ON BLOWING UP VERSAL DISCRIMINANTS

PIOTR JAWORSKI

Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
E-mail: jwptxa@mimuw.edu.pl

Abstract. It is well-known that the versal deformations of nonsimple singularities depend on moduli. The first step in deeper understanding of this phenomenon is to determine the versal discriminant, which roughly speaking is an obstacle for analytic triviality of an unfolding or deformation along the moduli.

The goal of this paper is to describe the versal discriminant of $Z_{k,0}$ and $Q_{k,0}$ singularities basing on the fact that the deformations of these singularities may be obtained as blowing ups of certain deformations of $J_{k,0}$ singularities.

1. Introduction. Let

$$f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$$

be a germ of an analytic function with an isolated critical point at the origin. Let

$$F : U \to \mathbb{C}, \qquad (0,0) \in U \subset \mathbb{C}^n \times \Lambda$$

be an analytic deformation of f ($f(x) = F(x,0)_0$), which is miniversal for right equivalence. Obviously $F_\lambda(x) = F(x, \lambda)$ is versal for V-equivalence (contact equivalence).

Let T be the moduli set, i.e. the subset of Λ consisting of such λ that $F_\lambda(x)$ has a critical point p of multiplicity $\mu = \mu(f)$ and $F_\lambda(p) = 0$.

Assume that T is smooth. Let $\pi : (\Lambda, 0) \to (T, 0)$ be an analytic projection (transversal to T). The *versal discriminant* V of the deformation F_λ relative to the projection π is the subset of the fiber $\pi^{-1}(0)$ consisting of such parameters λ that the deformation

$$F_\lambda : \lambda \in \pi^{-1}(0)$$

is not infinitesimally V-versal at λ (i.e. it is not transversal to the strata of V-equivalence).

Now let the germ f be quasihomogeneous (weighted homogeneous). Then there exists a distinguished class of projections induced by quasihomogeneity. Indeed, let v be an associated quasihomogeneous weight. We consider the *quasihomogeneous* miniversal

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deformation:

$$F_\lambda = f + \sum_{i=1}^{\mu} \lambda_i e_i$$

where $e_1, \ldots, e_\mu$ is a quasihomogeneous base of the local algebra $\mathcal{O}_n/I_f$, where I_f is the ideal spanned by partial derivatives $\partial f/\partial x_i$ $i = 1, \ldots, n$ (compare [1] §8). We say that the parameter λ_i is *under-*, *over-*, or *diagonal* if the weight of e_i is less than, greater than or equal to the weight of f. In this case the moduli set T is a linear subspace of the base $\Lambda = \mathbb{C}^\mu$ spanned by over- and diagonal λ's. Moreover, there is a canonical projection π onto T—"forgetting" the underdiagonal λ's. For such a projection the restriction F_λ, $\lambda \in \pi^{-1}(0)$ is a part of the deformation consisting of underdiagonal terms, so called underdiagonal deformation (called also the deformation of negative weight). Since the quasihomogeneous germs are germs of polynomials, F_λ is defined globally and we may put for example $U = \mathbb{C}^n \times \mathbb{C}^\mu$. Nevertheless since the versal discriminant depends on the choice of the domain where the deformation is defined, we restrict ourselves to the domains U such that:

  • U splits; $U = \{(x, \lambda_1, \lambda') : \lambda_1 \in \mathbb{C}, (x, \lambda') \in U'\}$,
  • U is a neighborhood of $\mathbb{C}^n \times \pi^{-1}(0)$,
  • the factor algebra of analytic functions on U modulo the ideal generated by partial derivatives $\partial F_\lambda/\partial x_i$, $i = 1, \ldots, n$ is a free module over the ring of analytic functions on $\Pi(U)$ (the projection of U on Λ) generated by introduced above polynomials $e_1, \ldots, e_\mu$, i.e.

$$\mathcal{O}(U)/I_{F_\lambda} = \mathcal{O}(\Pi(U)) \otimes_{\mathbb{C}} \mathcal{O}_n/I_f.$$

In this paper we apply the results concerning the $J_{k,0}$ singularities (see [7]) and describe the versal discriminants, relative to the canonical projections onto the moduli sets, of quasihomogeneous miniversal deformations of $Z_{k,0}$ and $Q_{k,0}$ singularities, restricted to the domains U as above.

We consider the deformations of $J_{k,0}$ singularities with a critical point (resp. degenerated critical point) at the origin. Next we blow up the origin. The strict transformation of such a family of functions shows to be a deformation of $Z_{k,0}$ (resp. $Q_{k,0}$). Moreover we obtain the whole miniversal deformations in such a way. Furthermore the versal discriminant is mapped to the versal discriminant.

We would like to mention here that the notions similar to the versal discriminant (*versality discriminant, instability locus, . . .*) were investigated by many authors. The reader is referred for example to works of J. Damon or A. du Plessis and C. T. C. Wall—[2, 3, 4, 9, 10] or to the other papers on topological triviality—[8, 11].

2. Notation. We recall the basic definitions. Let $F(x, \lambda)$ be an analytic deformation of $f(x)$ defined on a domain U. An analytic vector field η defined on $\Pi(U)$ is called *liftable* if there exists an analytic vector field ξ defined on U such that

$$\xi = \sum \xi_i(x, \lambda)\frac{\partial}{\partial x_i} + \eta$$

and $\xi(F) = AF$, where $A(x, \lambda)$ is analytic. We call such a vector field ξ *a lifting* of η. Obviously the lifting is tangent to the zero set of F.

