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# CHAOS IN D0 BRANE DYNAMICS

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**Abstract.** We consider the classical and quantum dynamics of D0 branes within the Yang-Mills approximation. Using a simple ansatz we show that a classical trajectory exhibits a chaotic motion. Chaotic dynamics in $N = 2$ supersymmetric Yang-Mills theory is also discussed.

Recent advances in string theory have led to the discovery of dualities between five known superstring theories. These theories are expected to be obtained by taking various limits of the conjectured eleven-dimensional M-theory [19, 37]. At low energies/large distances M-theory is described by eleven-dimensional supergravity. Banks, Fischler, Shenker and Susskind [3] have proposed that M-theory in the infinite momentum frame is de-

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scribed in terms of a supersymmetric matrix model, the so called M(artix) theory. Moreover, the only dynamical degrees of freedom are Dirichlet zero-branes and the calculation of any physical quantity of M-theory can be reduced to a calculation in the matrix quantum mechanics. A system of $N$ Dirichlet zero-branes is described in terms of nine $N \times N$ Hermitian matrices $X_i, i = 1, ..., 9$ together with their fermionic superpartners. The action can be regarded as ten-dimensional $SU(N)$ supersymmetric Yang-Mills theory reduced to $(0+1)$ space-time dimensions:

$$S = \int dt Tr\left(\frac{1}{2}D_t X_i D_t X_i + \frac{1}{4}[X_i, X_j][X_i, X_j]\right) + (fermions), \tag{1}$$

where $D_t = \partial_t + iA_0$. The action (1) was considered in the theory of eleven-dimensional supermembranes in [11, 12, 15] and in the dynamics of D-particles in [10, 22]. In the original formulation [3] of the conjectured correspondence between M-theory and M(atrix) theory the large $N$ limit was assumed. A more recent formulation [35] is valid for finite $N$. The M(atrix) theory is interesting because it could provide a non-perturbative approach to quantum gravity. Therefore it is important to investigate exact properties of the model (1).

In this paper we study the bosonic part of the dynamical system (1). We show a complicated chaotic behavior of classical trajectories and discuss its quantization. The appearance of chaos in a classical system means that we cannot trust the ordinary perturbative analysis of the corresponding quantum system. We demonstrate that after the separation of "slow" and "fast" modes there is a singular contribution from the "slow" modes to the Hamiltonian of the "fast" modes (see formulae (15) and (16)). We also discuss a possibility of considering quantum chaos as a source of the holographic feature of the M(atrix) theory.

A classical dynamical system is defined by its phase space $P$, the dynamical flow $S_t$ and the invariant measure $\mu$. For the Hamiltonian system one considers the reduced phase space obtained by fixing the energy and other (if any) integrals of motion. The measure $\mu$ is the corresponding restriction of the Liouville measure $\prod dq_i dp_i$. The system is said to exhibit a chaotic or stochastic behavior if it is ergodic and moreover it is unstable, i.e. it has a positive Lyapunov exponent.

There are different levels of chaos and they can be specified for instance by $K$-property, central limit theorem, exponential decay of correlations etc. [2, 14, 18, 29, 33, 38]. Chaos is often quantified by computing Lyapunov exponents. If the dynamical system is defined by means of the system of differential equations $\dot{x}_i = F_i(x)$ then the Lyapunov exponent $\chi$ of a solution $x_i(t)$ is given by

$$\chi = \lim_{t\to\infty} \frac{1}{t} \log \frac{\rho(t)}{\rho(0)}, \tag{2}$$

where

$$\rho^2(t) = \sum_i (a_i^2(t) + \dot{a}_i^2(t))$$

and $a_i(t)$ is the solution of the equation $\dot{a} = F'(x)a$.

The equations of motion for the action (1) in the $A_0 = 0$ gauge read

$$\ddot{X}_i = [[X_j, X_i], X_j]. \tag{3}$$

By varying over $A_0$ one also gets the constraint

$$[X_i, \dot{X}_i] = 0. \tag{4}$$

The Hamiltonian for the bosonic part of the action (1) reads

$$H = Tr\left(\frac{1}{2}P_i^2 - \frac{1}{4}[X_i, X_j]^2\right), \tag{5}$$

where $P_i$ is the momentum conjugate to the $X_i$.

