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GROWTH AT INFINITY OF A POLYNOMIAL WITH A COMPACT ZERO SET

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1. Introduction. In the article we give the explicit bound for the growth at infinity of a polynomial with a compact set of zeros. Our aim is to prove the following theorem:

THEOREM 1. *Let $F \in \mathbf{R}[X_1, \ldots, X_n]$ be a polynomial of degree $d > 2$ such that the set $F^{-1}(0)$ is compact. Then there exist constants c, $R > 0$ such that*

$$|F(x)| \geq c|x|^{d-(d-1)^n} \qquad \textit{for all} \quad |x| > R.$$

Recall that we have a similar estimation in the complex case. Consider a polynomial map $H : \mathbf{C}^n \to \mathbf{C}^n$ of degree d such that $H^{-1}(0)$ is finite. Then, by Kollár's theorem, $|H(z)| \geq \text{const.}|z|^{d-d^n}$ for $|z| \gg 1$ (see [Ko]). Our theorem is a real counterpart of this inequality.

2. Two lemmas. The following lemmas will be used in the proof of the main theorem.

LEMMA 1. *Let $G : \mathbf{R}^n \to \mathbf{R}$ be a polynomial of positive degree d. Then there exists a linear automorphism $L : \mathbf{R}^n \to \mathbf{R}^n$ such that the polynomial $F = G \circ L$ satisfies the following conditions*:

(i) *All partial derivatives of F are of degree $d - 1$.*

(ii) *The sets $\Gamma_i = \{x \in \mathbf{R}^n \mid \partial F/\partial X_1(x) = \ldots = \partial F/\partial X_{i-1}(x) = \partial F/\partial X_{i+1}(x) = \ldots = \partial F/\partial X_n(x) = 0,\ \partial F/\partial X_i(x) \neq 0\}$ $(1 \leq i \leq n)$ are one-dimensional submanifolds of $\mathbf{R}^n$ whenever they are not empty*,

(iii) *For every $x \in \Gamma_i$ $(1 \leq i \leq n)$ the differentials $\mathrm{d}_x(\partial F/\partial X_1), \ldots, \mathrm{d}_x(\partial F/\partial X_{i-1})$, $\mathrm{d}_x(\partial F/\partial X_{i+1}), \ldots, \mathrm{d}_x(\partial F/\partial X_n)$ are linearly independent.*

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Proof. Let $GL(n)$ be the set of linear automorphisms of $\mathbf{R}^n$. We claim that

$$\{L \in GL(n) \mid \deg \frac{\partial G \circ L}{\partial X_1} = \ldots = \deg \frac{\partial G \circ L}{\partial X_n} = d-1\}$$

is a dense subset of $GL(n)$. Let $G_d(X_1, \ldots, X_n)$ be the leading form of the polynomial G that is the homogeneous polynomial of degree d for which $\deg(G - G_d) < d$. Consider a substitution $G_d \circ L$ where $L(X_1, \ldots, X_n) = (\sum_{i=1}^n l_1^i X_i, \ldots, \sum_{i=1}^n l_n^i X_i)$. We have $(G_d \circ L)(X_1, \ldots, X_n) = G_d(\sum_{i=1}^n l_1^i X_i, \ldots, \sum_{i=1}^n l_n^i X_i) = G_d(l_1^1, \ldots, l_n^1) X_1^d + \ldots \ldots + G_d(l_1^n, \ldots, l_n^n) X_n^d +$ other monomials. If $G_d(l_1^1, \ldots, l_n^1) \neq 0, \ldots, G_d(l_1^n, \ldots, l_n^n) \neq 0$, then all partial derivatives of $G \circ L$ are of degree $d-1$. Since the set $\{L \in GL(n) \mid G_d(l_1^1, \ldots, l_n^1) \neq 0, \ldots, G_d(l_1^n, \ldots, l_n^n) \neq 0\}$ is a complement of a proper algebraic set, it is open and dense in $GL(n)$. This proves the claim.

