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CONTROL ON WEAK ASYMPTOTIC ABELIANNESS WITH THE HELP OF THE CROSSED PRODUCT CONSTRUCTION

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Abstract. The crossed product construction is used to control in some examples the asymptotic behaviour of time evolution. How invariant states on a small algebra can be extended to invariant states on a larger algebra reduces to solving an eigenvalue problem. In some cases (the irrational rotation algebra) this eigenvalue problem has only trivial solutions and by reduction of the subalgebra control on all invariant states can be found.

1. Introduction. When one started to develop an ergodic theory in the framework of non-commuting systems [14] one had to assume some asymptotic behaviour of the evolution. There is a hierarchy of assumptions starting from G-abelianness through weak asymptotic abelianness leading to norm asymptotic abelianness. The purpose of these assumptions is twofold: on the one hand G-abelianness guarantees that an invariant state can be decomposed uniquely into extremal invariant states – as it is obviously true for abelian systems. On the other hand, it allows one to characterize equilibrium systems: these are those states where when the dynamics is locally disturbed, it is possible to find a state in the folium that is invariant under the perturbed dynamics [6]. Here noncommutativity of the observable algebra enters essentially. There is another description of equilibrium states: [10], we demand that after an adiabatical perturbation the system returns to its initial state. If we check what kind of asymptotic behaviour we need, we notice that in the first case weak asymptotic abelianness is sufficient, whereas in the second case scattering arguments were used that are based on the assumption of strong asymptotic abelianness.

If we turn to physical models checking whether the assumption of asymptotic abelianness is reasonable the only examples where we have good control on the asymptotic be-

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haviour already on the algebraic level are (quasi)free evolutions for Fermions or Bosons:

$$\tau_t\, a(f) = a(e^{iht}\, f).$$

For Fermions the anticommutators converge in norm to zero, which guarantees that the commutators of even elements (observables) converge in norm to zero. Here it is the attitude to consider only the even elements as relevant. This is not necessary because it can be shown that in every invariant state the odd elements are weakly asymptotic abelian, i.e. the expectation value of odd elements vanish. For Bosons, due to the unboundedness of the creation- and annihilation operators we remain with strong convergence of the corresponding Weyl operators in every representation. If we look for physical models with interaction we just find the XY-model [2] and the Luttinger model [9] for which we have some information for the long time behaviour: on a subalgebra we have norm resp. strong asymptotic abelianness but this does not carry over to the whole algebra and we have to wonder whether we can at least save weak asymptotic abelianness.

Our strategy is to use the crossed product construction to enlarge a suitable algebra. If we have sufficient control of the time evolution on the small algebra and sufficient control via the crossed product on the embedding then we are able to control the time evolution on the extended algebra: We have either to solve an algebraic cocycle equation or an eigenvalue equation to find the set of invariant states. In both cases, when there are only trivial solutions, i.e. if the corresponding crossed product symmetry is unbroken, or when one can find non-trivial solutions the system is weakly asymptotically abelian.

Finally we will apply the method to the known examples: We will extend the even Fermi algebra and we will show that there are only two extensions possible, one leading to the full Fermi algebra, the other leading to the XY-model and some generalizations [2, 11]. Also Bosonic free systems can be extended, allowing a large variety, one corresponding to the Luttinger model [9, 7]. Finally we shall show how for the CAT map on the irrational rotation algebra the crossed product construction not only allows to control the extension but in addition is so stringent that we can control the whole state [12].