It has been proved in [12] that the Hamiltonian of the supersymmetric matrix model has a continuous spectrum starting at zero. This result can be interpreted as a manifestation of the instability of the supermembranes against deformations into stringlike configurations. The possible instability of membranes is already evident from the classical consideration because the potential energy has valleys through which certain membrane configurations can escape to infinity without increasing the mass. For the bosonic membrane this classical instability is cured by quantum mechanics: the spectrum of the quantum Hamiltonian is actually discrete [24, 32]. However, the theory still become unstable if one introduces supersymmetry [12]. These effects can be seen in a toy bosonic model [5] with the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}x_1^2 x_2^2. \tag{6}$$

The quantum mechanical Hamiltonian (6) has a discrete spectrum [25, 24, 32]. However, its supersymmetric version has a continuous spectrum [12]. One observes that certain properties of the classical bosonic system are more similar to the properties of the quantum supersymmetric system rather then to the quantum bosonic system. Supersymmetry saves classics.

The dynamical system with the Hamiltonian (6) is very interesting because it exhibits a complicated chaotic behavior of trajectories. Classical and quantum chaos in the system (6) obtained as the reduction of the Yang-Mills theory has been considered in [5, 9, 25], see also [27, 28]. Chaos in the Einstein-Yang-Mills equations was discussed in [4, 16].

The potential in (6) has zero-directions, the valleys, $x_1 = 0$ and $x_2 = 0$. In these directions the particle can move without changing energy and the presence of these hyperbolic valleys is an origin of stochastic behavior of the particle. A typical trajectory is plotted in Fig. 1. We see that a movement of a particle is limited by four hyperbolas. The particle being positioned in one of the valleys starts to oscillate between two nearest hyperbolas and after a number of oscillations it changes the valley. So, during the time large enough one can see the particle in any of the four valleys. Therefore we cannot treat the dynamics perturbatively just making a linearisation around some fixed value of $x_1$ and $x_2$. To illustrate this instability we present in Fig. 2 the dependence of $\chi(t) = \frac{1}{t}\log\rho(t)/\rho(0)$ on $t$ for some initial data. We see that for $t$ large enough $\chi(t)$ goes to a fixed value, $\chi \approx 0.85$.

Note that the addition of fermions leads to appearing of the interaction with a fermionic current. The new system also has a tendency to produce a chaotic behavior (cf. [8]).

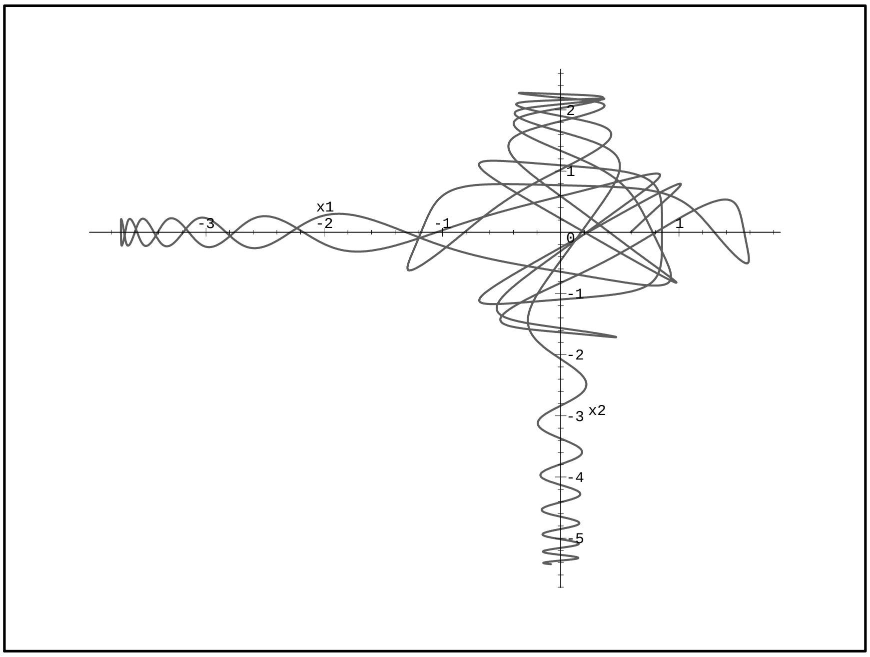


Fig. 1. Typical trajectory of the two-dimensional system with Hamiltonian (6)

