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QUANTUM SYMMETRIES IN NONCOMMUTATIVE C^*-SYSTEMS

MARCIN MARCINIAK

Institute of Mathematics, Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: matmm@univ.gda.pl

Abstract. We introduce the notion of a completely quantum C^*-system (A, G, α), i.e. a C^*-algebra A with an action α of a compact quantum group G. Spectral properties of completely quantum systems are investigated. In particular, it is shown that G-finite elements form the dense *-subalgebra $\mathcal{A}$ of A. Furthermore, properties of ergodic systems are studied. We prove that there exists a unique α-invariant state ω on A. Its properties are described by a family of modular operators $\{\sigma_z\}_{z\in\mathbb{C}}$ acting on $\mathcal{A}$. It turns out that ω is a KMS state provided that ω is faithful.

1. Introduction. By a classical space we mean a locally compact topological space X. Symmetries of a classical space are described by groups of homeomorphisms of X. More precisely, a system with a symmetry is a triple (X, G, τ) where X is a classical space, and $\tau = \{\tau_g\}_{g\in G}$ is a representation of locally compact topological group G in the group of homeomorphisms of X. Every system with a symmetry has the dual picture $(C(X), G, \alpha)$ where $C(X)$ is the C^*-algebra of complex continuous functions on X vanishing at infinity, and $\alpha = \{\alpha_g\}_{g\in G}$ is the action of G on the algebra $C(X)$ defined by $[\alpha_g(f)](x) = f(\tau_{g^{-1}}(x))$, $f \in C(X)$, $x \in X$. By the well-known Gelfand-Naimark theorem every triple (A, G, α) where A is an abelian C^*-algebra, and α is an action of a locally compact group G on A (*cf.* [9]), can be reconstructed from some classical system with a symmetry in the described above way. A natural generalization of a system with symmetry is that with A being an arbitrary (i.e. noncommutative in general) C^*-algebra.

The aim of this paper is to investigate basic questions of a description of symmetries of noncommutative systems. There are some reasons to believe that the class of topological (classical) groups is not sufficient to describe the relevant full symmetries. For a deeper discussions of this problem with the framework of algebraic quantum field theory we refer

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the reader to [4] and references given there (see also [3, 6], as well as [7]).

In our paper we will use the theory of compact quantum groups developed by Woronowicz in [13, 14]. In Section 2 we recall some basic definitions and results of this theory. Next, in Section 3 we formulate the notion of completely quantum system (A, G, α), where A is a unital C^*-algebra, G is a compact quantum group, and α is an action of G on A. Motivated by [1, 2, 10] we treat completely quantum C^*-systems as a noncommutative system with a generalized (quantum) symmetry. We analyse spectral properties of complete quantum systems. In particular, we investigate the structure of the set $\mathcal{A}$ of G-finite elements of (A, G, α). Section 4 is devoted to ergodic systems. We prove that if α is an ergodic action, then there exists a unique α-invariant state ω on A. It is possible to describe the modular properties of ω by means of the family $\{\sigma_z\}_{z\in\mathbb{C}}$ of linear operators on $\mathcal{A}$. It appears that if ω is faithful then it is a KMS state with respect to the group $\{\sigma_t\}_{t\in\mathbb{R}}$. It generalizes the results from [5] that if α is an ergodic action of a compact topological group then, the unique α-invariant state is a trace on A.

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2. Compact quantum groups. In this section we briefly recall basic definitions and properties of compact quantum groups defined by Woronowicz. We follow [13, 14].

If A is a C^*-algebra, and $X, Y \subset A$ are subsets of A, then we define $XY = \operatorname{span}\{xy : x \in X,\, y \in Y\}$.

Definition 2.1 ([14]). A *compact quantum group* is a pair $G = (C, \varDelta)$, where C is a separable unital C^*-algebra, and $\varDelta : C \longrightarrow C \otimes C$ is a unital *-homomorphism such that $(\varDelta \otimes \mathrm{id}_C)\varDelta(b) = (\mathrm{id}_C \otimes \varDelta)\varDelta(b)$ for every $b \in C$ and the subspaces $(C \otimes \mathbb{1}_C)\varDelta(C)$, $(\mathbb{1}_C \otimes C)\varDelta(C)$ are dense in $C \otimes C$.

Let us remind that if K is a finite dimensional linear space, then by a *representation* of G acting on K we mean a linear map $v : K \longrightarrow K \otimes C$ such that

$$(v \otimes \mathrm{id}_C)v = (\mathrm{id}_K \otimes \varDelta)v. \tag{2.1}$$

If $e_1, e_2, \ldots, e_d$ is a basis of K, then there are uniquely determined elements $v_{ij} \in C$, $i, j = 1, 2, \ldots, d$, such that $v(e_j) = \sum_{i=1}^d e_i \otimes v_{ij}$ for $j = 1, 2, \ldots, d$. The condition (2.1) implies $\varDelta(v_{ij}) = \sum_{k=1}^d v_{ik} \otimes v_{kj}$ for every $i, j = 1, 2, \ldots, d$. The matrix $[v_{ij}]_{i,j=1,2,\ldots,d} \in M_d(\mathbb{C}) \otimes C$ is called the *matrix of the representation* v in the basis $e_1, e_2, \ldots, e_d$.

Let v, w be representations of G acting on spaces K, L respectively. Then an operator $S : K \longrightarrow L$ is called an *intertwiner* between v and w if $(S \otimes \mathrm{id}_C)v = wS$. The set of all intertwiners between v and w is denoted by $\mathrm{Mor}(v, w)$. Representations v and w are called *equivalent* if $\mathrm{Mor}(v, w)$ contains an isomorphism of the linear spaces K and L.

A representation v acting on K is *irreducible* if $\mathrm{Mor}(v, v) = \{\lambda\mathbb{1}_{B(K)} : \lambda \in \mathbb{C}\}$, where $\mathbb{1}_{B(K)}$ denotes the identity operator on K.

If K is a finite dimensional Hilbert space, then a representation v acting on K is called *unitary* if for some orthonormal basis $e_1, e_2, \dots, e_d$ of K the matrix $[v_{ij}]$ of v in the basis is a unitary element of the algebra $M_d(\mathbb{C}) \otimes C$. This is equivalent to $\sum_{k=1}^d v_{ik} v_{jk}^* = \delta_{ij} \mathbb{1}_C = \sum_{k=1}^d v_{ki}^* v_{kj}$ for any $i, j = 1, 2, \dots, d$.