2. The crossed product construction. Assume $\mathcal{A}_e$ is a given algebra and α an automorphism on $\mathcal{A}_e$ with $\alpha^2 A = W^* A W$, $W \in \mathcal{A}_e$. We define the (generalized) crossed product as

$$\begin{aligned} \widehat{\mathcal{A}} &= \mathcal{A}_e \overset{\alpha}{\bowtie} Z^{(2)} \subset A_e \otimes M^2 \\ \widehat{\mathcal{A}} &= \left\{ \begin{pmatrix} A_1 & A_2 \\ \alpha A_2 W & \alpha A_1 \end{pmatrix}; A_1, A_2 \in \mathcal{A}_e \right\}. \end{aligned} \tag{1}$$

On $\widehat{\mathcal{A}}$ exists an automorphism γ with $\gamma^2 = 1$,

$$\gamma \begin{pmatrix} A_1 & A_2 \\ \alpha A_2 W & \alpha A_1 \end{pmatrix} = \begin{pmatrix} A_1 & -A_2 \\ -\alpha A_2 W & \alpha A_1 \end{pmatrix} \tag{2}$$

so that $\mathcal{A}_e = \{\widehat{A}; \widehat{A} \in \widehat{\mathcal{A}}, \gamma \widehat{A} = \widehat{A}\}$. Automorphisms that are inner conjugated to α, i.e. $\alpha_x A = x \alpha A x^*$, $x \in \mathcal{A}_e$ lead with $W_x = x\alpha(x)W$ to another algebra $\widehat{\mathcal{A}}_x$ that is unitarily related to $\widehat{\mathcal{A}}$ via

$$\widehat{A}_x = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \widehat{\mathcal{A}} \begin{pmatrix} 1 & 0 \\ 0 & x^* \end{pmatrix}.$$

PROPOSITION 1. *Assume τ is an automorphism on $\widehat{\mathcal{A}}_e$. τ can be extended to an automorphism $\widehat{\tau}$ on $\widehat{\mathcal{A}}$ iff $\tau\alpha\tau^{-1}\alpha^{-1}$ is an inner automorphism on $\mathcal{A}_e$, i.e. $\tau\alpha\tau^{-1}\alpha^{-1}A = VAV^*$, for some $V \in \mathcal{A}_e$. Then*

$$\widehat{\tau}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & V \\ \alpha V & 0 \end{pmatrix}.$$

In addition we can conclude $VV^* = V^*V = 1$, $\alpha V = V^*$.

Remark 1. If $\mathcal{A}_e$ has trivial center, then according to the last constraint V is unique up to a sign, which corresponds that with $\widehat{\tau}$, only $\widehat{\tau} \circ \gamma$ is another possible extension of τ. Notice that $\widehat{\tau} \circ \gamma = \gamma \circ \widehat{\tau}$, i.e. γ is a symmetry of $\widehat{\mathcal{A}}$ with respect to $\widehat{\tau}$.

We want to learn about the asymptotic behaviour of $\widehat{\tau}$ if we know the asymptotic behaviour of τ. Therefore we will assume in the following that τ is weakly asymptotically abelian and for simplicity that $W \equiv 1$.

PROPOSITION 2. *Assume ω is an extremal τ invariant state on $\mathcal{A}_e$.*

i) There exists a unique extension $\widehat{\omega}$ to $\widehat{\mathcal{A}}$ satisfying $\widehat{\omega} = \widehat{\omega} \circ \widehat{\tau} = \widehat{\omega} \circ \gamma$. With $\omega(A) = \langle \Omega | \pi(A) | \Omega \rangle$ in the GNS representation this state can be realized as

$$\left\langle \begin{array}{c} \Omega \\ 0 \end{array} \middle| \begin{array}{cc} \pi(A_1) & \pi(A_2) \\ \pi(\alpha A_2 W) & \pi(\alpha A_1) \end{array} \middle| \begin{array}{c} \Omega \\ 0 \end{array} \right\rangle.$$

ii) If ω_1 is another extension then

$$\omega_1 \leq 2\widehat{\omega} = \omega_1 + \omega_1 \circ \gamma.$$

iii) If π_ω and $\pi_\omega \circ \alpha$ are disjoint then

$$\pi_{\widehat{\omega}}(\widehat{\mathcal{A}})' = \left\{ \begin{pmatrix} A' & 0 \\ 0 & A' \end{pmatrix}, A' \in \pi_\omega(\mathcal{A}_e)' \right\}.$$

iv) If π_ω and $\pi_\omega \circ \alpha$ are unitarily equivalent, i.e.