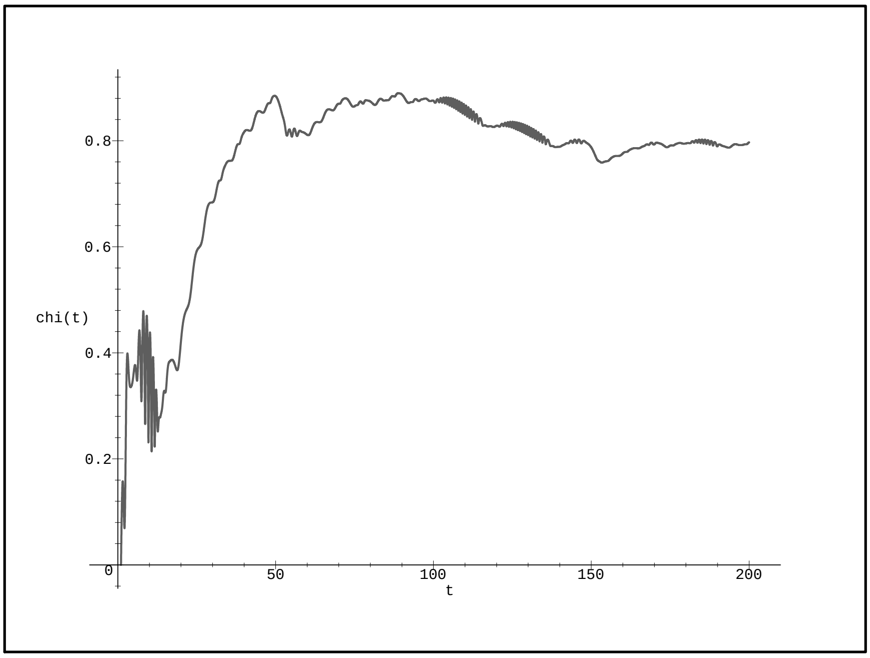


Fig. 2. Lyapunov exponent of the two-dimensional system with Hamiltonian (6)

The system (6) is, in fact, a particular case of the M(atrix) model (1). Let us consider the gauge group $SU(2)$ and take the following ansatz:

$$X_1 = x_1\sigma_1,\ X_2 = x_2\sigma_2,\ X_3 = x_3\sigma_3, \tag{7}$$

$$X_4 = ... = X_9 = 0,$$

where $x_\alpha = x_\alpha(t)$ are real valued functions of time, $\alpha = 1, 2, 3$ and $\sigma_\alpha$ are the Pauli matrices. Then the constraint (4) is satisfied and eqs. (3) are reduced to

$$\ddot{x}_\alpha = -2\Big(\sum_{\beta\neq\alpha} x_\beta^2\Big)x_\alpha. \tag{8}$$

Eqs (8) could be obtained from the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + (x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2). \tag{9}$$

The system (6) is obtained from (9) by seting $x_3 = 0$ and rescaling the coupling constant. The system (9) is an example of the dynamical system which exhibits a chaotic motion. It has been analyzed numerically in [9]. In Fig. 3 we show $(x_1, x_2)$ projection of a typical trajectory of the particle and in Fig. 4 we plot the corresponding $\chi(t)$. During the time large enough one can see the particle in any of the six valleys. By comparing Fig. 2 and Fig. 4 we see that in the three-dimensional case $\chi(t)$ tends to a constant value faster than in the two-dimensional case.

