SPLITTING THE CONSERVATION PROCESS INTO CREATION AND ANNIHILATION PARTS

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Abstract. The aim of this paper is the study of a non-commutative decomposition of the conservation process in quantum stochastic calculus. The probabilistic interpretation of this decomposition uses time changes, in contrast to the spatial shifts used in the interpretation of the creation and annihilation operators on Fock space.

1. Notation and preliminaries. Consider the Fock space $\Gamma(H) = \bigoplus_{n \geq 0} H^{\circ n}$ over a Hilbert space H, where $H^{\circ n}$ consists of the space of symmetric tensors in the tensor product $H^{\otimes n}$, endowed with the norm

$$\| \cdot \|_{H^{\circ n}}^2 = n! \, \| \cdot \|_{H^{\otimes n}}^2, \quad n \in \mathbf{N}.$$

The Malliavin gradient $\nabla^- : \Gamma(H) \to \Gamma(H) \otimes H$ is defined by transformation of the tensor $f^{\circ n}$, $f \in H$, into $\nabla^- f^{\circ n} = n f^{\circ(n-1)} \otimes f$, $n \in \mathbf{N}$, and the Skorohod integral $\nabla^+ : \Gamma(H) \otimes H \to \Gamma(H)$ satisfies $\nabla^+ f^{\circ n} \otimes g = f^{\circ n} \circ g$, $n \in \mathbf{N}$. Those operators are extended by polarization, linearity and closability to their respective domains in $\Gamma(H)$, $\Gamma(H) \otimes H$. We let $\Phi = \Gamma(L^2(\mathbf{R}_+))$ and denote by $\langle \cdot, \cdot \rangle$ the scalar product on Φ. We call $\mathcal{S}$ the set of elements of Φ whose chaos expansion is finite and involves only functions which are $\mathcal{C}^1$ with compact supports. The basic annihilation, creation and conservation operators a_h^-, a_h^+, a_h° are defined as

$$a_h^- F = (\nabla^- F, h)_H, \quad a_h^+ F = \nabla^+(F \otimes h), \quad a_h^\circ F = \nabla^+(h \nabla F), \quad F \in \mathcal{S}, \; h \in H.$$

We also let $a_t^\varepsilon = a_{1_{[0,t]}}^-$, $\varepsilon = -, +, \circ$. The process $(a_t^\circ)_{t \in \mathbf{R}_+}$ is called the conservation process, or gauge process. For $A \in \mathcal{B}(\mathbf{R}_+)$, let $\pi_A : L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ denote the projection operators defined as $\pi_A f = 1_A f$, $f \in L^2(\mathbf{R}_+)$, and let $\pi_{t]} = \pi_{[0,t]}$, $\pi_{[t} = \pi_{[t,\infty[}$,

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$t \in \mathbf{R}_+$. The exponential vector $\xi(f)$, $f \in L^2(\mathbf{R}_+)$, is defined as

$$\xi(f) = \sum_{n \in \mathbf{N}} \frac{1}{n!} f^{\circ n}.$$

We say that $F \in \Phi$ is $\mathcal{F}_A$-measurable if $\Gamma(\pi_A)F = F$, $A \in \mathcal{B}(\mathbf{R}_+)$. Let $\mathcal{S}([a,b[)$, $0 \leq a < b$, resp. $\mathcal{S}_{t]}$, $\mathcal{S}_{[t}$, denote the elements of $\mathcal{S}$ that are $\mathcal{F}_{[a,b[}$-measurable, resp. $\mathcal{F}_{t]}$-measurable, $\mathcal{F}_{[t}$-measurable. The second quantization $\Gamma(U) : \Phi \to \Phi$ of a bounded operator $U : L^2(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$ is defined as

$$\Gamma(U) = \bigoplus_{n \geq 0} U^{\circ n}.$$

The Fock space Φ is isomorphic to the tensor product $\Phi_{t]} \otimes \Phi_{[t}$, $t \in \mathbf{R}_+$, via the mapping $\xi(f) \mapsto \xi(\pi_{t]}f) \otimes \xi(\pi_{[t}f)$, with $\Phi_{t]} = \Gamma(L^2([0,t]))$ and $\Phi_{t]} = \Gamma(L^2([t,\infty[))$, $t \in \mathbf{R}_+$, cf. [5]. In this paper, tensor products between vector spaces are completed if and only if vector spaces are complete.