$$\pi_\omega(\alpha A) = X\pi_\omega(A)X^*, \quad X \in \mathcal{B}(\mathcal{H}_\omega), \quad X \text{ unitary}, \quad X^2 = 1,$$

then

$$\pi_{\widehat{\omega}}(\widehat{\mathcal{A}})' = \left\{ \begin{pmatrix} A_1' & A_2'X \\ A_2'X & A_1' \end{pmatrix}, A_1', A_2' \in \pi_\omega(\mathcal{A}_e)' \right\}.$$

Proof. See [11]. ■

THEOREM 3. *Let ω be a τ invariant state over $\mathcal{A}_e$, faithful on $\pi_\omega(\mathcal{A}_e)''$. Let U implement τ with*

$$\omega(A) = \langle \Omega | \pi_\omega(A) | \Omega \rangle, \quad \pi_\omega(\tau A) = U\pi_\omega(A)U^*, \quad U|\Omega\rangle = |\Omega\rangle.$$

There exists a $\widehat{\tau}$ invariant extension ω_1, $\omega_1 \neq \widehat{\omega}$, if one of the four equivalent conditions is satisfied

i) $\exists\, \widehat{T} \geq 0 \in \pi_{\widehat{\omega}}(\widehat{\mathcal{A}})'$, $\widehat{\tau}\widehat{T} =: \widehat{U}\widehat{T}\widehat{U}^* = \widehat{T}$.

ii) $\exists\, W$ *with* $\pi_\omega(\alpha A) = W\pi_\omega(A)W$ *and* $W = W^*$, $V = \tau(W)\cdot W$.

iii) $\exists\, B \in \pi_\omega(\mathcal{A}_e)'$ *and* $W_0 \in \mathcal{B}(\mathcal{H})$ *such that*

$$[W_0, U] = 0, \quad V = \tau(B)B^*, \quad \pi_\omega(\alpha A) = BW_0\pi_\omega(A)W_0^*B^*.$$

iv) $\exists\, |\psi\rangle$ *with* $U|\psi\rangle = V|\psi\rangle$.

Proof. Evidently one passes from (i) to (iii) by specifying the solution. For W satisfying (ii) we get (iv) by taking $|\psi\rangle = W|\Omega\rangle$. On the other hand, $|\psi\rangle$ implements $\omega\circ\alpha$ which gives the desired W. ∎

The same argument gives

THEOREM 4. *Assume ω is irreducible. Assume $U|\psi\rangle = V|\psi\rangle$ has a non-trivial solution. Then α is inner in $\pi_\omega(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$.*

THEOREM 5. *Assume U has apart from the eigenvector Ω continuous spectrum. Then V^*U can at most have an eigenvector to the eigenvalue 1. Otherwise the spectrum is continuous.*

This result suffices to guarantee weak asymptotic abelianness. Either V^*U has no eigenvector, then $\widehat{\omega}$ cannot be decomposed further into $\widehat{\tau}$ invariant states and in $\widehat{\omega}$ all expectation values $\widehat{\omega}(A\tau^n B)$ tend for $n\to\infty$ to $\widehat{\omega}(A)\widehat{\omega}(B)$. If, on the other hand, V^*U has an eigenvalue and thus (3,ii) has a solution $\overline{W}$, then $\widehat{\omega}$ can be decomposed by the projection operators

$$\frac{1}{2}\begin{pmatrix} 1 & \pm\overline{W} \\ \pm\overline{W} & 1 \end{pmatrix}$$

into extremal $\widehat{\tau}$ invariant states and this decomposition is unique. Then

$$\text{w-}\lim_{n\to\infty}(\widehat{\tau})^n\begin{pmatrix} 1 & \pm\overline{W} \\ \pm\overline{W} & 1 \end{pmatrix} = \left|\begin{array}{c}\Omega\\ \pm\psi\end{array}\right\rangle\left\langle\begin{array}{c}\Omega\\ \pm\psi\end{array}\right|$$

which guarantees clustering in extremally $\widehat{\tau}$ invariant states and thus weak asymptotic abelianness in $\widehat{\tau}$ invariant states.