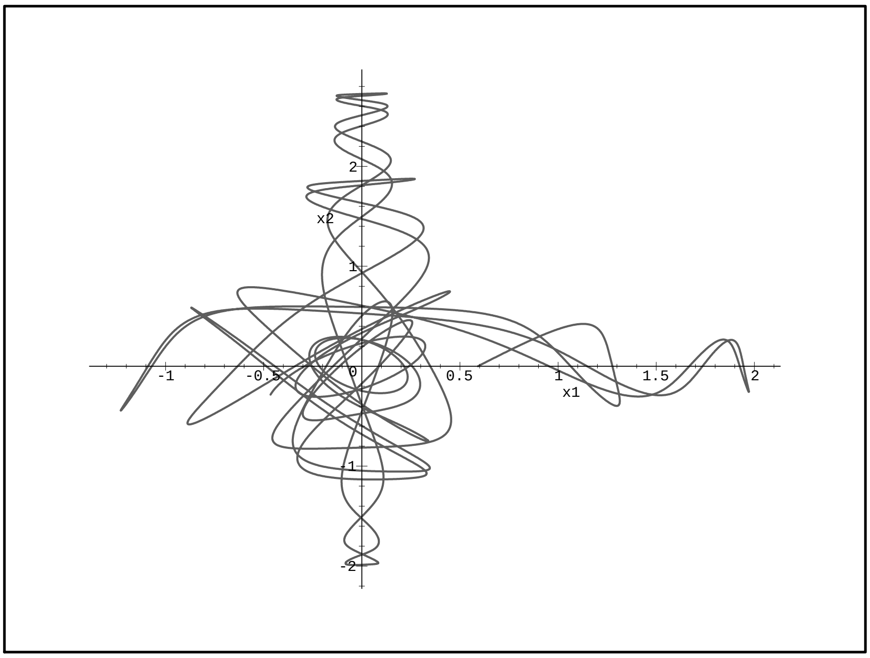


Fig. 3. Typical trajectory of the three-dimensional system with Hamiltonian (9)

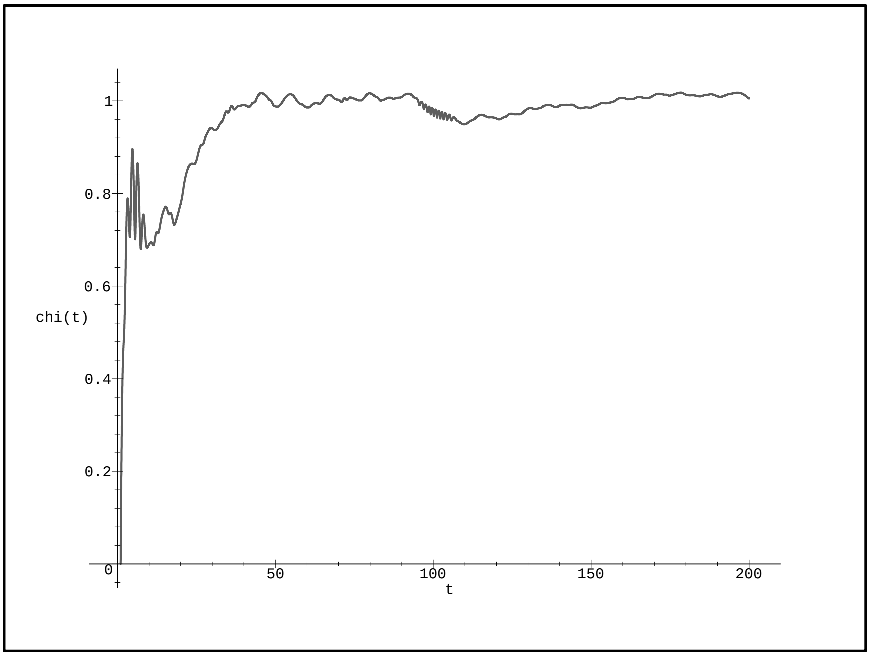


Fig. 4. Lyapunov exponent of the three-dimensional system with Hamiltonian (9)

The ansatz (7) can also be tested on su(3)

$$X_i = x_i T^i,\ i = 1, \dots, 8;\ X_9 = 0, \tag{10}$$

where $T^i$ are the su(3) generators. The Lagrangian equations of motion put the constraint on $x_i$

$$x_4^2 + x_5^2 - x_6^2 - x_7^2 = 0. \tag{11}$$

One of the possible solutions of (11) is $x_4 = x_6$, $x_5 = x_7$. It yields the system with six degrees of freedom with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{3} p_i^2 + \frac{1}{4}(p_4^2 + p_5^2) + \frac{1}{2}p_8^2 + \frac{1}{2}\sum_{1\le i<j\le 3} x_i^2 x_j^2 + \frac{1}{4}(x_4^2 + x_5^2)\sum_{1\le i\le 3} x_i^2$$
$$+\frac{5}{4}x_4^2 x_5^2 + \frac{1}{8}(x_4^4 + x_5^4) + \frac{3}{4}(x_4^2 + x_5^2)x_8^2, \tag{12}$$

where $p_i$ is the momentum conjugated to $x_i$. The estimation of the Lyapunov exponent by computer simulation exhibits the stochastic behaviour of the system. For the particular case $x_2 = x_3 = x_5 = x_8 = 0$ the Hamiltonian has the form

$$H = \frac{1}{2}p_1^2 + \frac{1}{4}p_4^2 + \frac{1}{2} + \frac{1}{4}x_4^2 x_1^2 + \frac{1}{8}x_4^4. \tag{13}$$

A typical trajectory for (13) is plotted in Fig. 5 and it can be seen that they are located in the valley $x_1 = 0$. The corresponding Lyapunov exponent is presented in Fig. 6.