We shall denote by $\mathcal{M}$ the $\mathcal{O}(\Pi(U))$ module of analytic liftable vector fields and by $\mathcal{M}(\lambda)$ the $\mathbb{C}$-linear space of their values at the point λ.

We recall that the tangent space to a V-equivalence stratum is spanned by liftable vector fields. Therefore we apply the following definition (compare the notion of a versality discriminant [2, 3, 4] and an instability locus [9]):

A point $\lambda \in \pi^{-1}(0)$ *belongs to the versal discriminant relative to the projection π if and only if* the module of analytic liftable vector fields $\mathcal{M}$ is not transversal to the fiber $\pi^{-1}(0)$ at λ

$$\mathcal{M}(\lambda) \oplus T_\lambda \pi^{-1}(0) \neq T_\lambda \Lambda.$$

3. The main results. In this paper we consider the quasihomogeneous analytic functions

$$f(x,y) = xy^3 + \beta y x^{2k-1} + \gamma x^{3k-2}, \quad k = 2, 3, \ldots,$$

and

$$g(x,y,z) = xz^2 + y^3 + \beta y x^{2k-2} + \gamma x^{3k-3} \quad k = 2, 3, \ldots,$$

where $4\beta^3 + 27\gamma^2 \neq 0$.

They have an isolated singular point at the origin. In Arnold's classification (see [1] §15) such singularities are called respectively $Z_{k-2,0}$ and $Q_{k-1,0}$ for $k > 2$ and X_9 and Q_8 for $k = 2$. To simplify statements we put $X_9 = Z_{0,0}$ and $Q_8 = Q_{1,0}$. We remark that the above singularities are classified by j-invariant:

$$j = \frac{4\beta^3}{4\beta^3 + 27\gamma^2},$$

the same which classifies the elliptic curves

$$z^2 = (y + \alpha x)^3 + \beta(y + \alpha x)x^2 + \gamma x^3$$

(compare [5] §IV.4), therefore they can be described by normal forms which depend only on one diagonal modulus.

Our aim is to describe the versal discriminant of these singularities relative to the projection π onto over- and diagonals;

$$\pi(\lambda_{\text{underdiagonal}}, \lambda_{\text{diagonal}}, \lambda_{\text{overdiagonal}}) = (\lambda_{\text{diagonal}}, \lambda_{\text{overdiagonal}}).$$

It shows that the versal discriminants do not depend on the choice of the particular quasihomogeneous deformation. Furthermore, with the exception when $k \geq 3$ and the singularity is harmonic or aharmonic ($j = 0, 1$), the *topological type* of the versal discriminant does not depend on the j-invariant.

Let $F(x, y, \lambda)$ and $G(x, y, z, \lambda)$ be miniversal quasihomogeneous deformations of f and g.

THEOREM 1. *The versal discriminant of $Z_{k-2,0}$, $k = 2, 3, \ldots$, consists of all parameters λ such that, after a substitution $y = y - a(x)$, F_λ is one of the following polynomials*:

$$
\begin{aligned}
&\text{A:}\quad xy^3 + ex^{2m-1}d(x)^2 y + x^{3m-2}d(x)^3, \quad &m = 1, 2, \ldots, k, \\
&\text{B:}\quad xy^3 + x^{m-1}b(x)y, \quad &m = 1, 2, \ldots, 2k, \\
&\text{C:}\quad xy^3 + x^{m-2}c(x), \quad &m = 2, 3, \ldots, 3k,
\end{aligned}
$$

where $27 + 4e^3 \neq 0$; $b(x)$, $c(x)$, $d(x)$ are polynomials of degree respectively $2k - m$, $3k - m$, $k - m$ with no more than $k - 2$ different nonzero roots; $a(x)$ is any polynomial of degree $k - 1$ or less.

THEOREM 2. *The versal discriminant of $Q_{k-1,0}$, $k = 2, 3, \ldots$, consists of all parameters λ such that, after a substitution $y = y - a(x)$, G_λ is one of the following polynomials:*

$$
\begin{array}{lll}
\text{A:} & xz^2 + y^3 + ex^{2m-2}d(x)^2 y + x^{3m-3}d(x)^3, & m = 1, 2, \ldots, k, \\
\text{B:} & xz^2 + y^3 + x^{m-2}b(x)y, & m = 2, 3 \ldots, 2k, \\
\text{C:} & xz^2 + y^3 + x^{m-3}c(x), & m = 3, 4, \ldots, 3k, \\
\text{D:} & xz^2 + 2az + y^3 + c_1(x), & a \neq 0,
\end{array}
$$

where $27 + 4e^3 \neq 0$; $b(x)$, $c(x)$, $d(x)$, $xc_1(x) - a^2$ are polynomials of degree respectively $2k - m$, $3k - m$, $k - m$, $3k - 2$ with no more than $k - 2$ different nonzero roots; $a(x)$ is any polynomial of degree $k - 1$ or less.

R e m a r k 1. The cases A, B, C, D from the above theorems occur for the quasihomogeneous singularities with j-invariant equal respectively to $27/(27 + 4e^3)$, 1, 0, 0.