Remark 2. For simplicity we assumed $\alpha^2 = 1$. Generalizations to $\alpha^n = 1$, $\gamma^n = 1$, are evident, if we assume $\widehat{\tau}\gamma = \gamma\widehat{\tau}$. Especially with $\tau\alpha^k\tau^{-1}\alpha^{-k} = \mathrm{ad}\, V_k$ we have to find solutions of

$$U|\psi_k\rangle = V_k|\psi_k\rangle$$

in order to construct invariant states that break the symmetry γ.

3. Extensions of the even Fermi algebra. We consider the even Fermi algebra on a one-dimensional lattice, i.e. even polynomials of creation- and annihilation operators. We are interested in the quasilocal structure of the algebra, i.e. we have an imbedding

$$\mathcal{A}_{e,\Lambda} \subset \mathcal{A}_{e,\bar{\Lambda}}$$

if $\Lambda \subset \bar{\Lambda}$ are subregions on the lattice. We notice that for $\mathcal{A}_{e,\Lambda}$ there exists exactly one crossed product extension, all automorphisms of $\mathcal{A}_{e,\Lambda}$ satisfying $\alpha^2 = 1$ are inner conjugated and there are no other non-inner automorphisms. We demand from our extension that

$$\widehat{\mathcal{A}}_{e,\Lambda} \subset \widehat{\mathcal{A}}_{e,\bar{\Lambda}}$$

i.e. if $\alpha_{\bar{\Lambda}}$ is an automorphism of $\mathcal{A}_{e,\bar{\Lambda}}$, reduced to $\mathcal{A}_\Lambda$ it has to coincide with α_Λ. On $\mathcal{A}_{e,0}$ there exists exactly one automorphism, implemented by $\sigma_0^x \approx a_0 + a_0^\dagger$. On $\mathcal{A}_{e,[0,1]}$ considered as subalgebra of the full matrix algebra $M_0^2 \times M_1^2$ we can extend it to the one implemented by $\sigma_0^x(\sigma_1^z)$ or σ_0^x. Continuing in this way $\alpha_{\bar{\Lambda}}$ has to be implemented by

$$\sigma_o^x \prod_{\substack{j\in\Lambda \\ j\neq 0}} (\sigma_j^z)^{k(j)} \qquad \text{with } k(j) \in \{0,1\}. \tag{3}$$

Space translations have to be extendable to the crossed product algebra, i.e. $\tau_\ell \alpha \tau_{-\ell} \alpha^{-1}$ correspond to

$$\sigma_0^x \prod_{j\neq 0} (\sigma_j^z)^{k(j)} \cdot \sigma_0^x \prod_{j\neq 0} (\sigma_j^z)^{k(j+\ell)} \tag{4}$$

and this operator has to be quasilocal, i.e.

$$\lim_{j\to+\infty} (k(j) - k(j+\ell)) = 0, \qquad \lim_{j\to-\infty} (k(j) - k(j+\ell)) = 0. \tag{5}$$

Therefore

$$\lim_{j\to+\infty} k(j) = 1 \text{ or } 0, \qquad \lim_{j\to-\infty} k(j) = 1 \text{ or } 0. \tag{6}$$

This gives exactly two possible extensions, namely $(1,0) \approx (0,1)$ corresponding to the Fermi algebra, whereas $(0,0) \approx (1,1)$ corresponds to the matrix algebra on the lattice.

