), Lecture Notes in Math. 959, Springer, Berlin-New York, 1982, 1–40.

[B_1] E. Becker, *On the real spectrum of a ring and its applications to semialgebraic geometry*, Bull. Amer. Math. Soc. (N.S.) 15 (1986), 19–60.

[B_2] E. Becker, *Partial orders on a field and valuation rings*, Comm. Algebra 7 (1979), 1933–1976.

[B_3] E. Becker, *Theory of Real Fields and Sums of Powers*, forthcoming book.

[BP] E. Becker and V. Powers, *Sums of powers in rings and the real holomorphy ring*, J. Reine Angew. Math. 480 (1996), 71–103.

[Be] R. Berr, *Basic principles for a morphological theory of semi-algebraic morphisms*, Habilitationsschrift, Univ. Dortmund, 1994.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, Ergeb. Math. Grenzgeb. (3) 12, Springer, Berlin, 1987.

[Bo] N. Bourbaki, *Algèbre commutative*, Chap. 1–7, Hermann, Paris, 1961–1965.

[BS] L. Bröcker, J.-H. Schinke, *On the L-adic Spectrum*, Schriftenreihe Math. Inst. Univ. Münster 2. Ser. 40, Münster, 1986.

[D] A. Dress, *Lotschnittebenen mit halbierbarem rechten Winkel*, Arch. Math. (Basel) 16 (1965), 388–392.

[Gi] R. Gilmer, *Two constructions of Prüfer domains*, J. Reine Angew. Math. 239/240 (1970), 153–162.

[Gr] J. Gräter, *Der allgemeine Approximationssatz für Manisbewertungen*, Monatsh. Math. 93 (1982), 277–288.

[G_1] M. Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math. 239/240 (1970), 55–67.

[G_2] M. Griffin, *Valuations and Prüfer rings*, Canad. J. Math. 26 (1974), 412–429.

[Huc] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York, 1988.

[Kn] M. Knebusch, *On valuation spectra*, this volume.

[KS] M. Knebusch and C. Scheiderer, *Einführung in die reelle Algebra*, Vieweg, Braunschweig, 1989.

[KZ] M. Knebusch and Digen Zhang, *Manis valuations and Prüfer extensions I*, Doc. Math. 1 (1996), 149–197.

[Ko] S. Kochen, *Integer valued rational functions over the p-adic numbers: A p-adic analogue of the theory of real fields*, in: Number Theory, Proc. Symp. Pure Math. XII, 57–73, Amer. Math. Soc., Providence, 1969.

[La] T. Y. Lam, *Orderings, Valuations and Quadratic Forms*, CBMS Regional Conference Series in Math. 52, Amer. Math. Soc., Providence, 1983.

[LM] M. Larsen and P. McCarthy, *Multiplicative Theory of Ideals*, Academic Press, New York-London, 1971.

[L] D. Lazard, *Autour de la platitude*, Bull. Soc. Math. France 97 (1969), 81–128.

[M] M. E. Manis, *Valuations on a commutative ring*, Proc. Amer. Math. Soc. 20 (1969), 193–198.

[Mar] M. Marshall, *Orderings and real places on commutative rings*, J. Algebra 140 (1991), 484–501.

[P] V. Powers, *Valuations and higher level orders in commutative rings*, J. Algebra 172 (1995), 255–272.

[Pt] M. Prechtel, *Universelle Vervollständigungen in der Kategorie der reell abgeschlossenen Räume*, Dissertation, Univ. Regensburg, 1992.

[PR] A. Prestel and P. Roquette, *Formally p-adic Fields*, Lecture Notes in Math. 1050, Springer, Berlin-New York, 1984.

[Rh] C. P. L. Rhodes, *Relative Prüfer pairs of commutative rings*, Comm. Algebra 19 (1991), 3423–3445.

[R] P. Roquette, *Bemerkungen zur Theorie der formal p-adischen Körper*, Beiträge Algebra Geom. 1 (1971), 177–193.

[Sa] P. Samuel, *La notion de place dans un anneau*, Bull. Soc. Math. France 85 (1957), 123–133.

[S] H.-W. Schülting, *Real holomorphy rings in real algebraic geometry*, in: Géométrie algébrique réelle et formes quadratiques, J.-L. Colliot-Théléne et al. (ed.), Lecture Notes in Math. 959, Springer, Berlin-New York, 1982, 433–442.

[Sch] N. Schwartz, *The Basic Theory of Real Closed Spaces*, Regensburger Math. Schriften 15, Regensburg, 1987.

[Sch_1] N. Schwartz, *The Basic Theory of Real Closed Spaces*, Mem. Amer. Math. Soc. 397 (1989).