For any $x = (x_1, \ldots, x_n)$ from $\mathbf{R}^n \setminus \{0\}$ we denote by $[x]$ the corresponding point $[x_1, \ldots, x_n]$ of the projective space $\mathbf{R}P^{n-1}$. Consider the map

$$[\operatorname{grad} G] : \mathbf{R}^n \setminus (\operatorname{grad} G)^{-1}(0) \to \mathbf{R}P^{n-1}.$$

From the semialgebraic version of Sard's lemma (see [BR], page 82) it follows that the set of regular values of this map contains an open subset $U \subset \mathbf{R}P^{n-1}$. The set $V = \{(v^1, \ldots, v^n) \in \mathbf{R}^n \times \ldots \times \mathbf{R}^n \mid \det(v_i^j) \neq 0,\ [v^i] \in U \text{ for } i = 1, \ldots, n\}$ is an open subset of $\mathbf{R}^n \times \ldots \times \mathbf{R}^n$. Each n-tuple $v = (v^1, \ldots, v^n)$ from this set yields a linear automorphism $A_v : \mathbf{R}^n \to \mathbf{R}^n$, $A_v(x) = (\langle v^1, x\rangle, \ldots, \langle v^n, x\rangle)$. Hence the set $\{A_v \in GL(n) \mid v \in V\}$ is open in $GL(n)$. Since $GL(n) \ni A \to A^{-1} \in GL(n)$ is an open map, $\{A_v^{-1} \in GL(n) \mid v \in V\}$ is also an open subset of $GL(n)$. Thus, there exists $v = (v^1, \ldots, v^n) \in V$ such that the automorphism $L = A_v^{-1}$ satisfies (i).

Let us define the polynomial $F = G \circ L$. Since $G = F \circ A_v$, $\operatorname{grad} G = A_v^T \circ \operatorname{grad} F \circ A_v$, where A_v^T is the adjoint of A_v. From this equation it follows that for any $w \in \mathbf{R}^n \setminus \{0\}$, $[w]$ is a regular value of $[\operatorname{grad} F]$ if and only if $[A_v^T(w)]$ is a regular value of $[\operatorname{grad} G]$. Let $e^1 = (1, \ldots, 0), \ldots, e^n = (0, \ldots, 1)$ form the standard basis of $\mathbf{R}^n$. Since $A_v^T(e^i) = v^i$ for $i = 1, \ldots, n$, we conclude that $[e^1], \ldots, [e^n]$ are regular values of $[\operatorname{grad} F]$. Applying the implicit function theorem to the map $[\operatorname{grad} F]$ we see that each of the sets $\Gamma_i = [\operatorname{grad} F]^{-1}([e^i])$ $(1 \le i \le n)$ is either a one-dimensional submanifold of $\mathbf{R}^n$ or is empty. This proves (ii).

We prove the third part of the lemma for Γ_n. To simplify the notation we write $\partial_i F$ for $\partial F/\partial X_i$. Because $[e^n]$ is a regular value of $[\operatorname{grad} F]$, $0 \in \mathbf{R}^{n-1}$ is a regular value of the map $\psi : \mathbf{R}^n \setminus (\partial_n F)^{-1}(0) \to \mathbf{R}^{n-1}$, $\psi = (\partial_1 F/\partial_n F, \ldots, \partial_{n-1} F/\partial_n F)$ that is, the map $[\operatorname{grad} F]$ written in the coordinates $\{[x_1, \ldots, x_n] \in \mathbf{R}P^{n-1} \mid x_n \neq 0\} \ni [x_1, \ldots, x_n] \to (x_1/x_n, \ldots, x_{n-1}/x_n) \in \mathbf{R}^{n-1}$. Therefore, for every $x \in \Gamma_n$ the differentials $\mathrm{d}_x(\partial_1 F/\partial_n F), \ldots, \mathrm{d}_x(\partial_{n-1} F/\partial_n F)$ are linearly independent. On the other hand, for $x \in \Gamma_n$ and $i = 1, \ldots, n-1$ we have $\mathrm{d}_x(\partial_i F/\partial_n F) = (1/\partial_n F)\mathrm{d}_x(\partial_i F)$, therefore the differentials $\mathrm{d}_x(\partial_1 F), \ldots, \mathrm{d}_x(\partial_{n-1} F)$ are also linearly independent. The proof for Γ_i, $i \neq n$ is similar. ∎