The asymptotic behaviour with respect to space translations is reflected for Fermions, (1,0):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widehat{\tau}_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\widehat{\tau}_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \ell \neq 0, \tag{7}$$

and for the lattice, (0,0):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widehat{\tau}_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \widehat{\tau}_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \ell \neq 0, \tag{8}$$

which results from

$$\sigma_0^x \prod_{j=1}^{\infty} \sigma_j^z \cdot \sigma_\ell^x \prod_{j=\ell+1}^{\infty} \sigma_j^z = -\sigma_\ell^x \prod_{j=\ell+1}^{\infty} \sigma_j^z \cdot \sigma_0^x \prod_{j=1}^{\infty} \sigma_j^z$$

or

$$\sigma_0^x \sigma_\ell^x = \sigma_\ell^x \sigma_0^x.$$

3.1. Symmetry breaking for Fermions. Let $\widehat{\tau}_t$ be a quasifree evolution $\widehat{\tau}_t a(f) = a(e^{iht} f)$ where h has absolutely continuous spectrum. Therefore τ_t is norm (and thus also weakly) asymptotically abelian on $\mathcal{A}_e$. Therefore we can apply Theorem 3.iv and look for a solution to

$$U_t|\psi\rangle = V_t|\psi\rangle \qquad \forall\, t$$

with $V_t = (a(e^{iht}f) + a^\dagger(e^{iht}f))(a(f) + a^\dagger(f))$. For appropriate choice of f with respect to h

$$\alpha_f\, V_t = -V_t \qquad |t| > t_0.$$

Therefore

$$\begin{aligned}\lim_{t\to\infty}\langle\psi|U_t|\psi_t\rangle &= \lim_{t\to\infty}\langle\psi|V_t|\psi\rangle = \lim_{t\to\infty}\langle V_t^*\psi|\psi\rangle = \lim_{t\to\infty}\langle\alpha V_t\psi|\psi\rangle \\ &= \lim_{t\to\infty} -\langle V_t\psi|\psi\rangle = \lim_{t\to\infty} -\langle\psi|U_t^*|\psi\rangle\end{aligned}$$

and thus $\langle\psi|\Omega\rangle = 0$. This implies [11] that $\psi = 0$.

3.2. XY model. We apply our method to the XY model with the following result: The algebra of the XY model is $\bigotimes_{\ell\in Z} M_\ell^2$ and the time evolution is determined by the Hamiltonian

$$H = -J\sum[(1+c)\sigma_j^x\sigma_{j+1}^x + (1-c)\sigma_j^y\sigma_{j+1}^y + 2\lambda\sigma_j^z], \quad c\in(0,1), \quad \lambda\in R.$$

With $\gamma\sigma_\ell^{x,y} = -\sigma_\ell^{x,y}$ the fixed point algebra of γ is equivalent to the even Fermi algebra and stable under time evolution. The corresponding Hamiltonian for the Fermi algebra is quasifree with continuous spectrum, corresponding to U. The evolution corresponding to V^*U is also quasifree, related to U by a scattering transformation connected with an odd Bogoliubov transformation for $|\lambda| < 1$ and an infinite Bogoliubov transformation otherwise [2]. For the lattice system an additional Bogoliubov transformation is needed. Together with the previous one it is inner for $|\lambda| < 1$. The symmetry γ can be broken iff the remaining scattering automorphism satisfies the cocycle property (Theorem 3.ii). This holds in the ground state (vacuum state) but not in KMS states or other faithful quasifree states. In all cases the system is weakly asymptotically abelian.

4. Algebraic Fermion Bosonization. We follow the model as presented in [1] and refer to [8, 4, 9, 5] for its connection to interacting Fermions.

The algebra we start with is the Weyl algebra $\mathcal{W}(C_0^\infty \times C_0^\infty)$ with commutation relation

$$W(f_1,g_1)W(f_2,g_2) = \exp\left[i\int(f_1'g_2 - f_2'g_1)\right] W(f_1+f_2, g_1+g_2).$$

We can extend the algebra either by extending the test function space or by applying a crossed product construction with the automorphism group

$$\alpha_{\widehat{f},\widehat{g}} W(f_1,g_1) = \exp\left[i\int(\widehat{f}'g_1 - f_1'\widehat{g})\right] W(f_1,g_1).$$