2. A remark on non-commutative stochastic integration. The goal of this section is to notice that in quantum stochastic integration, integrator processes need not be adapted. Therefore, the stochastic integral of adapted processes can be defined with respect to processes other than $(a_t^-)_{t \in \mathbf{R}_+}$, $(a_t^+)_{t \in \mathbf{R}_+}$ and $(a_t^\circ)_{t \in \mathbf{R}_+}$. We recall that a process $(X_t)_{t \in \mathbf{R}_+}$ of operators on Φ is said to be adapted if X_t is written as $Y_t \otimes I_d$ on $\Phi_{t]} \otimes \Phi_{[t}$, $t \in \mathbf{R}_+$, where I_d denotes the identity operator, cf. [1], [3]. If $U^- : \Phi \to \Phi \otimes L^2(\mathbf{R}_+)$ and $U^+ : \Phi \otimes L^2(\mathbf{R}_+) \to \Phi$ are operators densely defined on $\mathcal{S}$ and $\mathcal{S} \otimes L^2(\mathbf{R}_+)$, we let $u_h^-, u_h^+ : \Phi \to \Phi$, $h \in L^2(\mathbf{R}_+)$, be the operators defined on $\mathcal{S}$ as

$$u_h^- F = (h, U^- F)_{L^2(\mathbf{R}_+)}, \quad u_h^+ F = U^+(F \otimes h), \quad F \in \mathcal{S}, \quad h \in L^2(\mathbf{R}_+).$$

We also let $u_t^- = u_{\mathbf{1}_{[0,t]}}^-$, $u_t^+ = u_{\mathbf{1}_{[0,t]}}^+$, $t \in \mathbf{R}_+$, and note that $(u_t^-)_{t \in \mathbf{R}_+}$ is adapted if and only if $(u_t^+)_{t \in \mathbf{R}_+}$ is.

DEFINITION 1. Let $\mathcal{A}$ denote the class of couples (U^-, U^+) where $U^- : \Phi \to \Phi \otimes L^2(\mathbf{R}_+)$ and $U^+ : \Phi \otimes L^2(\mathbf{R}_+) \to \Phi$ are two closable and mutually adjoint operators densely defined on $\mathcal{S}$, that map $\mathcal{S}_{t]}$ into $\Phi_{t]}$ and $\mathcal{S}_{[t}$ into $\Phi_{[t}$, and such that

$$u_h^-(F \otimes G) = (u_h^- F) \otimes G + F \otimes (u_h^- G), \quad F \in \mathcal{S}_{t]}, \ G \in \mathcal{S}_{[t}, \ h \in L^2(\mathbf{R}_+), \ t \in \mathbf{R}_+.$$

We note that (∇^-, ∇^+) belongs to $\mathcal{A}$.

DEFINITION 2. A couple $(U^-, U^+) \in \mathcal{A}$ is said to be an extension of the stochastic integral if $\Gamma(\pi_{s]})U_t^- = \Gamma(\pi_{s]})\nabla_t^-$, $\quad 0 \leq s < t$, $t \in \mathbf{R}_+$.

The last property states that the adapted projections of $U^- F$ and ∇F coincide $\forall F \in \mathcal{S}$. Let $\mathcal{U}$ denote the set of elements of $\Phi \otimes L^2(\mathbf{R}_+)$ of the form $\sum_{i=1}^{i=n} F_i \otimes h_i$, with $h_1, \ldots, h_n \in \mathcal{C}_c^1(\mathbf{R}_+)$, and $F_1, \ldots, F_n \in \mathcal{S}$, $n \in \mathbf{N}$. The set of square-integrable adapted processes is the completion in $\Phi \otimes L^2(\mathbf{R}_+)$ of the set of adapted processes in $\mathcal{U}$. If Φ is identified to the L^2-space of a stochastic process $(Z_t)_{t \in \mathbf{R}_+}$ with stationary independent increments and $(U^-, U^+) \in \mathcal{A}$ is an extension of the stochastic integral, then U^+ coincides with the stochastic integral with respect to $(Z_t)_{t \in \mathbf{R}_+}$ on the square-integrable adapted processes.