A state h on the C^*-algebra is called a *Haar measure* if $(\mathrm{id}_C \otimes h)\Delta(b) = h(b)\mathbb{1}_C = (h \otimes \mathrm{id}_C)\Delta(b)$ for every $b \in C$. It is proved in [14] that every compact quantum group admits a unique Haar measure.

Let $\hat{G}$ denote the set of equivalence classes of unitary representations of the group G. For $\tau \in \hat{G}$ by $[u_{ik}^\tau]_{i,k=1,2,\dots,d_\tau}$ we denote the matrix of some representative u^τ of the class τ. By ι we denote the equivalence class of the *trivial representation*, i.e. $d_\iota = 1$ and $u_{11}^\iota = \mathbb{1}_C$. In [14] Woronowicz proved that the set $\{u_{ik}^\tau : \tau \in \hat{G},\ i, k = 1, 2, \dots, d_\tau\}$ forms a linear basis of a dense *-subalgebra $\mathcal{C}$ of C. Moreover, for every $\tau \in \hat{G}$ there is a unique invertible matrix $F_\tau \in M_{d_\tau}(\mathbb{C})$ such that $\operatorname{Tr} F_\tau = \operatorname{Tr} F_\tau^{-1}$ and the following Peter-Weyl-Woronowicz relations hold:

$$h(u_{ik}^\tau {u_{jl}^\sigma}^*) = \frac{1}{\operatorname{Tr} F_\tau} \delta_{\tau\sigma} \delta_{ij} (F_\tau)_{kl}, \qquad h({u_{ik}^\tau}^* u_{jl}^\sigma) = \frac{1}{\operatorname{Tr} F_\tau} \delta_{\tau\sigma} \delta_{kl} (F_\tau^{-1})_{ij} \tag{2.2}$$

for every $\tau, \sigma \in \hat{G}$, $i, k = 1, 2, \dots, d_\tau$ and $j, l = 1, 2, \dots, d_\sigma$.

Let $\tau \in \hat{G}$ and $i, k = 1, 2, \dots, d_\tau$. If $b \in C$ then let $\rho_{ik}^\tau(b) = \operatorname{Tr} F_\tau \sum_{p=1}^{d_\tau} (F_\tau)_{ip} h({u_{pk}^\tau}^* b)$ and $\rho^\tau(b) = \sum_{i=1}^{d_\tau} \rho_{ii}^\tau(b)$. Let also $D_{ik}^\tau : C \longrightarrow C$ be linear operators defined by the formula $D_{ik}^\tau b = (\mathrm{id}_C \otimes \rho_{ik}^\tau)\Delta(b)$ for every $b \in C$. From (2.2) we easily get

PROPOSITION 2.2. *If $\tau, \pi \in \hat{G}$, $i, k = 1, 2, \dots, d_\tau$, $j, l = 1, 2, \dots, d_\pi$ then*

(*a*) $\rho_{ik}^\tau(u_{jl}^\pi) = \delta_{\tau\pi} \delta_{ij} \delta_{kl}$, $\rho^\tau(u_{jl}^\pi) = \delta_{\tau\pi} \delta_{jl}$, $D_{ik}^\tau u_{jl}^\pi = \delta_{\tau\pi} \delta_{kl} u_{ji}^\tau$,
(*b*) $D_{ik}^\tau D_{jl}^\pi = \delta_{\tau\pi} \delta_{jk} D_{il}^\tau$; D_{ii}^τ *is a projection onto* $D_{ii}^\tau C = \operatorname{span}\{u_{1i}^\tau, u_{2i}^\tau, \dots, u_{d_\tau i}^\tau\}$,
(*c*) $h((D_{ik}^\tau b)^* c) = h(b^* (D_{ki}^\tau c))$ *for every* $b, c \in C$.

3. Completely quantum C^*-systems

DEFINITION 3.1 ([10]). Let A be a unital C^*-algebra and $G = (C, \Delta)$ be a compact quantum group. A unital *-homomorphism $\alpha : A \longrightarrow A \otimes C$ will be called an *action* of the group G on A if $(\alpha \otimes \mathrm{id}_C)\alpha(x) = (\mathrm{id}_A \otimes \Delta)\alpha(x)$ for every $x \in A$, and the subspace $(\mathbb{1}_A \otimes C)\alpha(A)$ is dense in $A \otimes C$.

The triple (A, G, α) will be called a *completely quantum C^*-system*.

DEFINITION 3.2. Let (A, G, α) be a completely quantum C^*-system. A *G-module* in (A, G, α) is a finite dimensional linear subspace $X \subset A$ such that $\alpha(X) \subset X \otimes_{\mathrm{alg}} C$, where $X \otimes_{\mathrm{alg}} C = \operatorname{span}\{x \otimes b : x \in X,\ b \in C\}$.

An element $x \in A$ is called *G-finite* if $x \in X$ for some G-module X.

Let us observe that if X is a G-module then $\alpha|_X : X \longrightarrow X \otimes_{\mathrm{alg}} C$ is a representation of G on the finite dimensional space X. Two G-modules X, Y are called *isomorphic* if $\alpha|_X$ and $\alpha|_Y$ are equivalent. A G-module X is called *simple* if $\alpha|_X$ is irreducible.

For every subset $E \subset \hat{G}$ let $M^\alpha(E)$ denote the closed subspace in A generated by elements of G-modules equivalent with u^τ for some $\tau \in E$. We will write $M^\alpha(\tau)$ instead of $M^\alpha(\{\tau\})$.

PROPOSITION 3.3. *Let* $P^\tau_{ik}(x) = (\mathrm{id}_A \otimes \rho^\tau_{ik})\alpha(x)$ *for every* $\tau \in \hat{G}$, $i, k = 1, 2, \ldots, d_\tau$, $x \in A$, *and let* $P^\tau(x) = \sum_{i=1}^{d_\tau} P^\tau_{ii}(x)$ *for every* $\tau \in \hat{G}$, $x \in A$. *Then for every* $\tau, \pi \in \hat{G}$, $i, k = 1, 2, \ldots, d_\tau$ *and* $j, l = 1, 2, \ldots, d_\pi$ *we have*

(*a*) $P^\tau_{ik} P^\pi_{jl} = \delta_{\tau\pi}\delta_{jk} P^\tau_{il}$; P^τ_{ii}, P^τ *are projections*;
(*b*) $\alpha(P^\tau_{ik}(x)) = (\mathrm{id}_A \otimes D^\tau_{ik})\alpha(x)$, $\alpha(P^\tau(x)) = (\mathrm{id}_A \otimes D^\tau)\alpha(x)$, *for* $x \in A$.