Further, we denote by $|x|$ the supremum norm $|x| = \max\{|x_1|, \ldots, |x_n|\}$ for $x = (x_1, \ldots, x_n)$. We will also use the following convention: *Using notation* $|x| \gg 1$ *we mean that the corresponding condition is satisfied for* $|x| > R$, *where* R *is sufficiently large.*

Lemma 2. *Let* $F \in \mathbf{R}[X_1, \ldots, X_n]$ *be a polynomial with a compact set of zeros and let* $K = \{x \in \mathbf{R}^n \mid \forall y \in \mathbf{R}^n \quad |y| = |x| \Rightarrow |F(y)| \ge |F(x)|\}$. *If* $A \subset K$ *is an unbounded semialgebraic set, then the following conditions are equivalent*:

(i) $|F(x)| \geq c|x|^{\alpha}$ *for* $|x| \gg 1$,
(ii) $|F(x)| \geq c|x|^{\alpha}$ *for* $|x| \gg 1$, $x \in A$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious. Assume that (ii) is true. Since $\{|x| \mid x \in A\}$ is an unbounded semialgebraic subset of $\mathbf{R}_+$, there exists a constant $R > 0$ such that $(R, \infty) \subset \{|x| \mid x \in A\}$. By (ii) we can choose R sufficiently large so that $|F(x)| \geq c|x|^{\alpha}$ for $|x| \geq R$, $x \in A$. Let $y \in \mathbf{R}^n$ be an arbitrary point with $|y| > R$. Then there exists $x \in A$ such that $|x| = |y|$. By (ii) and the definition of K we get $|F(y)| \geq |F(x)| \geq c|x|^{\alpha} = c|y|^{\alpha}$ which ends the proof. ∎

3. Proof of Theorem 1. The proof proceeds by induction on the number of variables. For polynomials in one variable the theorem is obvious. Assume that the theorem holds for polynomials in $n-1$ variables. We shall check that it is true for polynomials in n variables.

We shall perform some reductions:

If the theorem is true for a polynomial F, then it holds also for $F \circ L$, where $L : \mathbf{R}^n \to \mathbf{R}^n$ is a linear automorphism. Therefore, we can assume that F satisfies the conditions (i), (ii) and (iii) of Lemma 1.

The set $F^{-1}(0)$ is bounded. Hence $F(x) \neq 0$ for all $|x| > R$, where R is sufficiently large. Since for $n \geq 2$ the set $\{x \in \mathbf{R}^n \mid |x| > R\}$ is connected, a sign of F restricted to $\{x \in \mathbf{R}^n \mid |x| > R\}$ does not change. Without loss of generality we can assume that $F(x) > 0$ for $|x| > R$.

Let

$$K = \{x \in \mathbf{R}^n \mid \forall y \in \mathbf{R}^n \quad |y| = |x| \Rightarrow |F(y)| \geq |F(x)|\}.$$

First, we prove the theorem under the additional assumption that $K \cap (\operatorname{grad} F)^{-1}(0)$ is unbounded. Let A be an unbounded connected component of this set. Since $\operatorname{grad} F(x) = 0$ for $x \in A$, we conclude that $F|_A = c$ with some $c > 0$ (see [BR], Theorem 2.5.1). By Lemma 2 we get $|F(x)| \geq c|x|^0$ for $|x| \gg 1$ which ends the proof in this case.

Hence we may assume throughout the rest of the proof that $K \cap (\operatorname{grad} F)^{-1}(0)$ is bounded.