On $\mathcal{W}(C_0^\infty \times C_0^\infty)$ space translations are defined. They can be extended to

$$\widehat{\mathcal{A}}_{\widehat{f},\widehat{g}} = \mathcal{W} \overset{\alpha_{\widehat{f},\widehat{g}}}{\bowtie} Z$$

iff $\widehat{f}(x+y) - \widehat{f}(x) \in C_0^\infty$, $\widehat{g}(x+y) - \widehat{g}(x) \in C_0^\infty$. Therefore we can allow that

$$\lim_{x\to\infty}\widehat{f}(x) - \lim_{x\to-\infty}\widehat{f}(x) =: \widehat{f}(\infty) - \widehat{f}(-\infty) = \widehat{f}_0 \neq 0$$

and similarly $\widehat{g}(\infty) - \widehat{g}(-\infty) = \widehat{g}_0 \neq 0$. The additional unitary operators in $\widehat{\mathcal{A}}_{\widehat{f},\widehat{g}}$ (i.e. $W(\widehat{f},\widehat{g})$) are anticommuting for large distances iff

$$\begin{aligned}&\lim_{x\to\infty}\exp\left[i\int(\widehat{f}'(y+x)\widehat{g}(y) - \widehat{f}'(y)\widehat{g}(x+y))dy\right] = \\ &\quad = \exp\left[i(\widehat{f}(\infty) - \widehat{f}(-\infty))(\widehat{g}(-\infty) - \widehat{g}(\infty))\right] = -1.\end{aligned}$$

If we consider

$$\widehat{\mathcal{A}}_{\widehat{f};\widehat{g},\lambda} = W \overset{\alpha_{\widehat{f},0}}{\bowtie} Z \overset{\alpha_{0,\widehat{g}}}{\bowtie} Z$$

with $\widehat{f}(\infty) - \widehat{f}(-\infty) = \lambda\sqrt{\pi}$, $\widehat{g}(\infty) - \widehat{g}(-\infty) = \frac{1}{\lambda}\sqrt{\pi}$ we can find a one-parameter family of extensions with Fermionic (i.e. anticommuting) character.

If in addition to space translation we consider a time evolution (that results from the Luttinger model or from the Schwinger model)

$$\tau_t\, W(f,g) = W(f_t, g_t)$$

with

$$\widetilde{f}_t(p) = \exp\left[i\omega(|p|)t\right]\widetilde{f}(p), \qquad \widetilde{g}_t(p) = \exp\left[-i\omega(|p|)t\right]\widetilde{g}(p)$$

where $\omega(|p|) = |p|(1 - \widetilde{V}(p))^{1/2}$ or in the case of the Schwinger model

$$\omega_S(|p|) = |p|\left(1 + \frac{m^2}{p^2}\right)^{1/2}$$

we see that the automorphism τ_t is only extendable to $\widehat{\mathcal{A}}$ if

$$\lim \frac{\omega(|p|)}{|p|} < \infty.$$

In this case the time evolution $\widehat{\tau}_t$ on the extended algebra inherits the asymptotic commutation relations of the space translations.

The symmetry γ that acts on the $\widehat{\mathcal{A}}$ and leaves W pointwise invariant commutes by construction with space translation and time translation. Due to the anticommutativity the method of (9) can be applied and again symmetry breaking cannot occur, i.e. all invariant states have to be even. (We cannot find a non-trivial solution of Theorem 3.iv in the odd sectors.)

On our algebra $\mathcal{W}(C_0^\infty \times C_0^\infty)$ we can also define other automorphisms, namely e.g.