4. Blowing down the deformations. The goal of this section is to describe the link between deformations of $J_{k,0}$ singularities and $Z_{k-2,0}$ and $Q_{k-1,0}$. In order to limit the number of formulas we introduce two parameters $\hat{\gamma}$ and $\hat{\beta}$:

$$
(\hat{\gamma}, \hat{\beta}) = \begin{cases} (0,1) & \text{if } \gamma = 0 \\ (1,0) & \text{if } \gamma \neq 0. \end{cases}
$$

4.1. $Z_{k,0}$ case. We write down the miniversal deformation of $Z_{k-2,0}$ singularity

$$
f(x,y) = xy^3 + \beta y x^{2k-1} + \gamma x^{3k-2}, \quad 4\beta^3 + 27\gamma^2 \neq 0, \quad k = 2, 3, \ldots,
$$

in a form

$$
F(x,y,\lambda) = f(x,y) + \lambda_{2,0}y^2 + \sum_{i=0}^{3k-3} \lambda_{0,i}x^i + \sum_{i=0}^{2k-2} \lambda_{1,i}yx^i + \sum_{i=0}^{k-2} \lambda_{e,i}(\hat{\gamma}y + \hat{\beta}x^{k-1})x^{2k+i-1}
$$

where $(x,y,\lambda) \in \mathbb{C}^2 \times \mathbb{C}^{6k-3}$ (compare [1] §8.2 and §8.3).

Next we blow down the line $x = 0$. We put

$$
h(x,y) = x^2 f(x, y/x), \qquad F_1(x,y,\lambda) = x^2 F(x, y/x, \lambda).
$$

h has a $J_{k,0}$ singularity at the origin and F_1 is its deformation. Indeed:

$$
h(x,y) = y^3 + \beta y x^{2k} + \gamma x^{3k},
$$

$$
F_1(x,y,\lambda) = h(x,y) + \lambda_{2,0}y^2 + \sum_{i=2}^{3k-1} \lambda_{0,i-2}x^i + \sum_{i=1}^{2k-1} \lambda_{1,i-1}yx^i + \sum_{i=0}^{k-2} \lambda_{e,i}(\hat{\gamma}y + \hat{\beta}x^k)x^{2k+i}
$$

where $(x,y,\lambda) \in \mathbb{C}^2 \times \mathbb{C}^{6k-3}$. We remark that all functions $F_1(\cdot, \cdot, \lambda)$ have a critical point at the origin and the corresponding critical value is 0.

F_1 as every other deformation is equivalent to the one induced from the miniversal. If we select the miniversal deformation of h in a form

$$
H(x,y,\tau) = h(x,y) + \sum_{i=0}^{3k-2} \tau_{0,i}x^i + \sum_{i=0}^{2k-2} \tau_{1,i}yx^i + \sum_{i=0}^{k-1} \tau_{e,i}(\hat{\gamma}y + \hat{\beta}x^k)x^{2k+i-1}
$$

where $(x, y, \tau) \in \mathbb{C}^2 \times \mathbb{C}^{6k-2}$, then F_1 is induced by a shift

$$x = x' - x_0, \quad y = y' - y_0,$$

$$x_0 = \begin{cases} \dfrac{-\lambda_{0,3k-3}}{3k\gamma} & \text{if } \gamma \neq 0 \\[2mm] \dfrac{-\lambda_{1,2k-2}}{2k\beta} & \text{if } \gamma = 0, \end{cases}$$

$$y_0 = \frac{1}{3}\lambda_{2,0},$$

i.e.

$$F_1(x, y, \lambda) = H(x + x_0, y + y_0, \varphi(\lambda)).$$

We remark that φ is a polynomial mapping whose image coincides with the set of parameters τ such that the curve $H_\tau = 0$ has a singular point. Since its domain is smooth and it is injective in regular points, φ is a normalization of its image.

4.2. $Q_{k,0}$ case. We write down the miniversal deformation of $Q_{k-1,0}$ singularity

$$g(x, y) = xz^2 + y^3 + \beta yx^{2k-2} + \gamma x^{3k-3}, \quad 4\beta^3 + 27\gamma^2 \neq 0, \quad k = 2, 3, \ldots,$$

in a form

$$G(x, y, \lambda) = g(x, y, z) + \lambda_{1,1,0}yz + \lambda_{1,0,0}z$$
$$+ \sum_{i=0}^{3k-4} \lambda_{0,i}x^i + \sum_{i=0}^{2k-3} \lambda_{1,i}yx^i + \sum_{i=0}^{k-2} \lambda_{e,i}(\hat{\gamma}y + \hat{\beta}x^{k-1})x^{2k+i-2}$$

where $(x, y, z, \lambda) \in \mathbb{C}^3 \times \mathbb{C}^{6k-4}$.