Let us define

$$\begin{aligned} A_i &= \{x \in \mathbf{R}^n \mid |x_k| < |x_i| \quad \text{for } k \in \{1, \ldots, n\} \setminus \{i\}\}, \\ B_{i,j} &= \{x \in \mathbf{R}^n \mid x_i = x_j,\ |x_k| \leq |x_i| \quad \text{for } k = 1, \ldots, n\}, \\ C_{i,j} &= \{x \in \mathbf{R}^n \mid x_i = -x_j,\ |x_k| \leq |x_i| \quad \text{for } k = 1, \ldots, n\}. \end{aligned}$$

Since $\mathbf{R}^n = \bigcup A_i \cup \bigcup B_{i,j} \cup \bigcup C_{i,j}$, at least one of the sets $K \cap \bigcup A_i$, $K \cap \bigcup B_{i,j}$, $K \cap \bigcup C_{i,j}$ is unbounded. Let us consider three cases:

Case 1: $K \cap \bigcup B_{i,j}$ is unbounded. Then at least one of the sets $K \cap B_{i,j}$ $(1 \leq i < j \leq n)$ is unbounded. Without loss of generality we can assume that this is the set $K \cap B_{n-1,n}$.

Consider the polynomial $\tilde{F}(X_1, \ldots, X_{n-1}) = F(X_1, \ldots, X_{n-1}, X_{n-1})$ of degree $\tilde{d} \leq d$. By the inductive assumption we have $|\tilde{F}(\tilde{x})| \geq c|\tilde{x}|^{\tilde{d}-(\tilde{d}-1)^{n-1}}$ for $\tilde{x} \in \mathbf{R}^{n-1}$, $|\tilde{x}| \gg 1$.

If we take any $x \in B_{n-1,n}$, $x = (x_1, \ldots, x_{n-1}, x_{n-1})$ and if we set $\tilde{x} = (x_1, \ldots, x_{n-1})$, then $|\tilde{x}| = |x|$ and $\tilde{F}(\tilde{x}) = F(x)$. Hence $|F(x)| \geq c|x|^{\tilde{d}-(\tilde{d}-1)^{n-1}}$ for $|x| \gg 1$, $x \in B_{n-1,n}$. By Lemma 2 and by the inequality $\tilde{d}-(\tilde{d}-1)^{n-1} \geq d-(d-1)^n$ we get $|F(x)| \geq c|x|^{d-(d-1)^n}$ for $|x| \gg 1$.

Case 2: $K \cap \bigcup C_{i,j}$ is unbounded. The proof is analogous.

Case 3: $K \cap \bigcup A_i$ is unbounded. Then at least one of the sets $K \cap A_i$ $(1 \le i \le n)$ is unbounded. Without loss of generality we can assume that this is $K \cap A_n$.

Take $R > 0$ large enough so that $F(x) > 0$ for $|x| > R$ and let $y = (y_1, \ldots, y_n)$ be an arbitrary point in $K \cap A_n$ with $|y| > R$. Consider a function $f(x_1, \ldots, x_{n-1}) = F(x_1, \ldots, x_{n-1}, y_n)$ defined for $|x_i| < |y_n|$ $(1 \le i < n)$. Taking into account two points, $y = (y_1, \ldots, y_{n-1}, y_n)$ and $x = (x_1, \ldots, x_{n-1}, y_n)$, where $|x_i| < |y_n|$ $(1 \le i < n)$, we see that $|x| = |y|$, therefore $F(x) \ge F(y)$. Hence the point $(y_1, \ldots, y_{n-1})$ is a local minimum of f. Thus $\partial F/\partial X_1(y) = \ldots = \partial F/\partial X_{n-1}(y) = 0$.

Summarizing, we see that for all $y \in K \cap A_n$, $|y| \gg 1$ we have $\partial F/\partial X_1(y) = \ldots = \partial F/\partial X_{n-1}(y) = 0$, $\partial F/\partial X_n(y) \ne 0$. Moreover, from Lemma 1 it follows that $K \cap A_n$ is a one-dimensional semialgebraic manifold in a neighborhood of infinity. We want to find a parametrization of a branch at infinity of this set. To that end we employ complex algebraic geometry.