Proof. It is a simple consequence of Proposition 2.2. ∎

PROPOSITION 3.4 ([10, 8]). *For every* $\tau \in \hat{G}$ *and* $x \in A$ *the following conditions are equivalent*:

(*a*) $x \in M^\alpha(\tau)$;
(*b*) $x \in P^\tau A$;
(*c*) *there exist* $n \le d_\tau$ *and linearly independent* G*-modules* $X_1, X_2, \ldots, X_n$ *equivalent to* u^τ *such that* $x \in \bigoplus_{i=1}^n X_i$.

Remark 3.5. Definition 3.1 was first given in [10]. It was shown that $M^\alpha(\hat{G}) = A$ (*cf.* [10, Theorem 1.5]). Moreover, it was proved that for every $\tau \in \hat{G}$ there exist a set of indices $\mathcal{J}_\tau$ and G-modules X^τ_μ equivalent to u^τ such that $M^\alpha(\tau) = \bigoplus_{\mu \in \mathcal{J}_\tau} X^\tau_\mu$. In spite of the fact that X^τ_μ are not uniquely determined, the cardinality of $\mathcal{J}_\tau$ does not depend on the choice of these subspaces. This cardinality is called the multiplicity of u^τ in (A, G, α) and is denoted by c_τ.

As a consequence of the above remark and Proposition 3.4 we get

THEOREM 3.6. *Suppose that* (A, G, α) *is a completely quantum* C^**-system. Let* $\mathcal{A} = \{x \in A : \alpha(x) \in A \otimes_{\mathrm{alg}} \mathcal{C}\}$. *Then* $\mathcal{A}$ *is a dense* **-subalgebra of* A *invariant with respect to* α, *i.e.* $\alpha(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{C}$. *Moreover*

(*a*) $\mathcal{A} = \bigoplus_{\tau \in \hat{G}} M^\alpha(\tau)$,
(*b*) $(\mathrm{id}_A \otimes e)\alpha(x) = x$ *for* $x \in \mathcal{A}$,
(*c*) $(\mathrm{id}_A \otimes m)(\alpha \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \kappa)\alpha(x) = x \otimes \mathbb{1}_C$, *for* $x \in \mathcal{A}$,

where e *is the counit and* κ *is the coinverse of the group* G (*cf.* [14, Theorem 1.2]), *and* $m : \mathcal{C} \otimes_{\mathrm{alg}} \mathcal{C} \longrightarrow \mathcal{C}$ *is the multiplication map.*

Remark 3.7. In [14] it is proved that the *-algebra $\mathcal{C}$ spanned by the elements u^τ_{ij} with comultiplication Δ has the structure of a *-Hopf algebra. It follows from Theorem 3.6 that the *-algebra $\mathcal{A}$ of G-finite elements with α is a right $\mathcal{C}$-comodule (*cf.* [11]).

COROLLARY 3.8. *Let* (A, G, α) *be a completely quantum* C^**-system*, *and let* $\mathcal{A}$ *be the* **-subalgebra defined in Theorem 3.6. Then*, *for every* $x \in \mathcal{A}$, $\alpha(x) = 0$ *implies* $x = 0$.

Proof. Let $x \in \mathcal{A}$ and $\alpha(x) = 0$. From Proposition 3.4 we get $x = \sum_{j=1}^n x_j$, where $x_j \in X_j$, $j = 1, 2, \ldots, n$, and $X_1, X_2, \ldots, X_n$ are linearly independent simple G-modules. The system of vectors $\{\alpha(x_j) : j = 1, 2, \ldots, n\}$ is linearly independent, because the subspaces $X_j \otimes_{\mathrm{alg}} C$ are linearly independent in $A \otimes C$. But $\sum_j \alpha(x_j) = \alpha(x) = 0$, so $\alpha(x_j) = 0$ for every $j = 1, 2, \ldots, n$. For $j = 1, 2, \ldots, n$ let $x_{1j}, x_{2j}, \ldots, x_{d_\tau j}$ be a basis of X_j such that $\alpha(x_{ij}) = \sum_{k=1}^{d_\tau} x_{kj} \otimes u_{ki}$. Then $x_j = \sum_{i=1}^{d_\tau} \lambda_i x_{ij}$ for some $\lambda_i \in \mathbb{C}$, $i = 1, 2, \ldots, d_\tau$. So, $0 = \alpha(x_j) = \sum_i \lambda_i \sum_k x_{kj} \otimes u_{ki} = \sum_{i,k} \lambda_i x_{kj} \otimes u_{ki}$. The matrix elements u_{ki} are linearly independent, hence $\lambda_i x_{kj} = 0$ for every $k, i = 1, 2, \ldots, d_\tau$. The elements x_{kj} are nonzero, so $\lambda_i = 0$ for every $i = 1, 2, \ldots, d_\tau$. This implies $x_j = 0$ for every $j = 1, 2, \ldots, n$, and $x = 0$. ∎

4. Ergodic actions. Let (A, G, α) be a completely quantum C^*-system. By A^α we denote the fixed point subalgebra of A, namely $A^\alpha = \{x \in A : \alpha(x) = x \otimes \mathbb{1}_C\}$. Let us observe that $A^\alpha = M^\alpha(\iota)$, where ι is the trivial representation of G. Let $E_\alpha = P^\iota_{11}$.

PROPOSITION 4.1. *Let (A, G, α) be a completely quantum C^*-system. Then E_α is a projection with norm 1 onto the fixed point subalgebra A^α.*

Proof. By Proposition 3.3 E_α is a projection. Moreover, we have $\|E_\alpha(x)\| \le \|x\|$ because h is a state and α is a *-homomorphism, hence it has norm 1. ∎

DEFINITION 4.2. The action α is called *ergodic* if $A^\alpha = \mathbb{C}\mathbb{1}_A$. A state ω on the algebra A is called *α-invariant* if $(\omega \otimes \mathrm{id}_C)\alpha(x) = \omega(x)\mathbb{1}_C$ for $x \in A$.