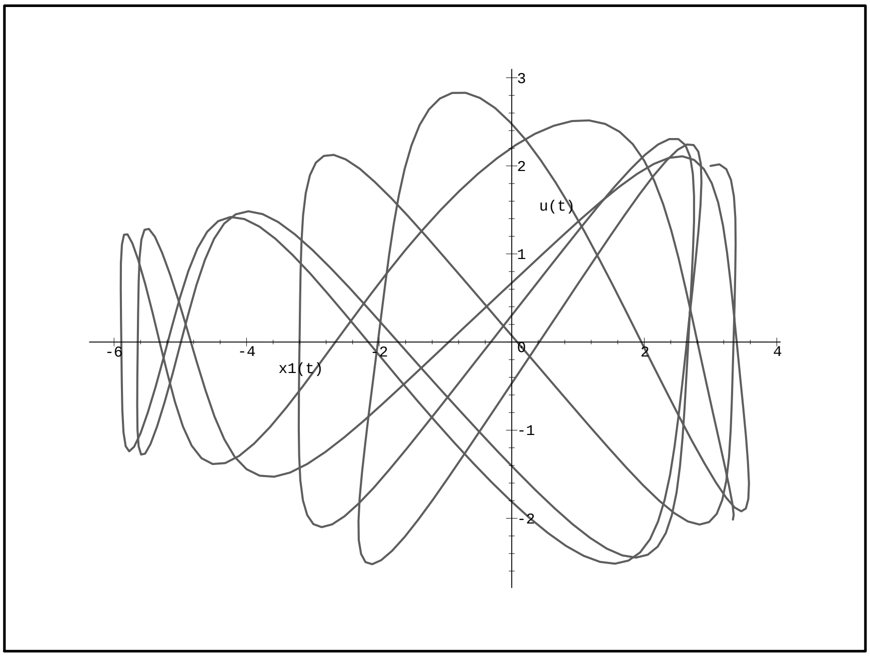


Fig. 5. Typical trajectory of the two-dimensional system with Hamiltonian (13)

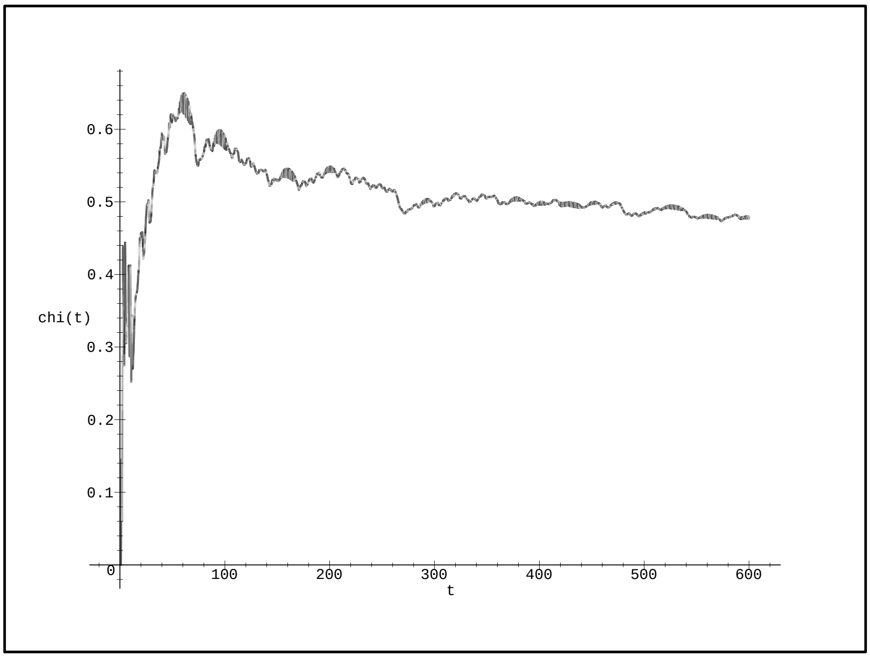


Fig. 6. The example of the Lyapunov exponent for the two-dimensional system, obtained in the su(3) case