First we blow down lines $x = 0$, $y = $ const. The transform of g is $f + z^2$, indeed

$$xg(x, y, zx^{-1}) = z^2 + xy^3 + \beta yx^{2k-1} + \gamma x^{3k-2}.$$

We put

$$G_1(x, y, z, \lambda) = xG(x, y, zx^{-1}, \lambda) = z^2 + f(x, y) + \lambda_{1,1,0}yz + \lambda_{1,0,0}z$$
$$+ \sum_{i=1}^{3k-3} \lambda_{0,i-1}x^i + \sum_{i=1}^{2k-2} \lambda_{1,i-1}yx^i + \sum_{i=0}^{k-2} \lambda_{e,i}(\hat{\gamma}y + \hat{\beta}x^{k-1})x^{2k+i-1}$$

where $(x, y, z, \lambda) \in \mathbb{C}^3 \times \mathbb{C}^{6k-4}$. G_1 is induced from $z^2 + F$ by a shift

$$z = z' - \frac{1}{2}(\lambda_{1,1,0}y + \lambda_{1,0,0}x).$$

Thus

$$G_1(x, y, z, \lambda) = F\big(x, y, \varphi_1(\lambda)\big) + \Big(z + \frac{1}{2}(\lambda_{1,1,0}y + \lambda_{1,0,0}x)\Big)^2.$$

The image of φ_1 consists of such parameters λ that the curve $F_\lambda = 0$ is tangent to the axis $x = 0$ or contains it. Furthermore φ_1 is a branched double covering.

Next we blow down the plane $x = 0$. Analogically as in the previous subsection the transform of $f + z^2$ is $h + z^2$ and of $F + z^2$ is $H + z^2$. Moreover the image of $\varphi \circ \varphi_1$ consists of such parameters τ that the curve $H_\tau = 0$ has a degenerated singular point.

5. Transformations of liftable vector fields. Let Φ be an analytic fibered morphism

$$\Phi : X \times \Lambda \longrightarrow Y \times \Omega, \quad \Phi(x, \lambda) = (\overline{\Phi}(x, \lambda), \varphi(\lambda)).$$

We shall denote the differential of Φ in the following way

$$D\Phi = \begin{pmatrix} D_x\overline{\Phi} & D_\lambda\overline{\Phi} \\ 0 & D_\lambda\varphi \end{pmatrix}.$$

Furthermore, let U be an open subset of $Y \times \Omega$, $F : U \longrightarrow \mathbb{C}$ an analytic function, η a liftable (with respect to F) vector field on $\Pi(U)$, and ξ its lifting.

$$\xi(y, \omega) = \overline{\xi}(y, \omega) + \eta(\omega) = \sum \overline{\xi}_i(y, \omega)\frac{\partial}{\partial y_i} + \sum \eta_i(\omega)\frac{\partial}{\partial \omega_i},$$

$$\xi(F)(y, \omega) = A(y, \omega)F(y, \omega).$$

We state here several results concerning $D\varphi^{-1}(\eta)$ and $D\Phi^{-1}(\xi)$.

5.1. Stabilization. Assume that $X = Y \times \mathbb{C}$, $\Lambda = \Omega$, $\overline{\Phi}$ is a projection $\overline{\Phi}(x', x_{n+1}) = x'$ and φ is an identity.

PROPOSITION 3. *η is liftable with respect to the function $F_1(x) = F(x') + x_{n+1}^2$ defined on $U \times \mathbb{C}$ and its lifting equals*

$$\xi_1(x, \omega) = \xi(x', \omega) + A(x', \omega)\frac{x_{n+1}}{2}\frac{\partial}{\partial x_{n+1}}.$$

Proof.

$$\xi_1(F_1) = \xi(F) + A\frac{x_{n+1}}{2}\frac{\partial}{\partial x_{n+1}}(x_{n+1}^2) = AF + Ax_{n+1}^2 = AF_1. \quad \blacksquare$$

5.2. Normalization. Assume that $\overline{\Phi}(x, \lambda)$ is a diffeomorphism for a fixed λ.

PROPOSITION 4. *If η_1 is such an analytic vector field that $D_\lambda\varphi(\eta_1) = \eta$ then it is liftable with respect to the function $F_1 = F \circ \Phi$ defined on $U_1 = \Phi^{-1}(U)$ and its lifting equals*

$$\xi_1(x, \lambda) = (D_x\overline{\Phi})^{-1}(\overline{\xi} - D_\lambda\overline{\Phi}(\eta_1)) + \eta_1.$$

Proof. We have

$$D\Phi(\xi_1)(F) = \Big(D_x\overline{\Phi}\big((D_x\overline{\Phi})^{-1}(\overline{\xi} - D_\lambda\overline{\Phi}(\eta_1)) + D_\lambda\overline{\Phi}(\eta_1)\big) + D_\lambda\varphi(\eta_1)\Big)(F)$$

$$= (\overline{\xi} - D_\lambda\overline{\Phi}(\eta_1) + D_\lambda\overline{\Phi}(\eta_1) + \eta)(F) = \xi(F) = AF.$$

Thus

$$\xi_1(F \circ \Phi)(x, \lambda) = D\Phi(\xi_1)(F)(\Phi(x, \lambda)) = A(\Phi(x, \lambda))F_1(x, \lambda). \quad \blacksquare$$

5.3. Blowing up. Assume that φ is an identity and $\overline{\Phi}$ is an affine blowing up which does not depend on ω

$$\Phi(x_1, x_2, \ldots, x_n, \omega) = (x_1, x_2x_1, \ldots, x_mx_1, x_{m+1}, \ldots, x_n, \omega).$$

Let F_1 be a strict transform of F

$$F_1(x, \omega) = x_1^{-p}F \circ \Phi(x, \omega).$$

PROPOSITION 5. *If ξ is tangent to the center of Φ then η is liftable with respect to F_1 and its lifting ξ_1 is tangent to the exceptional divisor.*