PROPOSITION 1. *A couple $(U^-, U^+) \in \mathcal{A}$ is an extension of the stochastic integral if and only if U^+ and ∇^+ coincide on the square-integrable adapted processes.*

P r o o f. Let $1_{[a,b[}G \in \mathcal{U}$ be a simple adapted process. We have

$$\langle U^+(1_{[a,b[}G), F\rangle = \int_0^\infty 1_{[a,b[}(s)\langle G, U_s^- F\rangle ds$$

$$= \int_0^\infty 1_{[a,b[}(s)\langle G, \nabla_s^- F\rangle ds = \langle \nabla^+(1_{[a,b[}G), F\rangle, \quad F \in \mathcal{S}.$$

This statement is extended by linearity and closability to the square-integrable adapted processes in $\Phi \otimes L^2(\mathbf{R}_+)$. ∎

R e m a r k 1. Let $(U^-, U^+) \in \mathcal{A}$ be an extension of the stochastic integral, and assume that $(u_t^-)_{t \in \mathbf{R}_+}$, $(u_t^+)_{t \in \mathbf{R}_+}$ are adapted. Then $(U^-, U^+) = (\nabla^-, \nabla^+)$.

P r o o f. We will show that for any $t \in \mathbf{R}_+$, $\Gamma(\pi_{s]})U_t^- = \Gamma(\pi_{s]})\nabla_t^-$, $s < t$, and $\Gamma(\pi_{[s})U_t^- = \Gamma(\pi_{[s})\nabla_t^-$, $s > t$, on $\mathcal{S}$. The first part is a consequence of the fact that (U^-, U^+) is an extension of the stochastic integral. Assume $t < a < s$ and let $F \in \mathcal{S}_{a]}$, $G \in \mathcal{S}_{[a}$. Then since $(u_t^-)_{t \in \mathbf{R}_+}$ and $(a_t^-)_{t \in \mathbf{R}_+}$ are adapted,

$$\Gamma(\pi_{[s})U_t^-(F \otimes G) = \Gamma(\pi_{[s})((U_t^- F) \otimes G) = \langle U_t^- F, 1\rangle \Gamma(\pi_{[s})G$$

$$= \langle \nabla_t^- F, 1\rangle \Gamma(\pi_{[s})G = \Gamma(\pi_{[s})\nabla_t^-(F \otimes G),$$

since $\Gamma(\pi_{[s})U_t^- F = \langle U_t^- F, 1\rangle = \langle \nabla_t^- F, 1\rangle = \Gamma(\pi_{[s})\nabla_t^- F$, $a < s$. Hence $U^- = \nabla^-$. ∎

As a consequence of this remark, if $(U^-, U^+) \in \mathcal{A}$ is an extension of the stochastic integral that differs from (∇^-, ∇^+), then $(u_t^-)_{t \in \mathbf{R}_+}$ and $(u_t^+)_{t \in \mathbf{R}_+}$ cannot be adapted.

R e m a r k 2. Let $(U^-, U^+) \in \mathcal{A}$. For the stochastic integral of simple adapted processes to be defined with respect to du_t^-, du_t^+, it is sufficient that

$$(u_t - u_s) = I_d \otimes (u_t - u_s) \quad on \quad \Phi_{s]} \otimes \Phi_{[s}, \ \forall s \geq t, \ \ t \in \mathbf{R}_+.$$

This condition is satisfied in particular if (U^-, U^+) is an extension of the stochastic integral. Hence quantum stochastic integrals of simple adapted operator processes $(u_t^-)_{t \in \mathbf{R}_+}$, $(u_t^+)_{t \in \mathbf{R}_+}$ can be defined without requiring the adaptedness of $(u_t^-)_{t \in \mathbf{R}_+}$, $(u_t^+)_{t \in \mathbf{R}_+}$.