Note that the ansatz (7) deals only with the matrix diagonal in (space, isotopic) indices and all other degrees of freedom are frozen. One can think that stochastic behavior is an artifact of this ansatz and instability will be cured by taking into account the fluctuation of the frozen degrees of freedom. To analyze this possibility we consider the model (1) with only two matrices and relaxed constraints. A relaxing of the Gauss law means that we consider an interaction of the "electric" field with some current. Denote: $X_1 = \phi_1$ and $X_2 = \phi_2$, then the action reads

$$S = \int dt \, \mathrm{Tr} \left( \frac{1}{2} \dot{\phi}_1^2 + \frac{1}{2} \dot{\phi}_2^2 + \frac{\lambda}{2} [\phi_1, \phi_2]^2 \right). \tag{14}$$

This action describes the dynamics of the classical vacuum moduli space in the $N = 2$ SUSY Yang-Mills theory [31].

The dynamics of "fast" modes in (14) can be reduced in special cases to the following two-dimensional Hamiltonian systems

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \lambda x_1^2 x_2^2 + \frac{(m-n)^2}{4(x_1 - x_2)^2} + \frac{(m+n)^2}{4(x_1 + x_2)^2}, \tag{15}$$

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \lambda x_1^2 x_2^2 + \frac{m^2}{2x_1^2}, \tag{16}$$

where $m$ and $n$ are some real constants. These constants describe a contribution from the "slow" modes to the Hamiltonian of the "fast" modes. We see that the "slow" modes are separated but they bring a singular term to the Hamiltonian of the "fast" modes. Note that an influence of the extra term $1/2x_1^2$ on the chaotic behavior of a two-dimensional system has been discussed in the recent paper [8]. Typical trajectories for the Hamiltonian (15) are plotted on Fig. 7 and for the Hamiltonian (16) on Fig. 8.

In these special cases one can recover the constraints by taking $m = 0$. In the case (16) one comes back to the ansatz (9) with $x_3 = 0$. But in the case (15) we get an extra repulsive potential. This potential describes a repulsive from walls located along two diagonals $x_1 = \pm x_2$ in the $x_1, x_2$ plane. A movement of a particle is limited by equipotential lines

$$\lambda x_1^2 x_2^2 + \frac{n^2(x_1^2 + x_2^2)}{2(x_1^2 - x_2^2)^2} = const,$$

and the particle oscillates in one of the four valleys (it chooses one of them according to the initial data). In each of four allowed domains the potential still has a direction of the instability. Therefore, the presence of the constraint does not affect the chaotic character of trajectories.

Different approaches have been developed to address the question what is the way in which classical chaos manifests itself in the properties of the corresponding quantum system [18, 38], [5, 9, 25], [1, 6, 7, 8, 17, 21, 23, 26, 27, 30, 34]. The simplest manifestation of classical chaos for a quantum system is the nature of spectral fluctuations of the energy levels. It was suggested that for chaotic systems the statistical properties of the spectrum should be that of the random matrix theory with the Wigner-Dyson distribution