P r o o f. Since ξ is tangent to the center $C = \{y_1 = \ldots = y_m = 0\}$ of the blowing up, $(D\Phi)^{-1}(\xi)$ is analytic. Indeed:

$$D\Phi\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial y_1} + \sum_{i=2}^{m} \frac{y_i}{y_1} \frac{\partial}{\partial y_i},$$

$$D\Phi\left(\frac{\partial}{\partial x_i}\right) = y_1 \frac{\partial}{\partial y_i}, \quad i = 2, \ldots, m,$$

$$D\Phi\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i}, \qquad i = m+1, \ldots, n.$$

Therefore

$$(D\Phi)^{-1}\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i}, \qquad i = m+1, \ldots, n,$$

$$(D\Phi)^{-1}\left(\frac{\partial}{\partial y_i}\right) = \frac{1}{x_1} \frac{\partial}{\partial x_i}, \quad i = 2, \ldots, m,$$

$$(D\Phi)^{-1}\left(\frac{\partial}{\partial y_1}\right) = \frac{\partial}{\partial x_1} - \sum_{i=2}^{m} \frac{x_i}{x_1} \frac{\partial}{\partial x_i}.$$

Furthermore, since ξ is tangent to $C = \{y_1 = \ldots = y_m = 0\}$, its coefficients $\overline{\xi_1}, \ldots, \overline{\xi_m}$ vanish at C. Hence

$$\overline{\xi_i}(\Phi(x,\omega)) = x_1 \cdot \hat{\xi}_i(x,\omega), \qquad i = 1, \ldots, m,$$

where $\hat{\xi}_i$ are analytic. We put

$$\xi_1 = (D\Phi)^{-1}(\xi) = x_1 \hat{\xi}_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^{m} (\hat{\xi}_i - x_i \hat{\xi}_1) \frac{\partial}{\partial x_i} + \sum_{i=m+1}^{n} \overline{\xi_i} \circ \Phi \frac{\partial}{\partial x_i} + \eta.$$

The first coefficient vanishes at the exceptional divisor $E = \{x_1 = 0\}$ hence ξ_1 is tangent to E. On the other hand

$$\xi_1(x_1^p F_1)(x,\omega) = \xi_1(F \circ \Phi)(x,\omega) = \xi(F)(\Phi(x,\omega)) = A(\Phi(x,\omega)) \cdot x_1^p F_1(x,\omega),$$

and

$$\xi_1(x_1^p F_1) = p x_1^{p-1} F_1 \xi_1(x_1) + x_1^p \xi_1(F_1) = p x_1^p \hat{\xi}_1 F_1 + x_1^p \xi_1(F_1).$$

Hence

$$\xi_1(F_1) = (A \circ \Phi - p\hat{\xi}_1) F_1. \quad \blacksquare$$

6. Blowing up a versal discriminant. In this section we will show that the versal discriminant of $Z_{k,0}$ (resp. $Q_k, 0$) is contained in the preimage of the versal discriminant of $J_{k,0}$.

Let $H(x,y,\tau)$ be a deformation of $J_{k,0}$ singularity introduced in Subsection 4.1. We remark that due to Proposition 3 every $H(x,y,\tau)$-liftable vector field is also $(z^2 + H(x,y,\tau))$-liftable. Thus both deformations have the same versal discriminant.

We denote by Σ_1 and by Σ_2 subsets of the domain U consisting of such elements (x,y,τ) that

$$H = 0, \quad \nabla H = 0$$

and

$$H = 0, \quad \nabla H = 0, \quad \det D^2 H = 0$$

and by Δ_1 resp. Δ_2 their projections, i.e. sets of such parameters τ that the curve $H_\tau = 0$ has a singular or resp. degenerated singular point. We remark that every liftable vector field is tangent to both Δ_1 and Δ_2 and its lifting is tangent to Σ_1 and Σ_2.

6.1. $Z_{k,0}$ case. Since Σ_1 is described by equations

$$0 = \tau_{0,0} + \sum_{i=1}^{3k-2} \tau_{0,i} x^i + \sum_{i=0}^{2k-2} \tau_{1,i} y x^i + \sum_{i=0}^{k-1} \tau_{e,i}(\hat{\gamma} y + \hat{\beta} x^k) x^{2k+i-1} + h(x,y),$$

$$0 = \tau_{0,1} + \sum_{i=2}^{3k-2} i\tau_{0,i} x^{i-1} + \sum_{i=1}^{2k-2} i\tau_{1,i} y x^{i-1}$$

$$+ \sum_{i=0}^{k-1} \tau_{e,i}\big((2k+i-1)\hat{\gamma} y + (3k+i-1)\hat{\beta} x^k\big) x^{2k+i-2} + \partial_x h(x,y),$$

$$0 = \tau_{1,0} + \sum_{i=1}^{2k-2} \tau_{1,i} x^i + \partial_y h(x,y),$$

it is a manifold. Furthermore over a generic point of Δ_1 lies just one point of Σ_1 hence the projection of Σ_1 onto Δ_1 is a normalization. The mapping φ introduced in Subsection 4.1 is a normalization as well, therefore, up to the diffeomorphism of normalizations, for any liftable vector field η we may put $(D\varphi)^{-1}(\eta) = \xi_{|\Sigma_1}$. Next, applying Propositions 4 and 5 we show that every H-liftable vector field corresponds to the F-liftable one. Furthermore the transformation described in 4.1 maps an underdiagonal deformation to an underdiagonal hence the image of a versal discriminant is contained in a versal discriminant.