Define $H_1 = \partial F/\partial X_1, \ldots, H_{n-1} = \partial F/\partial X_{n-1}$ and let $C = \{z \in \mathbf{C}^n \mid H_1(z) = \ldots = H_{n-1}(z) = 0\}$. Decompose C to the union of irreducible algebraic components $C = C_1 \cup \ldots \cup C_s$. Treating $\mathbf{R}^n$ as a subset of $\mathbf{C}^n$ we see that $K \cap A_n \cap C$ is unbounded. Hence there exists a component C_i such that $K \cap A_n \cap C_i$ is unbounded. For simplicity put $\Gamma = C_i$.

We will check that $\dim_{\mathbf{C}} \Gamma = 1$. By Lemma 1 there exists $x \in K \cap A_n \cap \Gamma$ for which the differentials $\mathrm{d}_x H_1, \ldots, \mathrm{d}_x H_{n-1}$ are linearly independent. Therefore, $\dim_{\mathbf{C}} \Gamma \le n - \operatorname{rank}(\Gamma, x) \le n - \operatorname{rank}(\mathrm{d}_x H_1, \ldots, \mathrm{d}_x H_{n-1}) = 1$ (see [BR], pages 122–135). Furthermore, Γ is unbounded, so $\dim_{\mathbf{C}} \Gamma = 1$.(1)

Next, we will check that $\deg \Gamma \le (d-1)^{n-1}$. Let us recall an invariant δ of algebraic sets introduced in Łojasiewicz's book ([Ło] pages 419–420): *Let $W = W_1 \cup \ldots \cup W_s$ be a decomposition of an algebraic set W to irreducible components. Then*, *by definition* $\delta(W) = \sum_{i=1}^{s} \deg W_i$. We will use the inequality $\delta(W \cap V) \le \delta(W)\delta(V)$. Applying this property to the set C we see that $\deg \Gamma \le \delta(C) = \delta(\{H_1 = 0\} \cap \ldots \cap \{H_{n-1} = 0\}) \le \prod_{i=1}^{n-1} \delta(\{H_i = 0\}) \le (d-1)^{n-1}$.

Further, we will consider $\mathbf{C}^n$ as a affine part of the projective space $\mathbf{C}P^n$. We will use the natural identification between $(x_1, \ldots, x_n) \in \mathbf{C}^n$ and $[1, x_1, \ldots, x_n] \in \mathbf{C}P^n$. With the use of this identification we can treat K, A_n and Γ as subsets of $\mathbf{C}P^n$.

Since $K \cap A_n \cap \Gamma$ is an unbounded set and $\mathbf{C}P^n$ is compact, there exists a point a in the hyperplane at infinity $\{[x_0, \ldots, x_n] \in \mathbf{C}P^n \mid x_0 = 0\}$ such that $a \in \operatorname{cl}(K \cap A_n \cap \Gamma)$.

The homogeneous coordinates of a can be chosen such that $a = [0, a_1, \ldots, a_{n-1}, 1]$. Indeed, for all $x \in A_n$ we have $|x_i| < |x_n|$ for $1 \le i < n$. Since $a \in \operatorname{cl}(A_n)$, $|a_i| \le |a_n|$ for $1 \le i < n$. Therefore, the last coordinate a_n does not vanish and by homogeneity we can assume that $a_n = 1$.

Let $\bar{\Gamma}$ be the projective closure of the curve Γ. Since $a \in \bar{\Gamma}$, according to [Ło] (pages 173–176) there exists a finite sequence of injective holomorphic parametrizations $\gamma_i : (D, 0) \to (\bar{\Gamma}, a)$ $(1 \le i \le l)$, where $D = \{t \in \mathbf{C} \mid |t| < \delta\}$, such that the curve $\bar{\Gamma}$ is the union $\gamma_1(D) \cup \ldots \cup \gamma_l(D)$ in some neighborhood of a. These parametrizations are of the form

$$\gamma_i(t) = [t^{d_i}, \gamma_{i,1}(t), \ldots, \gamma_{i,n-1}(t), 1].$$

Furthermore, we can assume that the real branches of $\bar{\Gamma}$ are parametrized such that $(t^{d_i}, \gamma_{i,1}(t), \ldots, \gamma_{i,n-1}(t)) \in \mathbf{R}^n$ if and only if $t \in \mathbf{R}$. This can be done by substituting

(1) If $\dim_{\mathbf{C}} \Gamma = 0$, then Γ would consist of one point.