PROPOSITION 4.3. *If α is an ergodic action of a compact quantum group G on a unital C^*-algebra A, then there exists a unique α-invariant state ω on A.*

Proof. Definition 4.2 and Proposition 4.1 imply that for every $x \in A$ there exists $\omega(x) \in \mathbb{C}$ such that $E_\alpha(x) = \omega(x)\mathbb{1}_A$. The map $\omega : A \longrightarrow \mathbb{C}$ is a continuous linear functional on A such that $\omega(\mathbb{1}_A) = 1$. The projection E_α is a positive map (*cf.* Proposition 4.1 and [12]), so ω is a state. Let us show that ω is α-invariant. If $x \in A$ then

$$\begin{aligned}
&\mathbb{1}_A \otimes (\omega \otimes \mathrm{id}_C)\alpha(x) = \\
&\quad = (E_\alpha \otimes \mathrm{id}_C)\alpha(x) = (\mathrm{id}_A \otimes h \otimes \mathrm{id}_C)(\alpha \otimes \mathrm{id}_C)\alpha(x) = [\mathrm{id}_A \otimes (h \otimes \mathrm{id}_C)\Delta]\alpha(x) \\
&\quad = (\mathrm{id}_A \otimes h)\alpha(x) \otimes \mathbb{1}_C = E_\alpha(x) \otimes \mathbb{1}_C = \mathbb{1}_A \otimes \omega(x)\mathbb{1}_C.
\end{aligned}$$

The fourth equality follows from properties of the Haar measure h. Suppose that ω' is another α-invariant state on A. Then for every $x \in A$ the α-invariance of ω implies

$$\begin{aligned}
\omega'(x) &= h(\omega'(x)\mathbb{1}_C) = (\mathrm{id}_{\mathbb{C}} \otimes h)(\omega' \otimes \mathrm{id}_C)\alpha(x) = (\omega' \otimes \mathrm{id}_{\mathbb{C}})(\mathrm{id}_A \otimes h)\alpha(x) \\
&= \omega'(E_\alpha(x)) = \omega'(\omega(x)\mathbb{1}_A) = \omega(x)\omega'(\mathbb{1}_A) = \omega(x),
\end{aligned}$$

so the state ω is uniqely determined. ∎

LEMMA 4.4. *Let ϕ be a faithful positive linear functional on a *-algebra $\mathcal{B}$. If $n \in \mathbb{N}$ and $b_1, b_2, \ldots, b_n \in \mathcal{B}$ are linearly independent then the matrix $[\phi(b_i^* b_j)]_{i,j=1,2,\ldots,n}$ is strictly positive definite.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. Then $\sum_{i,j} \phi(b_i^* b_j)\overline{\lambda_i}\lambda_j = \phi\left((\sum_i \lambda_i b_i)^*(\sum_j \lambda_j b_j)\right) \ge 0$, hence the matrix $[\phi(b_i^* b_j)]$ is positive definite. If $\sum_{i,j} \phi(b_i^* b_j)\overline{\lambda_i}\lambda_j = 0$, then $\sum_i \lambda_i b_i = 0$ because the state ϕ is faithful. The system $b_1, b_2, \ldots, b_n$ is linearly independent, so $\lambda_i = 0$ for each $i = 1, 2, \ldots, n$. ∎

PROPOSITION 4.5. *Let (A, G, α) be a completely quantum C^*-system with an ergodic action α, and let $\mathcal{A}$ be the *-subalgebra of G-finite elements and let ω be the α-invariant state described in Proposition 4.3. For every $x \in \mathcal{A}$, if $\omega(x^*x) = 0$ then $x = 0$.*

P r o o f. Let $x \in \mathcal{A}$. As in the proof of Proposition 4.3 we get

$$\omega(x^*x)\mathbb{1}_A = (\mathrm{id}_A \otimes h)\alpha(x^*x) = (\mathrm{id}_A \otimes h)(\alpha(x)^*\alpha(x)). \tag{4.1}$$

Theorem 3.6 yields $\alpha(x) \in \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{C}$. Hence, there are $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in \mathcal{A}$, $b_1, b_2, \ldots, b_n \in \mathcal{C}$ such that $b_1, b_2, \ldots, b_n$ are linearly independent and

$$\alpha(x) = \sum_{i=1}^{n} x_i \otimes b_i. \tag{4.2}$$

Suppose that $\omega(x^*x) = 0$. Then (4.1) implies

$$\sum_{i,j} h(b_i^* b_j) x_i^* x_j = 0. \tag{4.3}$$

The state h is faithful on $\mathcal{C}$ (*cf.* [13, Theorem 4.2.5]). Due to Lemma 4.4 and properties of positive definite matrices there are constants $a_1, a_2, \ldots, a_n > 0$ and a unitary matrix $[\gamma_{ij}]_{i,j=1,2,\ldots,n}$ such that $h(b_i^* b_j) = \sum_k \gamma_{ik} a_k \overline{\gamma_{jk}}$. (4.3) yields $0 = \sum_{i,j} \sum_k \gamma_{ik} a_k \overline{\gamma_{jk}} x_i^* x_j = \sum_k a_k \left(\sum_i \gamma_{ik} x_i^*\right) \left(\sum_i \gamma_{ik} x_i^*\right)^*$, so $\sum_i \gamma_{ik} x_i^* = 0$ for every $k = 1, 2, \ldots, n$. Consequently, $0 = \sum_k \overline{\gamma_{jk}} \sum_i \gamma_{ik} x_i^* = \sum_i \left(\sum_k \overline{\gamma_{jk}} \gamma_{ik}\right) x_i^* = \sum_i \delta_{ij} x_i^* = x_j^*$ for every $j = 1, 2, \ldots, n$. Combining this result with (4.2) we get $\alpha(x) = 0$. Now, Corollary 3.8 implies $x = 0$. ∎

PROPOSITION 4.6. *Suppose (A, G, α) is a completely quantum C^*-system with an ergodic action α. Then for every $\tau \in \hat{G}$ and $i, k = 1, 2, \ldots, d_\tau$ we have $\omega((P_{ik}^\tau x)^* y) = \omega(x^*(P_{ki}^\tau y))$ for $x, y \in A$.*