6.2. $Q_{k,0}$ case. Since the domain of φ_1 is smooth and the codimension of the set of its critical points is 2, $(D\varphi_1)^{-1}$ of any analytic vector field tangent to the image of φ_1 is analytic. This is our case since for every H-liftable vector field η, $(D\varphi)^{-1}(\eta)$ is tangent to the preimage $(\varphi)^{-1}(\Delta_2)$ which coincides with the image of φ_1.

Next, applying Propositions 4 and 5 we show that every H-liftable vector field corresponds to the G-liftable one. Furthermore the transformation described in Subsection 4.2 maps an underdiagonal deformation to an underdiagonal hence the image of a versal discriminant is contained in a versal discriminant.

6.3. Conclusions. Let F, G, H be miniversal deformations of respectively $J_{k,0}$, $Z_{k,0}$ and $Q_{k-1,0}$ singularity as in Section 4. As is shown in [7], the versal discriminant of $J_{k,0}$ singularity consists of such parameters λ that, after a substitution $y = y - a(x)$, H_λ is one of the following polynomials:

$$\text{A:} \quad y^3 + ed(x)^2 y + d(x)^3,$$
$$\text{B:} \quad y^3 + b(x)y,$$
$$\text{C:} \quad y^3 + c(x),$$

where $27 + 4e^3 \neq 0$; $b(x)$, $c(x)$, $d(x)$ are polynomials of degree respectively $2k$, $3k$, k with no more than $k-1$ different roots; $a(x)$ is any polynomial of degree k or less.

In [6] the reader can find the classification of singular points of zero sets of the above polynomials.

Shifting the roots of respectively $d(x)$, $b(x)$ or $c(x)$ to zero and applying to the above polynomials A, B and C transformations described in Section 4 we obtain

COROLLARY 6. *The versal discriminant of $Z_{k,0}$ is contained in a set consisting of such parameters λ that, after a substitution $y = y - a(x)$, F_λ is one of the following polynomials*:

$$\text{A: } xy^3 + ex^{2m-1}d(x)^2y + x^{3m-2}d(x)^3,$$
$$\text{B: } xy^3 + x^{m-1}b(x)y,$$
$$\text{C: } xy^3 + x^{m-2}c(x),$$

where $27 + 4e^3 \neq 0$; $b(x)$, $c(x)$, $d(x)$ are polynomials of degree respectively $2k-m$, $3k-m$, $k-m$ with no more than $k-2$ different nonzero roots; $a(x)$ is any polynomial of degree $k-1$ or less.

COROLLARY 7. *The versal discriminant of $Q_{k,0}$ is contained in a set consisting of such parameters λ that, after a substitution $y = y - a(x)$, G_λ is one of the following polynomials*:

$$\text{A: } xz^2 + y^3 + ex^{2m-2}d(x)^2y + x^{3m-3}d(x)^3,$$
$$\text{B: } xz^2 + y^3 + x^{m-2}b(x)y,$$
$$\text{C: } xz^2 + y^3 + x^{m-3}c(x),$$
$$\text{D: } xz^2 + 2az + y^3 + c_1(x), \qquad a \neq 0,$$

where $27 + 4e^3 \neq 0$; $b(x)$, $c(x)$, $d(x)$, $xc_1(x) - a^2$ are polynomials of degree respectively $2k-m$, $3k-m$, $k-m$, $3k-3$ with no more than $k-2$ different nonzero roots; $a(x)$ is any polynomial of degree $k-1$ or less.

We describe the last case D in more details. We start with a polynomial

$$H = z^2 + y^3 + c(x)x^2, \quad c(0) = -a^2 \neq 0.$$

After the substitution $z = z' + ax$ we obtain

$$z^2 + 2azx + y^3 + c_1(x)x^2, \quad c_1(x) = (c(x) + a^2)/x.$$

After weighted blowing up $(x, y, z) \mapsto (x, xy, x^2z)$ we get

$$x^3\big(xz^2 + 2az + y^3 + c_1(x)\big).$$

From the estimate for $J_{k,0}$ singularities we know that $c(x)x^2$ has no more than $k-1$ different roots. Hence $c(x) = xc_1(x) - a^2$ has no more than $k-2$ different nonzero roots.

7. Final estimates. To finish the proof of the theorems it is enough to show that the estimate on the number of different nonzero roots of the polynomials $b(x)$, $c(x)$, $d(x)$ and $4xc_1(x) - 1$ is strict. We apply the same method as in [7].

Let $e_1, \ldots, e_\mu$ be a quasihomogeneous basis of the factor algebra $\mathbb{C}[x,y]/I_{F_\lambda}$ (resp. $\mathbb{C}[x,y]/I_{G_\lambda}$). We remark that a point $\lambda \in \pi^{-1}(0)$ does not belong to the versal discriminant relative to the projection π if and only if the polynomials e_iF_λ (resp. e_iG_λ) and the underdiagonal base elements generate the whole algebra $\mathbb{C}[x,y]/I_{F_\lambda}$ (resp. $\mathbb{C}[x,y]/I_{G_\lambda}$).

