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# MEASURES CONNECTED WITH BARGMANN'S REPRESENTATION OF THE $q$-COMMUTATION RELATION FOR $q > 1$

ILONA KRÓLAK

*Institute of Mathematics, Wrocław University*
*pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland*
*E-mail: ikrol@math.uni.wroc.pl*

**Abstract.** Classical Bargmann's representation is given by operators acting on the space of holomorphic functions with scalar product $\langle z^n, z^k\rangle_q = \delta_{n,k}[n]_q! = F(z^n\overline{z}^k)$. We consider the problem of representing the functional $F$ as a measure. We prove the existence of such a measure for $q > 1$ and investigate some of its properties like uniqueness and radiality.

**1. Introduction.** The $q$-commutation relations were introduced by Greenberg [5]. Bożejko and Speicher [3] investigated it as interpolation between bosonic and fermionic case. Extensions to relations of the form $aa^+ = f(a^+a)$ have also been studied. One dimensional case, which is the main object of this paper has been investigated by Bargmann [2]. We shall make use of the language of $q$-calculus, which is over a century old (Gasper & Rahman 1990, Jackson 1910).

We recall some basic notation here:

A natural number $n$ has the following $q$-deformation

$$[n]_q = 1 + q + q^2 + q^3 + \ldots + q^{n-1}.$$

The $q$-factorials, $q$-binomial coefficients and $q$-shifted factorial are defined as:

$$[n]_q! = [1]_q[2]_q \ldots [n]_q, \qquad \binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

$$(a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j) \quad \text{with} \quad (a;q)_0 = 1,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n \quad \text{for} \quad |q| < 1.$$

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We will deal with the Fock representation of the commutation relation:

$$aa^{+} - qa^{+}a = \mathrm{Id}.$$

It is known (see [2]) that for $0 \le q < 1$ this representation can be identified with representation by operators which act on functions of complex variable. This representation is given by:

$$(a^{+}f)(z) = zf(z), \qquad (af)(z) = D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{z(q-1)} & \text{if } z \neq 0; \\ f'(0) & \text{if } z = 0. \end{cases}$$

The representation space $H^2(\mu^q)$ is the completion of the space of analytic functions on $D_q = \{z \in \mathbf{C} \text{ and } |z|^2 < \frac{1}{1-q}\}$ with respect to the inner product:

$$\langle z^n, z^k \rangle_q = \delta_{n,k}[n]_q! = \int_{\mathbf{C}} z^n \overline{z}^k \mu^q(dz) \tag{$*$}$$

where $\mu^q(dz) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q,q)_k} \lambda_{r_k}(dz)$, $r_k = \frac{q^{k/2}}{\sqrt{1-q}}$, and $\lambda_r$ is normalized Lebesgue measure on the circle with radius $r$ (see [7]).

This is called Bargmann's representation of the $q$–harmonic oscillator. As $q$ tends to 1, $\mu^q$ will tend to the Gauss measure on $\mathbf{C}$.

In both cases (for $0 \le q < 1$ and $q = 1$) the measure $\mu^q(dz)$ is unique and radial. However, we will prove that for $q > 1$ there are many measures satisfying $(*)$ and some of them are not radial.

**2. The existence of $\mu^q(dz)$.** For our further considerations $q$ will be greater than 1 and fixed. We look for probability measures which satisfy the following condition:

$$\int_{\mathbf{C}} z^n \overline{z}^k \mu^q(dz) = \delta_{n,k}[n]_q!, \qquad n,k \in \mathbf{N}. \tag{$\diamond\diamond\diamond$}$$

Using polar coordinates we can write: $\mu(dz) = \mu_r(d\varphi)\nu(dr)$, where $\int_0^{2\pi} \mu_r(d\varphi) = 1$.

PROPOSITION 1. *There is one to one correspondence between radial solutions of* $(\diamond\diamond\diamond)$ *and solutions of the Stieltjes moment problem*:

$$\int_0^{\infty} x^k \overline{\nu}(dx) = [k]_q!, \qquad k = 0,1,2.... \tag{1}$$

Proof. We split $\nu = \nu_1 + \nu_2$ where $\nu_1$ is the singular part of the measure $\nu$, $\nu_2$ is the absolutely continous part of this measure. The same we do for $\overline{\nu}$. Now change of variables gives $\nu_1(x) = \overline{\nu_1}(x^2)$, and $\nu_2(x) = \overline{\nu_2}(x^2)2x$. ■

The next facts for $-1 < q < 1$ were proved in [8]. We present, without proof, the modified versions of the result for $q > 1$.