P r o o f. Let $\tau \in \hat{G}$, $i, k = 1, 2, \ldots, d_\tau$ and $x, y \in A$. Then

$$\begin{aligned}
&\omega((P_{ik}^\tau x)^* y)\mathbb{1}_A = \\
&= (\mathrm{id}_A \otimes h)\alpha((P_{ik}^\tau x)^* y) = (\mathrm{id}_A \otimes h)(\alpha(P_{ik}^\tau x)^* \alpha(y)) \\
&= (\mathrm{id}_A \otimes h)((\mathrm{id}_A \otimes D_{ik}^\tau)\alpha(x)^* \alpha(y)) = (\mathrm{id}_A \otimes h)(\alpha(x)^* (\mathrm{id}_A \otimes D_{ki}^\tau)\alpha(y)) \\
&= (\mathrm{id}_A \otimes h)(\alpha(x^*)\alpha(P_{ki}^\tau y)) = (\mathrm{id}_A \otimes h)(\alpha(x^*(P_{ki}^\tau y)) = \omega(x^*(P_{ki}^\tau y))\mathbb{1}_A.
\end{aligned}$$

The first equality follows from the proof of Proposition 4.3, the third from Proposition 3.3.(b), the fourth from Proposition 2.2.(c), the fifth from Proposition 3.3.(b), and the last equality from the proof of Proposition 4.3. ∎

PROPOSITION 4.7. *Let (A, G, α) be a completely quantum C^*-system with an ergodic action α and let $\tau \in \hat{G}$. Then for every $N \in \mathbb{N}$ such that $N \leq \dim P_{11}^\tau A$ there exists a set of indices $\mathcal{I}$ with cardinality N and there are elements $x_{\mu i} \in M^\alpha(\tau)$, $\mu \in \mathcal{I}$, $i = 1, 2, \ldots, d_\tau$, such that*

$$\omega(x_{\mu i}^* x_{\nu j}) = \delta_{\mu\nu}\delta_{ij}, \tag{4.4}$$

and

$$\alpha(x_{\mu i}) = \sum_{k=1}^{d_\tau} x_{\mu k} \otimes u_{ki}^\tau \tag{4.5}$$

for every $\mu, \nu \in \mathcal{I}$ and $i, j = 1, 2, \ldots, d_\tau$.

P r o o f. Let $(\cdot, \cdot)$ be a sesquilinear form on A which is defined by the formula $(x, y) = \omega(x^* y)$ for $x, y \in A$. Theorem 3.6.(a) implies that $P_{11}^\tau A \subset M^\alpha(\tau) \subset \mathcal{A}$, so using Proposition 4.5 one concludes that $(\cdot, \cdot)$ is a scalar product on $P_{11}^\tau A$. We assumed that

$N \le \dim P^\tau_{11}A$, hence there is an orthonormal system $\{x_{\mu 1} : \mu \in \mathcal{I}\}$ of elements of $P^\tau_{11}A$, where $\mathcal{I}$ is a set of indices such that $\operatorname{card}\mathcal{I} = N$. Let us define $x_{\mu i} = P^\tau_{i1}x_{\mu 1}$ for $\mu \in \mathcal{I}$ and $i = 1, 2, \ldots, d_\tau$. Then we have $\omega(x^*_{\mu i}x_{\nu j}) = \omega((P^\tau_{i1}x_{\mu 1})^*(P^\tau_{j1}x_{\nu 1})) = \omega(x^*_{\mu 1}(P^\tau_{1i}P^\tau_{j1}x_{\nu 1})) = \delta_{ij}\omega(x^*_{\mu 1}x_{\nu 1}) = \delta_{ij}\delta_{\mu\nu}$, where the first equality follows from Proposition 4.6, the third from Proposition 3.3.(a), and the last equality follows from orthonormality of the system $\{x_{\mu 1} : \mu \in \mathcal{I}\}$. Hence, (4.4) follows.

Proposition 3.3.(b) yields $\alpha(x_{\mu 1}) = \alpha(P^\tau_{11}x_{\mu 1}) = (\mathrm{id}_A \otimes D^\tau_{11})\alpha(x_{\mu 1})$. So, from Proposition 2.2.(c) we have $\alpha(x_{\mu 1}) \subset A \otimes_{\mathrm{alg}} \operatorname{span}\{u^\tau_{11}, u^\tau_{21}, \ldots, u^\tau_{d_\tau 1}\}$. Therefore there are elements $y_1, y_2, \ldots, y_{d_\tau} \in A$ such that $\alpha(x_{\mu 1}) = \sum_{k=1}^{d_\tau} y_k \otimes u^\tau_{k1}$. If $i = 1, 2, \ldots, d_\tau$ then $x_{\mu i} = P^\tau_{i1}x_{\mu 1} = (\mathrm{id}_A \otimes \rho^\tau_{i1})\alpha(x_{\mu 1}) = \sum_{k=1}^{d_\tau} y_k\rho^\tau_{i1}(u^\tau_{k1}) = \sum_{k=1}^{d_\tau} y_k\delta_{ik} = y_i$. Hence, $\alpha(x_{\mu i}) = \alpha(P^\tau_{i1}x_{\mu 1}) = (\mathrm{id}_A \otimes D^\tau_{i1})\alpha(x_{\mu 1}) = \sum_{k=1}^{d_\tau} x_{\mu k} \otimes D^\tau_{i1}(u^\tau_{k1}) = \sum_{k=1}^{d_\tau} x_{\mu k} \otimes u^\tau_{ki}$, and (4.5) is proved. ∎

PROPOSITION 4.8. *Suppose α is an ergodic action of a compact quantum group $G = (C, \Delta)$ on a unital C^*-algebra A and $\tau \in \hat{G}$. Then $c_\tau \le \operatorname{Tr} F_\tau$.*

Proof. Let N, $\mathcal{I}$, $x_{\mu i}$, $\mu \in \mathcal{I}$, $i = 1, 2, ..., d_\tau$ be as in Proposition 4.7. Let $d = d_\tau$, $F = F_\tau$ and $M = \operatorname{Tr} F_\tau = \operatorname{Tr} F_\tau^{-1}$. Firstly, we will show that for $\mu, \nu \in \mathcal{I}$, the element $r_{\mu\nu} = \sum_{i,j=1}^d (F^{-1})_{ij}x^*_{\mu i}x_{\nu j}$ is a fixed point. Indeed, using (4.5) and (2.2) we get

$$\begin{aligned} E_\alpha(r_{\mu\nu}) &= \sum_{i,j,k,l}(F^{-1})_{ij}x^*_{\mu k}x_{\nu l}h({u^\tau_{ki}}^* u^\tau_{lj}) = \sum_{i,j,k,l}(F^{-1})_{ij}x^*_{\mu k}x_{\nu l}\frac{1}{M}\delta_{ij}(F^{-1})_{kl} \\ &= \frac{1}{M}\sum_i (F^{-1})_{ii}\sum_{k,l}(F^{-1})_{kl}x^*_{\mu k}x_{\nu l} = \sum_{k,l}(F^{-1})_{kl}x^*_{\mu k}x_{\nu l} = r_{\mu\nu}. \end{aligned}$$