P r o o f. Let

$$X = \sum_{i=1}^{i=n} X_i 1_{[t_i, t_{i+1}[}$$

be a simple adapted operator process. The composition $(u_t^- - u_s^-)X_s, t \geq s$, is well-defined by

$$(u_t^- - u_s^-)X_s(F \otimes G) = X_s F \otimes (u_t^- - u_s^-)G, \quad t \geq s,$$

on $\Phi_{s]} \otimes \Phi_{[s}$. Consequently, the integral of X with respect to du_t^- is well-defined as

$$\int_0^\infty X_s du_s^- = \sum_{i=1}^{i=n} X_i(u_{t_{i+1}}^- - u_{t_i}^-).$$

And $\int_0^\infty X_s du_s^+$ is defined by duality as

$$\int\limits_0^\infty X_s du_s^+ = \sum_{i=1}^{i=n} X_i(u_{t_{i+1}}^+ - u_{t_i}^+),$$

with

$$\Big\langle \int\limits_0^\infty X_s du_s^- F, G \Big\rangle = \Big\langle F, \int\limits_0^\infty X_s^* du_s^+ G \Big\rangle, \quad F, G \in \mathcal{S}.$$

We may also write

$$\int\limits_0^\infty X_s du_s^- F = \int\limits_0^\infty U_s^- X_s F ds, \quad \int\limits_0^\infty X_s du_s^+ F = U^+(XF), \quad F \in \mathcal{S}. \ \blacksquare$$

The notion of extension of the stochastic integral on Fock space also allows to extend the Clark representation formula, cf. [2, 8].

PROPOSITION 2. *Let $(U^-, U^+) \in \mathcal{A}$ be an extension of the stochastic integral on Fock space. Then the mapping $F \mapsto (\Gamma(\pi_{s]}) U_s^- F)_{s \in \mathbf{R}_+}$ is continuous from Φ to $\Phi \otimes L^2(\mathbf{R}_+)$, and any $F \in \Phi$ can be represented as*

$$F = \langle F, 1 \rangle + U^+(\Gamma(\pi_{\cdot]}) U_\cdot^- F).$$

P r o o f. For $F \in \mathcal{S}$, $\Gamma(\pi_{t]}) U_t^- F$ is defined as

$$\Gamma(\pi_{t]}) U_t^- F = \lim_{s \uparrow t} \Gamma(\pi_{s]}) U_t^- F = \lim_{s \uparrow t} \Gamma(\pi_{s]}) \nabla_t^- F = \Gamma(\pi_{t]}) \nabla_t^- F, \quad t \in \mathbf{R}_+.$$

The continuity result and the representation formula are well-known facts in case $(U^-, U^+) = (\nabla^-, \nabla^+)$, cf. [12], and they immediately extend to U^- from its definition and Proposition 1. $\blacksquare$

3. A decomposition of the conservation operator. The main goal of this section is to introduce a non-commutative decomposition of the number operator on Fock space and study its probabilistic interpretations. Let $\overset{\circ}{h}$, denote the function defined as $\overset{\circ}{h}(t) = \int_0^t h(s) ds$, $h \in L^2(\mathbf{R}_+)$, and let $f'(t) = \frac{d}{dt} f(t)$, $t \in \mathbf{R}_+$, $f \in \mathcal{C}_c^1(\mathbf{R}_+)$.

DEFINITION 3. We define the linear operators $\nabla^\ominus : \Phi \to \Phi \otimes L^2(\mathbf{R}_+)$ on $\mathcal{S}$ and $\nabla^\oplus : \Phi \otimes L^2(\mathbf{R}_+) \to \Phi$ on $\mathcal{U}$ by

$$\nabla_t^\ominus f^{\circ n} = -n(\pi_{[t} f') \circ f^{\circ(n-1)}, \quad t \in \mathbf{R}_+, \ f \in \mathcal{C}_c^1(\mathbf{R}_+), \quad n \in \mathbf{N},$$

and

$$\nabla^\oplus(f^{\circ n} \otimes h) = n \left(f \overset{\circ}{h} \right)' \circ f^{\circ(n-1)}, \quad f, h \in \mathcal{C}_c^1(\mathbf{R}_+),$$

and by polarization of these expressions.