7.1. $Z_{k,0}$ case

LEMMA 8. *The point λ belongs to versal discriminant if and only if F_λ is of type A, B or C and respectively the polynomial $xd(x)$, $xb(x)$, or $xc(x)$ has at most $k-1$ different roots.*

P r o o f. In $Z_{k,0}$ case it is enough to consider polynomials F_λ, xF_λ, ..., $x^m F_\lambda$, ..., yF_λ, xyF_λ, ..., $x^m yF_\lambda$, ... and $y^2 F_\lambda$.

We recall the main points of the proof of the $J_{k,0}$ case (see [7]). The polynomial H (of type A,B or C) belongs to the versal discriminant if and only if it fulfils two relations:

- For every $k \geq 0$ there is a polynomial $r_k(x)$ such that the polynomial

$$W_k(x, y) = x^k \big(3H(x, y) - yH_y(x, y)\big) - r_k(x)H_x(x, y)$$

multiplied by x is still underdiagonal, i.e. qdeg $W_k < 3k - 1$.

- There are two relatively prime polynomials $p(x)$ and $q(x)$ such that degree of p is not greater than $k - 2$, degree of q is not greater than $k - 1$, the x-coordinates of singular points of H are roots of q and

$$p(x)(3H(x, y) - yH_y(x, y)) = q(x)H_x.$$

The deformation F is obtained from H by shifting a singular point to the origin and blowing up. Both operations preserve the above relations. Note that the quasidegree of y is decreasing by one and qdeg $f = 3k - 2$.

Substituting

$$q(x) = xq_1(x), \quad H(x, xy) = x^2 F(x, y),$$

$$H_x(x, xy) = 2xF(x, y) + x^2 F_x(x, y) - xyF_y(x, y), \quad H_y(x, xy) = xF_y(x, y),$$

we obtain

$$W_k(x, xy) = x^k \big(3x^2 F(x, y) - x^2 yF_y(x, y)\big)$$
$$- r_k(x)\big(2xF(x, y) + x^2 F_x(x, y) - xyF_y(x, y)\big).$$

The right-hand side can be divided by x. Thus $W_k(x, xy)/x$ is an underdiagonal polynomial (its quasidegree is less than $3k - 2$).

In such a way we show that for $k = 0, 1, \ldots$, $(3x^{k+1} - 2r_k(x))F(x, y)$ is equivalent to an underdiagonal polynomial modulo the gradient ideal. Since the same is true for F itself (consider $(3k - 2)F - (k - 1)yF_y - xF_x$), we get that the class of any $x^k F$ is underdiagonal.

From the second relation we obtain

$$p(x)(3x^2 F(x, y) - x^2 yF_y(x, y)) = xq_1(x)(2xF(x, y) + x^2 F_x(x, y) - xyF_y(x, y)).$$

We divide both sides by x^2 and deduce that

$$(3p(x) - 2q_1(x))F(x, y)$$

is equivalent to 0 in the factor algebra. Since $\deg(3p(x) - 2q_1(x)) < k - 1$, the polynomials yF, xyF, ... and the underdiagonal base elements do not generate the whole factor algebra. Indeed, the dimension of the subspace spanned by yF, xyF, ... is smaller than the number of diagonal and overdiagonal base elements.

To finish the proof we show that $y^2 F$ is also equivalent to underdiagonal polynomial. This follows from the above and the following relations:

A: $\big(3y^2 + ex^{2m-1}d(x)^2\big)F = \big(y^3 + ex^{2m-1}d(x)^2 y + x^{3m-2}d(x)^3\big)F_y,$

B: $\big(y^2 + (x^{m-1}b(x))'\big)F = \big(xy^2 + x^{m-1}b(x)\big)F_x,$

C: $m > 2, \qquad 3y^2 F = \big(y + x^{m-3}c(x)\big)F_y,$

D: $m = 2, \qquad 3y^2 F = \big(y + (c(x) - c(0))/x\big)F_y + c(0)y^2.$ ∎

7.2. $Q_{k,0}$ case

LEMMA 9. *The point λ belongs to the versal discriminant if and only if G_λ is of type A, B, C or D and respectively the polynomial $xd(x)$, $xb(x)$, $xc(x)$ or $x(xc_1(x) - a^2)$ has at most $k - 1$ different roots.*

P r o o f. In $Q_{k,0}$ case we have to consider the following polynomials: F_λ, xF_λ, ..., ..., $x^m F_\lambda$, ..., yF_λ, xyF_λ, ..., $x^m yF_\lambda$, ..., zF_λ and zyF_λ.

Cases A, B, C. We repeat the same arguments as above. The deformation G is obtained from F by stabilization and blowing up. Note that the quasidegree of z is decreasing by one and qdeg $g = 3k - 3$.

We substitute

$$F(x, y) = xG(x, y, z) - \frac{1}{2}zG_z(x, y, z), \quad F_y(x, y) = xG_y(x, y, z),$$

$$F_x(x, y) = G(x, y, z) + xG_x(x, y, z) - zG_z(x, y, z)$$

and obtain

$$W_k(x, xy) = x^k(3x^3 G(x, y, z) - x^3 yG_y(x, y, z) - \frac{3}{2}x^3 zG_z(x, y, z))$$

$$- r_k(x)(3x^2 G(x, y, z) + x^3 G_x(x, y, z) - x^2 yG_y(x, y, z) - 2x^2 zG_z(x, y, z)).$$

The right-hand side can be divided by x^2. Thus $x^{-2}W_k(x, xy)$ is an underdiagonal polynomial (qdeg $< 3k - 3$).