$$P(E|\Delta E) = A|\Delta E|^{\alpha} \exp[-B(\Delta E)^2], \tag{17}$$

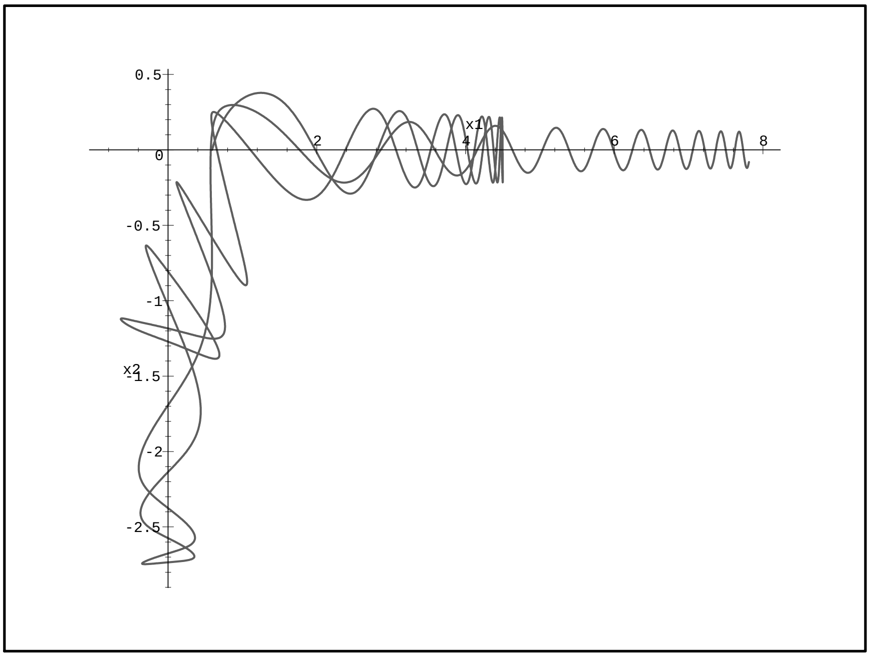


Fig. 7. Typical trajectory of the system with Hamiltonian (15), $m = -n$

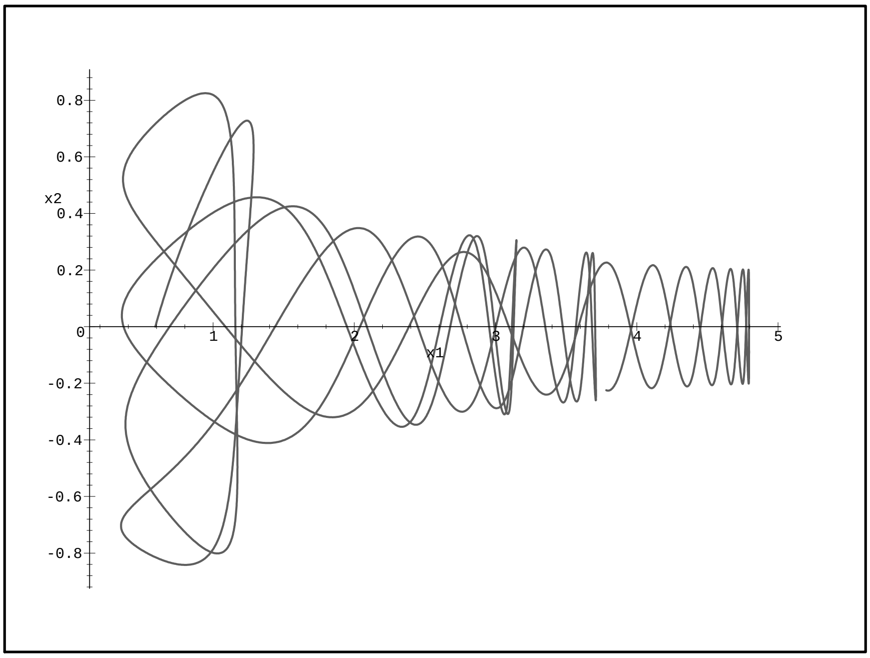


Fig. 8. Typical trajectory of the system with Hamiltonian (16)

where $\alpha > 0$, whereas the quantum version of a classically integrable system is described by the Poisson distribution

$$P(E|\Delta E) = a \exp[-b|\Delta E|] \tag{18}$$

The crucial difference between (17) and (18) is the behavior for $\Delta E \to 0$. For the Wigner-Dyson distribution (17) one has $P(E|\Delta E) \to 0$ if $\Delta E \to 0$, i.e. the density of the energy levels at the small scale vanishes. This is interpreted as the repulsive feature of the energy eigenvalues for quantum chaotic system. One can try to relate this feature with the holographic feature of M(atrix) theory. The holographic principle follows from the general consideration involving the Bekenstein-'t Hooft bound on the entropy of a spatial region. A holographic theory contains only degrees of freedom which carry the smallest unit of longitudinal momentum [3]. The transverse density of partons is bounded to about one parton per transverse Planck area, in this sense the holographic theory is repulsive. The partons form a kind of incompressive fluid. The repulsive feature of the holographic theory seems to be related with the repulsive feature of the distribution of the energy levels for quantum chaotic system. Quantum chaos is a source of the holographic feature of M-theory. A connection between the large $N$ limit, random matrix theory and entropy of black holes is discussed in [13, 20, 36].

In conclusion, we have shown that M(atrix) theory exhibits the classical chaotic motion and we have argued that quantum chaos is a source of the holographic feature. The statistical properties of the spectrum and other features of quantum chaos in M-theory require a further investigation.

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