The operators $\nabla^\ominus$ and $\nabla^\oplus$ are unbounded, closable, densely defined, and adjoint of each other. We note that $(\nabla^- + \nabla^\ominus, \nabla^+ + \nabla^\oplus) \in \mathcal{A}$ and is an extension of the stochastic integral. Let also $a_h^\ominus$, $a_h^\oplus$ be defined on $\mathcal{S}$ as

$$a_h^\ominus F = (\nabla^\ominus F, h)_{L^2(\mathbf{R}_+)}, \quad a_h^\oplus F = \nabla^\oplus(F \otimes h), \quad n \in \mathbf{N}, \ F \in \mathcal{S}, \ h \in L^2(\mathbf{R}_+),$$

and let $a_t^\ominus = a_{1_{[0,t]}}^\ominus$, $a_t^\oplus = a_{1_{[0,t]}}^\oplus$, $t \in \mathbf{R}_+$. The operator $a_h^\ominus$ is adjoint to $a_h^\oplus$, $h \in L^2(\mathbf{R}_+)$. Moreover, $a_h^\ominus$ and $a_h^\oplus$ can be defined as the differential second quantizations of $f \mapsto \overset{\circ}{h} f'$

and $f \mapsto (f\mathring{h})'$ as

$$a_h^\ominus = d\Gamma(\mathring{h}\partial), \quad a_h^\oplus = d\Gamma(\partial\mathring{h}).$$

The following proposition is a simple consequence of the derivation rule of the product of functions of real variable.

PROPOSITION 3. *The operator a_h° admits the non-commutative decomposition*

$$a_h^\circ = a_h^\ominus + a_h^\oplus, \quad h \in L^2(\mathbf{R}_+), \tag{1}$$

into the sum of two adjoint operators.

The operator $a_h^\ominus + a_h^\oplus$ has a Poisson law with parameter $\| h \|_{L^2(\mathbf{R}_+)}^2$ in the state $\xi(h)\exp(-\frac{1}{2} \| h \|_{L^2(\mathbf{R}_+)}^2)$, and $\left(a_t^\ominus + a_t^\oplus\right)_{t\in\mathbf{R}_+}$ is a non-commutative Poisson process in this state, cf. e.g. [1].

PROPOSITION 4. *We have the commutation relations*

$$[\nabla_s^-, \nabla_t^\ominus]\xi(f) = -(\pi_{[t}f')(s)\xi(f), \quad [a_h^+, \nabla_t^\ominus]\xi(f) = (\pi_{[t}h') \circ \xi(f),$$

$f, h \in \mathcal{C}_c^1(\mathbf{R}_+)$, $s, t \in \mathbf{R}_+$.

P r o o f. We have $\nabla_s^- \nabla_t^\ominus f^{\circ n} = -n(\pi_{[t}f')(s)f^{\circ(n-1)} - n(n-1)f(s)(\pi_{[t}f') \circ f^{\circ(n-2)}$, and $\nabla_t^\ominus \nabla_s^- f^{\circ n} = -n(n-1)f(s)(\pi_{[t}f') \circ f^{\circ(n-2)}$. On the other hand, $a_h^+ \nabla_t^\ominus f^{\circ n} = -na_h^+(\pi_{[t}f') \circ f^{\circ(n-1)}$, and $\nabla_t^\ominus a_h^+ f^{\circ n} = \nabla_t^\ominus h \circ f^{\circ n} = -n(\pi_{[t}f') \circ f^{\circ(n-1)} \circ h - (\pi_{[t}h') \circ f^{\circ n}$. ∎