PROPOSITION 2. *If* $f(x) = (D_qF)(x)$, *the limit* $\lim_{x\to+\infty} F(x) = F(\infty)$ *exists*, *and*

$$\int_0^{\infty} f(x)d_q^a(x) := \sum_{k=-\infty}^{\infty} f(aq^k)q^k a(q-1)$$

*exists for every* $a > 0$, *then* $\int_0^{\infty} f(x)d_q^a(x) = F(\infty) - F(0)$.

PROPOSITION 3. *If* $f(x) = \exp_q(x) := \sum_{s=0}^{\infty} \frac{x^s}{[s]_q!}$, *then* $(D_q f)(x) = f(x)$.

Remark 1. Note that

$$\int_0^\infty D_q\left(\frac{x^n}{\exp_q(x)}\right) d_q^a(x) = \begin{cases} \dfrac{x^n}{\exp_q(x)}\Big|_0^\infty = -\delta_{0,n}, \\ [n]_q \displaystyle\int_0^\infty \frac{x^{n-1}}{exp_q(qx)} d_q^a(x) - \int_0^\infty \frac{x^n}{exp_q(qx)} d_q^a(x). \end{cases}$$

COROLLARY 1. *As* $\overline{\nu}(dx)$ *we can take*

$$\overline{\nu}^a(dx) = a(q-1) \sum_{k=-\infty}^{\infty} \frac{q^k}{exp_q(q^{k+1}a)} \delta_{q^k a}, \qquad \textit{where } a \in [1, q].$$

Remark 2. For each point $t \in R_+$ there exists $a \in [1, q)$ such that $x \in \operatorname{supp} \overline{\nu}^a(dx)$.

**3. Characterization of radial measures.** The problem to find measures such that $\int_{-\infty}^{\infty} x^k \overline{\nu}(dx) = m_k$ for an arbitrary sequence $(m_k)$ is called the moment problem. We will be interested only in solutions whose support is beyond the negative axis. It may happen that the measure doesn't exist or exists but is not unique. Calculations from the previous section show that we have to deal with an indeterminate moment problem. In this case we can associate with the problem two sequences of polynomials. They are solutions of a three term recurence relation:

$$\omega_{n+1}(x) = (x - \alpha_{n+1})\omega_n(x) - \beta_n \omega_{n-1}, \qquad \beta_n > 0, \ \alpha_n \in R$$

with initial conditions

$$Q_0(x) = 1, \quad Q_1(x) = x - \alpha_1, \quad P_0(x) = 0, \quad P_1(x) = \beta_0.$$

From the theory of moments we know that $Q_n(x)$ defined above are orthogonal with respect to $\overline{\nu}(dx)$, i.e. $\int_R Q_n(x) Q_m(x) \overline{\nu}(dx) = 0$ if $m \neq n$ (see [1]).

THEOREM 1. *If the moment problem* $\int_{-\infty}^{\infty} x^k \theta(dx) = m_k$ *is indeterminate then the set of probability measures* $\theta(dx\,, \sigma)$ *that solve the problem can be indexed by functions* $\sigma(z)$ *analytic in the upper half plane* (Im$z > 0$) *and satisfying* Im$\sigma(z) \leq 0$ *for* Im$z > 0$. *Furthermore there exist entire functions* $A(z), B(z), C(z), D(z)$ *such that*

$$\int_{-\infty}^{\infty} \frac{\theta(dx\,,\,\sigma)}{z-x} = \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)}.$$

$A, B, C, D$ *are uniform limits on compact subsets of* $\mathbf{C}$ *of* $A_n, B_n, C_n, D_n$ *respectively, where*

$$\begin{aligned} A_{n+1}(z) &= [P_{n+1}(z)P_n(0) - P_{n+1}(0)P_n(z)]\, c_n, \\ B_{n+1}(z) &= [Q_{n+1}(z)P_n(0) - P_{n+1}(0)Q_n(z)]\, c_n, \\ C_{n+1}(z) &= [P_{n+1}(z)Q_n(0) - Q_{n+1}(0)P_n(z)]\, c_n, \\ D_{n+1}(z) &= [Q_{n+1}(z)Q_n(0) - Q_{n+1}(0)Q_n(z)]\, c_n, \end{aligned}$$

*with* $c_n = (\beta_1 \ldots \beta_n)^{-1}$ *(For details see [1]).*

Remark 3. It can be proved that $c_n = \int_0^\infty |Q_n(x)|^2 \theta(dx)$.