$\gamma_i(\xi_i t)$, where ξ_i is an appropriate d_i-th root of unity and by shrinking δ if necessary (see [Mi] or [Du] for the details).

Let $H = H(X_0, \ldots, X_n)$ be the homogenization of the polynomial $\partial F/\partial X_n$. Recall that it means that H is a homogeneous polynomial of degree $\deg H = \deg \partial F/\partial X_n$ such that $H(1, X_1, \ldots, X_n) = \partial F/\partial X_n(X_1, \ldots, X_n)$. We can calculate the intersection multiplicity of the curve $\bar{\Gamma}$ and the hypersurface $\{H = 0\}$ at a using the formula

$$\iota_a(\bar{\Gamma}, \{H = 0\}) = \sum_{i=1}^{l} \operatorname{ord}_0(H \circ \gamma_i)$$

(see [Sh], pages 190–194). By Bézout's theorem $\iota_a(\bar{\Gamma}, \{H = 0\}) \le (\deg \bar{\Gamma})(\deg H) \le (d-1)^n$. Hence $\operatorname{ord}_0(H \circ \gamma_i) \le (d-1)^n$ for $i = 1, \ldots, l$.

One has $a \in \operatorname{cl}(K \cap A_n \cap \Gamma)$. Hence there exists i $(1 \le i \le l)$ such that $a \in \operatorname{cl}(K \cap A_n \cap \gamma_i(D))$. Since γ_i is a proper map, $0 \in \operatorname{cl}(\gamma^{-1}(K \cap A_n))$. Furthermore, we see by the definition of γ_i that $\gamma^{-1}(K \cap A_n)$ is a semianalytic subset of $\mathbf{R}$. Therefore there exists $\epsilon > 0$ such that $\gamma_i((0, \epsilon)) \subset K \cap A_n$ or $\gamma_i((-\epsilon, 0)) \subset K \cap A_n$ (see [BM] for the definition and basic properties of semianalytic sets). In the rest of the proof we assume the former case (the proof for the case $\gamma_i((-\epsilon, 0)) \subset K \cap A_n$ is similar). We will again treat K, A_n and Γ as subsets of $\mathbf{C}^n$.

Set the following meromorphic map

$$\phi : \{t \in \mathbf{C} \mid 0 < |t| < \epsilon\} \ni t \to (\gamma_{i,1}(t)/t^{d_i}, \ldots, \gamma_{i,n-1}(t)/t^{d_i}, 1/t^{d_i}) \in \mathbf{C}^n.$$

Notice that $\phi(\{t \in \mathbf{C} \mid 0 < |t| < \epsilon\}) \subset \Gamma$ and that $\phi((0, \epsilon))$ is an unbounded semialgebraic subset of $K \cap A_n$.

We estimate the order of $F \circ \phi$ at zero. Either $\operatorname{ord}_0(F \circ \phi) = 0$ or by the equation

$$(F \circ \phi)' = \left(\frac{\partial F}{\partial X_1} \circ \phi\right) \phi_1' + \ldots + \left(\frac{\partial F}{\partial X_n} \circ \phi\right) \phi_n' = \left(\frac{\partial F}{\partial X_n} \circ \phi\right) \phi_n'$$

we have $\operatorname{ord}_0(F \circ \phi) = \operatorname{ord}_0(\partial F/\partial X_n \circ \phi) - d_i$.