4. Probabilistic interpretations of the decomposition

4.1. Wiener space interpretation. Let $(W, L^2(\mathbf{R}_+), \mu)$ denote the classical Wiener space, with Brownian motion $(B_t)_{t\in\mathbf{R}_+}$. We work here under the identification of $L^2(W, \mu)$ and Φ provided by the multiple stochastic integrals with respect to $(B_t)_{t\in\mathbf{R}_+}$. We are interested here in the properties of $\nabla^\ominus$ under the above identification. Recall that on the Wiener space, ∇^- is identified to a derivation operator which satisfies

$$(\nabla^- F, h)_{L^2(\mathbf{R}_+)} = \lim_{\varepsilon\to 0} \frac{F\left(B. + \int_0^. h(s)ds\right) - F}{\varepsilon}, \quad F \in \mathcal{S}, \ h \in L^2(\mathbf{R}_+),$$

cf. e.g. [6]. For $h \in L^2(\mathbf{R}_+)$, with $\sup_{x\in\mathbf{R}_+} | h(x) | < 1$, let $\nu_h(t) = t + \mathring{h}_t$, $t \in \mathbf{R}_+$.

DEFINITION 4. We define a mapping $\mathcal{T}_h : W \to W$, $t, \varepsilon \in \mathbf{R}_+$, as

$$\mathcal{T}_h(\omega) = \omega \circ \nu_h^{-1}, \quad h \in L^2(\mathbf{R}_+), \ \sup_{x\in\mathbf{R}_+} | h(x) | < 1.$$

The transformation $\mathcal{T}_h$ acts on the trajectory of $(B_s)_{s\in\mathbf{R}_+}$ by change of time, or by perturbation of its predictable quadratic variation. Although $\mathcal{T}_h$ is not absolutely continuous with respect to the Wiener measure, the functional $F \circ \mathcal{T}_h$ is well-defined for $F \in \mathcal{S}$, since elements of $\mathcal{S}$ can be defined for every trajectory of $(B_t)_{t\in\mathbf{R}_+}$, using the product formula for multiple stochastic integrals and the relation $\int_0^\infty f_s dB_s = -\int_0^\infty f'(s)B_s ds$, a.s., $f \in \mathcal{C}_c^1(\mathbf{R}_+)$.

PROPOSITION 5 ([9]). *We have for $F \in \mathcal{S}$*

$$\int_0^\infty h_t \left(\nabla_t^\ominus + \frac{1}{2}\nabla_t^- \nabla_t^-\right) F dt = -\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(F \circ \mathcal{T}_{\varepsilon h} - F). \tag{2}$$

4.2. Poisson space interpretation. We now consider a standard Poisson process on a probability space (B, P). The Poisson multiple stochastic integrals provide an isometric isomorphism between $L^2(B)$ and Φ, cf. [11]. From [7], ∇^- satisfies

$$\nabla_t^- f(T_1, \ldots, T_n) = \sum_{k=1}^{k=n} 1_{[T_{k-1}, T_k[}(t) \left(f(T_1, \ldots, T_k, t, T_{k+1}, \ldots, T_{n-1}) - f(T_1, \ldots, T_n) \right),$$

$$(3)$$

$t \in \mathbf{R}_+$, $f \in \mathcal{C}_c^1(\mathbf{R}_+^{n+1})$, which amounts to perturbing the trajectories of $(N_t)_{t \in \mathbf{R}_+}$ by addition of a jump at time t. The operator ∇^+ coincides with the Poisson stochastic integral on the square-integrable adapted processes. The operator $\nabla_t^- + \nabla_t^\ominus$ on Poisson space, as $\nabla_t^\ominus + \frac{1}{2}\nabla_t^- \nabla_t^-$ on the Wiener space, is defined by an infinitesimal time change on the Poisson process.

PROPOSITION 6 ([9]). *We have*

$$((\nabla^\ominus + \nabla^-)F, h)_{L^2(\mathbf{R}_+)} = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(F \circ \mathcal{T}_{\varepsilon h} - F), \quad F \in \mathcal{S}, \tag{4}$$

where the transformation $\mathcal{T}_h$ is defined as in Def. 4, i.e. it acts on the Poisson process trajectories via the time change ν_h.