The above calculations and Proposition 4.1 imply that $r_{\mu\nu} \in A^\alpha$. So, $r_{\mu\nu} = \omega(r_{\mu\nu})\mathbb{1}_A$, where we have used the ergodicity of α. On the other hand, relations (4.4) imply $\omega(r_{\mu\nu}) = \sum_{i,j}(F^{-1})_{ij}\omega(x^*_{\mu i}x_{\nu j}) = \sum_{i,j}(F^{-1})_{ij}\delta_{\mu\nu}\delta_{ij} = \delta_{\mu\nu}\sum_i (F^{-1})_{ii} = \delta_{\mu\nu}M$. Hence

$$\sum_{i,j}(F^{-1})_{ij}x^*_{\mu i}x_{\nu j} = \delta_{\mu\nu}M\mathbb{1}_A. \tag{4.6}$$

Secondly, let $s_{\mu\nu} = \sum_{i=1}^d x_{\mu i}x^*_{\nu i}$ with $\mu, \nu \in \mathcal{I}$. Simple computations making use of (4.5) and unitarity of $[u^\tau_{ij}]$ show that $\alpha(s_{\mu\nu}) = s_{\mu\nu} \otimes \mathbb{1}_A$. So $s_{\mu\nu} \in A^\alpha$. Hence, there are constants $\lambda_{\mu\nu} \in \mathbb{C}$ such that

$$s_{\mu\nu} = \lambda_{\mu\nu}M\mathbb{1}_A. \tag{4.7}$$

The matrix $[\lambda_{\mu\nu}]_{\mu,\nu\in\mathcal{I}} \in M_N(\mathbb{C})$ is selfadjoint (*cf.* (4.7)). Therefore, one can find elements $x'_{\mu i}$, $\mu \in \mathcal{I}$, $i = 1, 2, \ldots, d$, such that the relations (4.4), (4.5) hold for the system $\{x'_{\mu i}\}$, and additionally

$$\sum_i x'_{\mu i}{x'_{\nu i}}^* = \delta_{\mu\nu}\lambda_\mu M\mathbb{1}_A, \tag{4.8}$$

where λ_μ, $\mu \in \mathcal{I}$, are eigenvalues of the matrix $[\lambda_{\mu\nu}]$. For simplicity, we will write $x_{\mu i}$ instead of $x'_{\mu i}$.

Let $a = [\sum_{\mu\in\mathcal{I}} x_{\mu i}x^*_{\mu j}]_{i,j=1,2,\ldots,d} \in M_d(\mathbb{C}) \otimes A$. Then, (4.6) implies

$$\sum_{k,l} a_{ik}(F^{-1})_{kl}a_{lj} = \sum_{k,l}\sum_{\mu,\nu} x_{\mu i}x^*_{\mu k}(F^{-1})_{kl}x_{\nu l}x^*_{\nu j} = M\sum_\mu x_{\mu i}x^*_{\mu j} = Ma_{ij}.$$

Therefore $a(F^{-1}\otimes \mathbb{1}_A)a = Ma$. It means that the element $M^{-1}(F^{-\frac{1}{2}}\otimes \mathbb{1}_A)a(F^{-\frac{1}{2}}\otimes \mathbb{1}_A)$ is a projector in the algebra $M_d(\mathbb{C})\otimes A$. So, $0\le a\le M(F\otimes \mathbb{1}_A)$. If φ is the normalized trace on the algebra $M_d(\mathbb{C})$ then $(\varphi\otimes\omega)(a)\le M(\varphi\otimes\omega)(F\otimes\mathbb{1}_A) = M\varphi(F) = M^2d^{-1}$. But $(\varphi\otimes\omega)(a) = d^{-1}\sum_\mu \omega\left(\sum_i x_{\mu i}x_{\mu i}^*\right) = Md^{-1}\sum_\mu \lambda_\mu$. Therefore we arrived at

$$\sum_\mu \lambda_\mu \le M. \tag{4.9}$$

Thirdly, let us observe that due to the fact that $x_{\mu i}$ are not zero, (4.8) implies that $\lambda_\mu > 0$ for every $\mu\in\mathcal{I}$. Let $y_{\mu i} = \lambda_\mu^{-\frac{1}{2}}x_{\mu i}^*$ for $\mu\in\mathcal{I}$, $i = 1,2,\ldots,d$. (4.5) leads to $\alpha(y_{\mu i}) = \sum_{k=1}^d y_{\mu k}\otimes {u_{ki}^\tau}^*$. It is easy to check that the elements $\sum_{i=1}^d y_{\mu i}^* y_{\nu i}$, $\mu,\nu\in\mathcal{I}$, are fixed points. Moreover, the relations (4.8) imply

$$\sum_i y_{\mu i}^* y_{\nu i} = \delta_{\mu\nu}M\mathbb{1}_A. \tag{4.10}$$

Fourthly, suppose $b = [\sum_{\mu\in\mathcal{I}} y_{\mu i}y_{\mu j}^*]_{i,j=1,2,\ldots,d}\in M_d(\mathbb{C})\otimes A$. Then, taking into account (4.10) one can check that $b^2 = Mb$. Hence, $b = Mp$ where p is a projector in $M_d(\mathbb{C})\otimes A$, and $0\le b\le M\mathbb{1}_A$. If φ is again the normalized trace on $M_d(\mathbb{C})$, then $(\varphi\otimes\omega)(b) = d^{-1}\sum_i\omega(\sum_\mu y_{\mu i}y_{\mu i}^*) = d^{-1}\sum_i\sum_\mu \lambda_\mu^{-1}\omega(x_{\mu i}^*x_{\mu i}) = \sum_\mu\lambda_\mu^{-1}$. This leads to

$$\sum_\mu \lambda_\mu^{-1}\le M. \tag{4.11}$$

Finally, for every $\lambda > 0$ the inequality $\lambda+\lambda^{-1}\ge 2$ holds. So, the inequalities (4.9) and (4.11) give $2N\le\sum_\mu(\lambda_\mu+\lambda_\mu^{-1})\le 2M$. Therefore $N\le M$. Recall that N is the number of elements of any finite orthonormal system in $P_{11}^\tau A$. Consequently $\dim P_{11}^\tau A\le M$. ■