$$\alpha_{a,b}\, W(f,g) = e^{iaf(0)+ibg(0)} W(f,g).$$

The advantage of this automorphism is that it is strictly local (compare Section 3), the disadvantage that $\sigma_x \alpha_{a,b} \sigma_{-x} \alpha_{a,b}^{-1}$ is not an inner automorphism and therefore cannot be extended to $\mathcal{W} \overset{\alpha_{a,b}}{\bowtie} Z$ if we choose $W(C_0^\infty \times C_0^\infty)$. But $\alpha_{a,b}$ can be obtained as $\alpha_{a,b} = \lim \alpha_{f_n,g_n}$ where f_n, g_n are smooth functions with $\lim f_n(x) = a\Theta(x)$, $\lim g_n(x) = -b\Theta(x)$. If we have a covariant representation of $\mathcal{W}(C_0^\infty \times C_0^\infty)$ according to the Riesz extension theorem [13] this limiting procedure can be used to construct in an appropriate extended Hilbert space

$$\widehat{\mathcal{A}} = \bigcup_x U_x(\mathcal{W} \overset{\alpha_{a,b}}{\bowtie} Z) U_x^* = W \overset{\alpha}{\bowtie} G.$$

If we consider a representation where U_x belongs to $\pi_\omega(\mathcal{W})''$ no extension of the Hilbert space is necessary. The same is true if the weak closure of $\pi_\omega(\mathcal{W})''$ allows step function. In all these cases $a \cdot b = \pi$ guarantees weak asymptotic abelianness on the odd elements and strong asymptotic abelianness for the even elements.

5. The CAT map. We consider the irrational rotation algebra built by unitary operators $W_\Theta(n_1,n_2)$, $n_1,n_2 \in Z$ that satisfy the Weyl relation

$$W_\Theta(n_1,n_2)W_\Theta(\bar n_1,\bar n_2) = e^{i\pi\Theta(n_1\bar n_2 - n_2\bar n_1)}W_\Theta(n_1+\bar n_1, n_2+\bar n_2).$$

If Θ is irrational then the algebra has trivial center. On $\mathcal{A}_\Theta$ a matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in Z$, ad $-bc = 1$, defines an automorphism α_T

$$\alpha_T\, W_\Theta(\vec n) = W_\Theta(T\vec n).$$

$\mathcal{A}_{\Theta k^2} = \{W_\Theta(n_1k, n_2k)\}$ is a subalgebra of $\mathcal{A}_\Theta$, stable under α_T. It is the fixpoint algebra under the automorphism group

$$\gamma_{k_1,k_2}W_\Theta(n_1,n_2) = e^{2\pi i(n_1k_1+n_2k_2)/k}W_\Theta(n_1,n_2).$$

For given k there always exists some ℓ [3] such that

$$\gamma_{k_1,k_2}\,\alpha_T^\ell = \alpha_T^\ell\,\gamma_{k_1,k_2} \qquad \forall\, k_1,k_2.$$

$\mathcal{A}_\Theta$ can be obtained from $\mathcal{A}_{\Theta k^2}$ as generalized crossed product with the automorphisms implemented by $W_\Theta(1,0)$ and $W_\Theta(0,1)$ [12]. The construction (9) finds its analogue

$$U_T^\ell|\vec k\rangle = W(f_\ell(\vec k)|\vec k\rangle$$

with

$$\alpha_{\vec k}\; W(f_\ell(\vec k)) = e^{ic(\ell,\vec k)\Theta}W(f_\ell(\vec k)).$$

$c(\ell,\vec k)$ is a rapidly varying integer. If we apply the theorem of Van der Corput of the theory of uniform distributions to

$$e^{ic(\ell,\vec k)\Theta}\;\langle\psi_{\vec k}|U^\ell|\psi_{\vec k}\rangle$$

we see [12] that with probability 1 on Θ this term has 0 as invariant mean over ℓ. As in (9) we conclude

$$\langle\psi_{\vec k}|\Omega\rangle = 0, \quad |\psi_{\vec k}\rangle = 0, \quad \omega_\Theta(W(k_1,k_2)) = 0 \quad \text{for } 0 < k_1,k_2 < k.$$

Since k can be chosen arbitrarily, with probability 1 on Θ

$$\omega_\Theta(W(k_1,k_2)) = 0, \qquad k_1 \neq 0,\; k_2 \neq 0.$$

This shows that for these Θ the only state invariant under α_T is the tracial state.

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