In such a way we show that for $k = 0, 1, \ldots$, $(x^{k+1} - r_k(x))G(x, y, z)$ is equivalent to an underdiagonal polynomial modulo the gradient ideal. Since the same is true for G itself (consider $(3k - 3)G - (k - 1)yG_y - xG_x - (3/2k - 1)zG_z$), we get that the class of any $x^k G$ is underdiagonal.

From the second relation we obtain:

$$p(x)(3x^3 G(x, y, z) - x^3 yG_y(x, y, z) - \frac{3}{2}x^3 zG_z(x, y, z))$$

$$= xq_1(x)(3x^2 G(x, y, z) + x^3 G_x(x, y, z) - x^2 yG_y(x, y, z) - 2x^2 zG_z(x, y, z)).$$

We divide both sides by x^3 and deduce that

$$(p(x) - q_1(x))G(x, y)$$

is equivalent to 0 in the factor algebra. Since $\deg(p(x) - q_1(x)) < k - 1$, the polynomials yG, xyG, ... and the underdiagonal base elements do not generate the whole factor algebra. Indeed, the dimension of the subspace spanned by yG, xyG, ... is smaller than the number of diagonal and overdiagonal base elements.

To finish the proof it is enough to notice that since zx belongs to the gradient ideal, zG and zyG are equivalent to underdiagonal polynomials.

Case D. In this case we have

$$G = xz^2 + 2az + y^3 + c_1(x),$$

$$G_y = 3y^2, \quad G_x = z^2 + c_1'(x), \quad G_z = 2xz + 2a.$$

We put

$$\tilde{G} = G - zG_z - \frac{1}{3}yG_y + xG_x = xc_1'(x) + c_1(x),$$

$$D = x^2 G_x - \frac{1}{2}(xz - a)G_z = x^2 c_1'(x) + a^2.$$

Since $\deg D = 3k - 2$, using D, G_x and G_z we show that for any nonnegative k $x^k\tilde{G}$ is equivalent to an underdiagonal polynomial.

Furthermore let $d_1(x)$ be the greatest common divisor of $xc_1(x) - a^2$ and its derivative $xc_1'(x) + c_1(x)$. We put $xc_1'(x) + c_1(x) = d_0 d_1$ and $xc_1'(x) = d_2 d_1$. Then

$$(d_2 - xd_0)\tilde{G} = (d_2 - xd_0)d_0 d_1 = d_0 D_1.$$

Hence if the degree of $(d_2 - xd_0)$ is smaller than $k - 1$ then the polynomials $y\tilde{G}$, $xy\tilde{G}$, ... and the underdiagonal base elements do not generate the whole factor algebra.

To finish the proof it is enough to notice that z modulo gradient ideal is equivalent to the polynomial $xc_1'(x)/a$. Thus zG and zyG belong in the factor algebra to subspaces spanned by $x^k\tilde{G}$ and $yx^k\tilde{G}$. ∎

References

[1] V. I. A r n o l′d, S. M. G u s e ĭ n - Z a d e and A. N. V a r c h e n k o, *Singularities of Differentiable Maps*, vol. 1, Birkhäuser, Boston, 1985.

[2] J. D a m o n, *On the Pham example and the universal topological stratification of singularities*, in: Singularities, Banach Center Publ. 20, PWN–Polish Scientific Publishers, Warszawa, 1988, 161–167.

[3] J. D a m o n, *A-equivalence and the equivalence of sections of images and discriminants*, in: Singularity Theory and its Applications, Part 1 (Coventry 1988/1989), Lecture Notes in Math. 1492, Springer, Berlin, 1991, 93–121.

[4] J. D a m o n, A. G a l l i g o, *Universal topological stratification for the Pham example*, Bull. Soc. Math. France 121 (1993), 153–181.

[5] R. H a r t s h o r n e, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer, New York, 1977.

[6] P. J a w o r s k i, *Decompositions of hypersurface singularities of type $J_{k,0}$*, Ann. Polon. Math. 59 (1994), 117–131.

[7] P. J a w o r s k i, *On the versal discriminant of the $J_{k,0}$ singularities*, Ann. Polon. Math. 63 (1996), 89–99.

[8] E. L o o i j e n g a, *Semi-universal deformation of a simple elliptic hypersurface singularity, I: Unimodularity*, Topology 16 (1977), 257–262.

[9] A. d u P l e s s i s, C. T. C. W a l l, *Topological stability*, in: Singularities (Lille, 1991), London Math. Soc. Lecture Note Ser. 201, Cambridge Univ. Press, Cambridge, 1994, 351–362.

[10] A. d u P l e s s i s, C. T. C. W a l l, *The Geometry of Topological Stability*, London Math. Soc. Monogr. (N.S.) 9, Oxford Sci. Publ., Oxford Univ. Press, New York, 1995.

[11] K. W i r t h m ü l l e r, *Universell topologische triviale Deformationen*, Ph.D. thesis, University of Regensburg, 1979.