R e m a r k 3. Relations (2) and (4) imply, both in the Wiener and Poisson interpretations, the following product rule for $\nabla^\ominus$:

$$\nabla^\ominus(FG) = F\nabla^\ominus G + G\nabla^\ominus F - \nabla^- F\nabla^- G, \quad F, G \in \mathcal{S}.$$

As a consequence of this remark, a duality argument shows that Relation (1) can be extended to random $h \in \mathcal{U}$ as:

$$a_h^\circ + \nabla^\oplus(h) = a_h^\ominus + a_h^\oplus, \quad h \in L^2(\mathbf{R}_+),$$

where $\nabla^\oplus(h)$ is identified to a multiplication operator, in both the Wiener and Poisson interpretations of Φ. It is easy to check that $\nabla^\oplus(h) = 0$ if h is adapted, since $(\nabla^- + \nabla^\ominus, \nabla^+ + \nabla^\oplus) \in \mathcal{A}$ is an extension of the stochastic integral.

5. Non-commutative stochastic integration and Itô calculus. From Section 3, the integral of simple adapted processes with respect to the operator processes $(a_t^\ominus)_{t \in \mathbf{R}_+}$ and $(a_t^\oplus)_{t \in \mathbf{R}_+}$ are well-defined. However, the processes $(a_t^\ominus)_{t \in \mathbf{R}_+}$ and $(a_t^\oplus)_{t \in \mathbf{R}_+}$ are not adapted, hence iterated stochastic integrals of simple adapted processes with respect to $(a_t^\varepsilon)_{t \in \mathbf{R}_+}$, $\varepsilon = \ominus, \oplus$ have to be defined as anticipating stochastic integrals, and $a_t^\ominus$, $a_t^\oplus$ do not commute in general with the differential $du_t^\pm$. Our construction of anticipating quantum stochastic integration parallels that of [4] but concerns $(a_t^\ominus)_{t \in \mathbf{R}_+}$ and $(a_t^\oplus)_{t \in \mathbf{R}_+}$. For $U^- = \nabla^-, \nabla^\ominus$, the integrals $\int_0^t du_s^- a_s^\varepsilon$, $\int_0^t a_s^\varepsilon du_s^+$, $\varepsilon = \ominus, \oplus$, are defined on $\mathcal{S}$ as

$$\int_0^t du_s^- a_s^\varepsilon F = \int_0^t U_s^- a_s^\varepsilon F, \quad \int_0^t du_s^+ a_s^\varepsilon F = U^+(1_{[0,t]}(\cdot)a_{\cdot}^\varepsilon F), \quad F \in \mathcal{S}.$$

The integrals $\int_0^t a_s^\varepsilon du_s^+$, $\int_0^t a_s^\varepsilon du_s^-$ are defined by duality from $\int_0^t du_s^- a_s^\varepsilon$, $\int_0^t du_s^+ a_s^{*\varepsilon}$, with $*\varepsilon = \oplus, \ominus$ respectively if $\varepsilon = \ominus, \oplus$.

PROPOSITION 7. *Let $(U^-, U^+) \in \mathcal{A}$ and $(V^-, V^+) \in \mathcal{A}$ be extensions of the stochastic integral. We have the multiplication table*

$\cdot$	dt	du_t^-	dv_t^+
dt	*0*	*0*	*0*
dv_t^+	*0*	*0*	*0*
du_t^-	*0*	*0*	$du_t^- \cdot dv_t^+$

(5)

The non-vanishing product $du_t^- \cdot dv_t^+$ is given as

$$(da_t^{\ominus} + da_t^{\oplus}) \cdot (da_t^{\ominus} + da_t^{\oplus}) = da_t^{\circ}, \qquad da_t^- \cdot (da_t^+ + da_t^{\oplus}) = dt + da_t^-,$$

$$(da_t^- + da_t^{\ominus}) \cdot da_t^+ = dt + da_t^+, \qquad da_t^- \cdot da_t^+ = dt,$$

respectively if $(U^-, V^+) = (\nabla^- + \nabla^{\ominus}, \nabla^+ + \nabla^{\oplus})$, $(U^-, V^+) = (\nabla^-, \nabla^+ + \nabla^{\oplus})$, $(U^-, V^+) = (\nabla^- + \nabla^{\ominus}, \nabla^+)$, $(U^-, V^+) = (\nabla^-, \nabla^+)$.