COROLLARY 4.9. *Suppose α is an ergodic action of a compact quantum group $G = (C,\Delta)$ on a unital C^*-algebra A. Then $c_\tau = \operatorname{card}\mathcal{J}_\tau < \infty$ for every $\tau\in\hat{G}$* (*cf. Remark 3.5*). *Moreover, there is a basis $\{x_{\mu i}^\tau : \tau\in\hat{G},\ \mu\in\mathcal{J}_\tau,\ i = 1,2,\ldots,d_\tau\}$ of the linear space $\mathcal{A}$ and positive constats λ_μ^τ, $\tau\in\hat{G}$, $\mu\in\mathcal{J}_\tau$, such that $\alpha(x_{\mu i}^\tau) = \sum_{j=1}^{d_\tau} x_{\mu k}^\tau\otimes u_{ki}^\tau$, and*

$$\omega({x_{\mu i}^\tau}^* x_{\nu j}^\pi) = \delta_{\tau\pi}\delta_{\mu\nu}\delta_{ij},\qquad \omega(x_{\mu i}^\tau {x_{\nu j}^\pi}^*) = \delta_{\tau\pi}\delta_{\mu\nu}\lambda_\mu^\tau(F_\tau)_{ij} \tag{4.12}$$

for every $\tau,\pi\in\hat{G}$, $\mu\in\mathcal{J}_\tau$, $\nu\in\mathcal{J}_\pi$, $i = 1,2,\ldots,d_\tau$, $j = 1,2,\ldots,d_\pi$.

Proof. For given $\tau\in\hat{G}$ let $\mathcal{J}_\tau$ be a set of indices such that $\operatorname{card}\mathcal{J}_\tau = \dim P_{11}^\tau A$. Let $\{x_{\mu i}^\tau : \mu\in\mathcal{J}_\tau,\ i = 1,2,\ldots,d_\tau\}$ be the system constructed in Proposition 4.7 for $N = \dim P_{11}^\tau A$. In order to prove (4.12) it is enough to show the second equality. But this follows from straightforward calculations based on α-invariance of ω, (4.5), (2.2) and (4.8). ■

Now we are in a position to describe the modular properties of the state ω. In this description we will use the notion of a holomorphic function of exponential growth on the upper halfplane. Let us recall that f is such a function provided that there exist positive constants C, M such that $|f(z)|\le C\mathrm{e}^{M\operatorname{Im} z}$ for every $z\in\mathbb{C}$ with $\operatorname{Im} z > 0$.

THEOREM 4.10. *Let α be an ergodic action of a compact quantum group $G = (C,\Delta)$ on a unital C^*-algebra A, $\mathcal{A}$ its *-subalgebra of G-finite elements, and let ω be the unique*

α-invariant state on A. Then there exists a family $\{\sigma_z\}_{z\in\mathbb{C}}$ of linear maps $\sigma_z : \mathcal{A} \longrightarrow \mathcal{A}$ such that:

(*i*) *For every linear functional ϕ on $\mathcal{A}$ and $x \in \mathcal{A}$ the function $f_x^\phi(z) \stackrel{\mathrm{df}}{=} \phi(\sigma_z(x))$ is a holomorphic function with exponential growth on the upper halfplane*;

(*ii*) $\sigma_z(\mathbb{1}_A) = \mathbb{1}_A$ *for every* $z \in \mathbb{C}$;

(*iii*) $\sigma_z(\sigma_{z'}(x)) = \sigma_{z+z'}(x)$ *for every* $x \in \mathcal{A}$, $z, z' \in \mathbb{C}$. *Moreover* $\sigma_0 = \mathrm{id}_{\mathcal{A}}$;

(*iv*) *For every $z \in \mathbb{C}$ and $x, y \in \mathcal{A}$ we have $\sigma_z(xy) = \sigma_z(x)\sigma_z(y)$ and $\sigma_z(x^*) = \sigma_{\overline{z}}(x)^*$*;

(*v*) $\omega(xy) = \omega(y\sigma_i(x))$ *for every* $x \in \mathcal{A}$ *and* $y \in A$.

Proof. Let $\{x_{\mu i}^\tau : \tau \in \hat{G},\ \mu \in \mathcal{J}_\tau,\ i = 1, 2, \ldots, d_\tau\}$ be the system of elements of $\mathcal{A}$ described in Corollary 4.9. The constants λ_μ^τ are positive numbers and matrices F_τ are strictly positive definite. Therefore, for every $z \in \mathbb{C}$ we can define $(\lambda_\mu^\tau)^z = \mathrm{e}^{z\log\lambda_\mu^\tau}$ and $F_\tau^z = \mathrm{e}^{z\log F_\tau}$. Let $z \in \mathbb{C}$. We define

$$\sigma_z(x_{\mu i}^\tau) = (\lambda_\mu^\tau)^{-iz}\sum_{j=1}^{d_\tau}(F_\tau^{-iz})_{ij}x_{\mu j}^\tau \tag{4.13}$$

for every $\tau \in \hat{G}$, $\mu \in \mathcal{J}_\tau$, $i = 1, 2, \ldots, d_\tau$.

Let $\mathcal{W}$ denote the class of holomorphic functions of exponential growth on the upper halfplane. Let us observe that if $\lambda > 0$ then $|\lambda^{-iz}| = \lambda^{\mathrm{Im}\,z}$. Hence, the function $\mathbb{C} \ni z \mapsto \lambda^{-iz} \in \mathbb{C}$ is an element of $\mathcal{W}$. Thus, for any functional ϕ the function $f^\phi_{x_{\mu i}^\tau}(z) = (\lambda_\mu^\tau)^{-iz}\sum_j(F_\tau^{-iz})_{ij}\phi(x_{\mu j}^\tau)$ is a linear combination of elements of $\mathcal{W}$. Due to the fact that $\mathcal{W}$ has the structure of a complex linear space we infer that $f^\phi_{x_{\mu i}^\tau} \in \mathcal{W}$. We observe that for every ϕ and $x \in \mathcal{A}$ the function f_x^ϕ is a linear combination of functions $f^\phi_{x_{\mu i}^\tau}$. So f_x^ϕ is also an element of $\mathcal{W}$. This ends the proof of (i).

(ii) follows from the fact that $\mathbb{1}_A = x_{\mu 1}^\iota$, where ι is the trivial representation of G, μ is the only element of $\mathcal{J}_\iota$, $\lambda_\mu^\iota = 1$ and $F_\iota = 1$. (4.13) and rules of matrix calculations lead to (iii).