On the other hand we have

$$\begin{aligned}\frac{\partial F}{\partial X_n}(\phi(t)) &= H(1, \gamma_{i,1}(t)/t^{d_i}, \ldots, \gamma_{i,n-1}(t)/t^{d_i}, 1/t^{d_i}) \\ &= t^{-d_i \deg H} H(t^{t_i}, \gamma_{i,1}(t), \ldots, \gamma_{i,n-1}(t), 1) = t^{-d_i \deg H} H(\gamma_i(t)).\end{aligned}$$

Since $\deg H = d-1$ and $\operatorname{ord}_0(H \circ \gamma_i) \le (d-1)^n$, we conclude that $\operatorname{ord}_0(\partial F/\partial X_n \circ \phi) \le (d-1)^n - d_i(d-1)$. By this inequality and the preceding equalities we have $\operatorname{ord}_0(F \circ \phi) \le (d-1)^n - d_i d$ or $\operatorname{ord}_0(F \circ \phi) = 0$.

Remark. If $f, g : \{t \in \mathbf{C} \mid 0 < |t| < \epsilon\} \to \mathbf{C}$, $f \ne 0$, $g \ne 0$ are meromorphic functions, then there exist constants $c, \epsilon_1 > 0$ such that $|f(t)| \ge c|g(t)|^{\operatorname{ord}_0 f/\operatorname{ord}_0 g}$ for all $t \in \mathbf{C}$, $0 < |t| < \epsilon_1$.

The proof of this fact is simple and is left to the reader. By the remark and by the fact that $\phi((0, \epsilon)) \subset A_n$ implies $|\phi(t)| = |\phi_n(t)|$ for $t \in (0, \epsilon)$, we obtain an inequality

$$|F(\phi(t))| \ge c|\phi(t)|^{\operatorname{ord}_0(F \circ \phi)/\operatorname{ord}_0(\phi_n)} \quad \text{for} \quad t \in (0, \epsilon_1)$$

with some positive constants c, ϵ_1. By Lemma 2 we have

$$|F(x)| \ge c|x|^{\operatorname{ord}_0(F \circ \phi)/\operatorname{ord}_0(\phi_n)} \quad \text{for} \quad |x| \gg 1.$$

Moreover, the exponent $\operatorname{ord}_0(F\circ\phi)/\operatorname{ord}_0(\phi_n) \geq ((d-1)^n - d_i d)/(-d_i) = d-(d-1)^n/d_i \geq d-(d-1)^n$ or is equal zero and thus

$$|F(x)| \geq c|x|^{d-(d-1)^n} \quad \text{for} \quad |x| \gg 1. \ \blacksquare$$

4. Concluding remarks. In the course of the proof we have found a parametrization ϕ of the set K at infinity such that $|F(x)| \geq c|x|^{\operatorname{ord}_0(F\circ\phi)/\operatorname{ord}_0(\phi_n)}$ for $|x| \gg 1$. By a slight modification of the proof one can show that the number $\operatorname{ord}_0(F\circ\phi)/\operatorname{ord}_0(\phi_n)$ is the Łojasiewicz exponent at infinity for the polynomial F, i.e. the largest exponent α for which the estimate $|F(x)| \geq \text{const.}|x|^\alpha$ is true for $|x| \gg 1$.

We have checked that $\operatorname{ord}_0(F\circ\phi) \leq (d-1)^n - d\operatorname{ord}_0(\phi_n)$ or $\operatorname{ord}_0(F\circ\phi) = 0$. One can also prove the inequality $(d-1)^{n-1} \leq \operatorname{ord}_0(\phi_n) < 0$. As a result, there is only a finite number of fractions which can be the Łojasiewicz exponents for polynomials of fixed number of variables n and of fixed degree d.

So far I have not found a polynomial F for which the Łojasiewicz exponent $L_\infty(F) = d-(d-1)^n$. For example for the polynomial $F(X_1,\ldots,X_n) = (X_2X_1^{m-1}-1)^2 + (X_3-X_2^m)^2+\ldots+(X_n-X_{n-1}^m)^2+X_n^{2m}$ of degree $d=2m$ we have $L_\infty(F) = d-(1/2^{n-1})d^n$. This suggests that Theorem 1 could be essentially sharpened.

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