Proof. The following relations hold:

$$\langle u_t^+ F, v_t^- G \rangle = \int_0^t \langle u_s^+ F, V_s^- G \rangle ds + \langle F, \int_0^t U_s^- v_s^- G ds \rangle, \tag{6}$$

$$\langle v_t^- G, u_t^- F \rangle = \int_0^t \int_0^s \langle V_z^- G, U_s^- F \rangle dz ds + \int_0^t \int_0^u \langle V_z^- G, U_s^- F \rangle dz ds, \tag{7}$$

$F, G \in \mathcal{S}$. Let X, Y be simple adapted processes such that $\mathcal{S} \subset Dom(X_s Y_s)$, $s \in \mathbf{R}_+$. By linearity and adaptedness of $(X_t)_{t \in \mathbf{R}_+}$, $(Y_t)_{t \in \mathbf{R}_+}$, (6) implies

$$\left\langle \int_0^t Y_s dv_s^+ G, \int_0^t X_s^* du_s^- F \right\rangle$$

$$= \int_0^t \left\langle \int_0^s Y_z dv_z^+ G, X_s^* U_s^- F \right\rangle ds + \int_0^t \left\langle Y_s G, V_s^- \int_0^s X_z^* du_z^- F \right\rangle ds,$$

and by duality,

$$\left\langle \int_0^t Y_s du_s^- G, \int_0^t X_s^* dv_s^+ F \right\rangle$$

$$= \int_0^t \left\langle U_s^- G, Y_s^* \int_0^s X_z^* dv_z^+ F \right\rangle ds + \int_0^t \left\langle V_s^- X_s \int_0^s Y_z du_z^- G, F \right\rangle ds,$$

for $F, G \in \mathcal{S}$. Those relations are written as

$$\int_0^t X_s du_s^+ \int_0^t Y_s dv_s^+ = \int_0^t du_s^+ X_s \left(\int_0^s Y_z dv_z^+ \right) + \int_0^t \left(\int_0^s X_z du_z^+ \right) Y_s dv_s^+,$$

$$\int_0^t X_s dv_s^- \int_0^t Y_s du_s^- = \int_0^t \left(\int_0^s X_z dv_z^- \right) Y_s du_s^- + \int_0^t dv_s^- X_s \left(\int_0^s Y_z du_z^- \right),$$

hence $du_t^+ \cdot dv_t^+ = 0$, and $du_t^- \cdot dv_t^- = 0$. We have from (7)

$$\Big\langle \int_0^t Y_s dv_s^- G, \int_0^t X_s^* du_s^- F \Big\rangle = \int_0^t \Big\langle \int_0^s Y_z dv_z^- G, X_s^* U_s^- F \Big\rangle ds + \int_0^t \Big\langle Y_s V_s^- G, \int_0^s X_z^* du_z^- F \Big\rangle ds,$$

hence

$$\int_0^t X_s du_s^+ \int_0^t Y_s dv_s^- = \int_0^t du_s^+ X_s \Big(\int_0^s Y_z dv_z^- \Big) + \int_0^t \Big(\int_0^s X_z du_z^+ \Big) Y_s dv_s^-,$$

which means that $du_t^+ \cdot dv_t^- = 0$. The non-zero terms are explicitly computed in [10]. We notice here that they can also be obtained formally from the well-known relations

$$d(a_t^{\ominus} + a_t^{\oplus}) \cdot d(a_t^{\ominus} + a_t^{\oplus}) = da_t^{\ominus} + da_t^{\oplus}, \quad da_t^- \cdot d(a_t^{\ominus} + a_t^{\oplus}) = da_t^-, \quad d(a_t^{\ominus} + a_t^{\oplus}) \cdot da_t^+ = da_t^+. \quad \blacksquare$$

The Itô multiplication table now gives the commutation relations

$$d[a_t^{\ominus}, a_t^+] = da_t^- + [a_t^{\ominus}, da_t^+], \qquad d[a_t^-, a_t^{\oplus}] = da_t^+ + [da_t^-, a_t^{\oplus}],$$

and

$$d[a_t^{\ominus}, a_t^{\oplus}] = da_t^{\circ} + [a_t^{\ominus}, da_t^{\oplus}] + [da_t^{\ominus}, a_t^{\oplus}].$$

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