To prove (v) we observe that (4.13) implies $\sigma_i(x_{\mu i}^\tau) = \lambda_\mu^\tau\sum_j(F_\tau)_{ij}x_{\mu j}^\tau$. Therefore, it is enough to prove the equality of (v) for $x = x_{\mu i}^\tau$ and $y = {x_{\nu j}^\pi}^*$. But this follows from (4.12).

It remains to prove (iv). Let $x, y \in \mathcal{A}$. Firstly, take $z = i$. Using (v) for every $w \in \mathcal{A}$ we get $\omega(w\sigma_i(xy)) = \omega(xyw) = \omega(yw\sigma_i(x)) = \omega(w\sigma_i(x)\sigma_i(y))$. Puting $w = (\sigma_i(xy) - \sigma_i(x)\sigma_i(y))^*$ in this equlity we are led to

$$\omega((\sigma_i(xy) - \sigma_i(x)\sigma_i(y))^*(\sigma_i(xy) - \sigma_i(x)\sigma_i(y))) = 0.$$

Taking into account Proposition 4.5 we get the first equality of (iv) for $z = i$. On the other hand, we infer from (iii) and (v) that for a given $x \in \mathcal{A}$ we have $\omega(w\sigma_i(x^*)) = \omega(x^*w) = \overline{\omega(w^*x)} = \overline{\omega(\sigma_{-i}(x)w^*)} = \omega(w\sigma_{-i}(x)^*)$ for any $w \in \mathcal{A}$. A similar argument leads to the second equality of (iv) for $z = i$. Combining this result with (iii) we can show by induction that both equalities hold for $z = ki$ where $k = 1, 2, \ldots$. Let ϕ be any functional and let $x, y \in \mathcal{A}$. Let $f_1(z) = \phi(\sigma_z(xy))$ and $f_2(z) = \phi(\sigma_z(x)\sigma_z(y))$ where $z \in \mathbb{C}$. Both functions are elements of $\mathcal{W}$. The above considerations also imply $f_1(ki) = f_2(ki)$ for $k = 1, 2, \ldots$. Now, let us observe that the functions $g_j(z) = f_j(iz)$, $j = 1, 2$, fulfil the assumptions of

Lemma 5.5 in [13]. So $g_1 \equiv g_2$, and consequently $\phi(\sigma_z(xy)) = \phi(\sigma_z(x)\sigma_z(y))$ for every $z \in \mathbb{C}$. As ϕ is arbitrary, the first equality is proved. Similarly the second equality follows and the Theorem is proved. ∎

THEOREM 4.11. *Let (A, G, α) be as in the previous theorem. Suppose ω is faithful. Then there exists a one-parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of automorphisms of the algebra A such that ω is a KMS state with respect to this group.*

Proof. Let $\{\sigma_z\}_{z\in\mathbb{C}}$ be the family constructed in the previous theorem. Obviously, the family $\{\sigma_t\}_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms of *-algebra $\mathcal{A}$. Let $\{x^\tau_{\mu i}\}$ be the basis described in Corollary 4.9. Taking into account (4.12) and (4.13) we check that $\omega(\sigma_t(x^\tau_{\mu i})) = \omega(x^\tau_{\mu i})$ for every $t \in \mathbb{R}$ and $x^\tau_{\mu i}$. Let $(H_\omega, \pi_\omega, \Omega_\omega)$ be the GNS representation of the system (A, ω). The faithfulness of ω implies the same property of π_ω, i.e. the representation $\pi_\omega : A \longrightarrow B(H_\omega)$ is faithful. The group $\{\sigma_t\}_{t\in\mathbb{R}}$ is implemented by a group of unitary operators $\{V_t\}_{t\in\mathbb{R}}$ because ω is σ_t-invariant. The representation π_ω is faithful, so the mappings σ_t have extensions on the whole algebra A. By Theorem 4.10 the *-algebra is contained in the *-algebra of analytic elements of the group $\{\sigma_t\}_{t\in\mathbb{R}}$. Moreover, points (iii) and (v) of Theorem 4.10 yield $\omega(\sigma_z(x)y) = \omega(y\sigma_{z+i}(x))$ for every $z \in \mathbb{C}$, $x \in \mathcal{A}$, $y \in A$. The *-algebra $\mathcal{A}$ is dense in A, so we can complete our proof by applying Theorem 8.12.3 from [9]. ∎

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References

[1] D. Bernard and G. Felder, *Quantum group symmetries in two-dimensional lattice quantum field theory*, Nucl. Phys. B 365 (1991), 98-120.

[2] M. Fannes, B. Nachtergaele and R. F. Werner, *Quantum Spin Chains with Quantum Group Symmetry*, Commun. Math. Phys. 174 (1996), 477-507.

[3] K. Fredenhagen, K.-H. Rehren and B. Schroer, *Superselection Sectors with Braid Group Statistics and Exchange Algebras* I, Commun. Math. Phys. 125 (1989), 201-226, II, Rev. Math. Phys. - Special Issue (1992), 113-157.

[4] R. Haag, *Local Quantum Physics*, Springer-Verlag, Berlin Heidelberg 1992.

[5] R. Høeg-Krohn, M. B. Landstad and E. Størmer, *Compact ergodic groups of automorphisms*, Ann. Math. 114 (1981), 75-86.

[6] G. Mack and V. Schomerus, *Quasi Hopf quantum symmetry in quantum theory*, Nucl. Phys. B 370 (1992), 185-230.

[7] M. Marciniak, *Actions of compact quantum groups on C^*-algebras*, Proc. Amer. Math. Soc. 126 (1998), 607-616.

[8] M. Marciniak, *Quantum symmetries in noncommutative dynamical systems*, Gdańsk University, 1997 (in Polish).

[9] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, Academic Press, London 1979.

[10] P. Podleś, *Symmetries of Quantum Spaces. Subgroups and Quotient Spaces of Quantum $SU(2)$ and $SO(3)$ Groups*, Commun. Math. Phys. 170 (1995), 1-20.

[11] M. E. Sweedler, *Hopf algebras*, W.A. Benjamin, Inc., New York 1969.

[12] M. Takesaki, *Theory of operator algebras*, Springer Verlag, Berlin Heidelberg New York 1979.

[13] S. L. Woronowicz, *Compact Matrix Pseudogroups*, Commun. Math. Phys. 111 (1987), 613-665.

[14] S. L. Woronowicz, *Compact quantum groups*, preprint 1994.