Now we try to apply Theorem 1 to our problem.

PROPOSITION 4. *If* $Q_n$ *are polynomials associated with the Stieltjes moment problem* $\int_0^\infty x^n\overline{v}(dx) = [n]_q!$, *then*

$$Q_n(x) = D_q^{(n)}\left(x^n \frac{1}{exp_q(q^{-n+1}x)}\right) exp_q(qx) q^{\frac{n(n-1)}{2}}(-1)^n.$$

Proof. For $m < n$ we have

$$\int_0^\infty D_q\left(D_q^{n-1}\left(\frac{x^n}{exp(q^{-n+1}x)}x^m\right)\right)\mu^q(dx) = 0.$$

Now we can prove by induction that $\int Q_n(x)x^m\mu^q(dx) = 0$ for $m < n$. We use similar calculations as in Remark 1. ∎

PROPOSITION 5. *We have the following formula*

$$Q_n(x) = \sum_{k=0}^n (-1)^{n+k}\binom{n}{k}_q \frac{[n]_q!}{[k]_q!} q^{\frac{(n-k)(n-k-1)}{2}} x^k.$$

Remark 4. Analogously we can calculate $c_n = \int Q_n(x)x^n\overline{\nu}(dx)$.

PROPOSITION 6. *If* $s = \frac{1}{q}$, *then*

$$D_{n+1}(z) = Cz\,\frac{1-s^{n+1}}{1-s}\sum_{k=0}^n \frac{(s^{-n};s)_k s^{\frac{k(k-1)}{2}}(1-s)^k(s^{n+2}sx)^k}{(s^2;s)_k(s;s)_k} = zL_n.$$

PROPOSITION 7 ([9]). *If* $s = \frac{1}{q}$, *then*

$$H = \sum_{n=0}^\infty r^n L_n(z) = \frac{(s^2;q)_\infty}{(r;s)_\infty}\sum_{k=0}^\infty \frac{s^{k^2+k}[-(1-s)zsr]^k}{(s:s)_k(rs^2;s)_k}.$$

The application of Darboux method to $L_n$ gives

$$D(z) = MzH(r)(1-r)|_{r=1}.$$

COROLLARY 2. *There is a measure* $\overline{\nu}(dx)$ *which is discrete and has mass points at the zeros of* $D(z)$. *General theory says that if* $\sigma(z) = const$ *then* $\operatorname{supp}\theta(dx,\sigma) \subseteq R_+ \cup \{0\}$ *if and only if* $\sigma = +\infty$.

**4. Characterizations of non-radial measures.** In this section we consider the problem of existence of non-radial measures which give the Bargmann's representation for $q > 1$. It turns out that such measures exist because the moment problem $\int x^k\overline{\nu}(dx) = [n]_q!$ does not have a unique solution.

THEOREM 2. *A non-radial measure satisfying*

$$\int_{\mathbf{C}} z^n\overline{z}^k\mu(dz) = \delta_{k,n}m_n$$

*exists if and only if there exists a measure* $\nu$ *on* $R_+$ *with the following properties*:

$$\int_0^\infty x^{2k}\nu(dx) = m_k, \qquad k = 1, 2, \ldots.$$

*There exists* $f(x) \neq 0$ $\nu$*-a.e. such that*

(*a*) $\forall\, k \in \mathbf{N} :\ \int_0^\infty f(x)x^{2k}\nu(dx) = 0$,

(*b*) $\exists\, N \in \mathbf{N}\ :\ |f(x)| \leq x^N$ *on* $R_+$.

Proof. The proof of sufficiency is based on the following construction: We split $\mu_x(d\varphi) = \mu_x^{''}(d\varphi) + \mu_x^{'}(\varphi)d\varphi$ where $\mu_x^{'}$ is defined by

$$\mu'_x(\varphi) = \begin{cases} \dfrac{|f(x)|}{\pi x^N} & \text{for } \frac{2\pi k}{N} \leq \varphi < \frac{2\pi k+\pi}{N}, \qquad \text{if } f(x) > 0,\\ \dfrac{|f(x)|}{\pi x^N} & \text{for } \frac{2\pi k+\pi}{N} \leq \varphi < \frac{2\pi(k+1)}{N}, \quad \text{if } f(x) < 0,\\ 0 & \text{otherwise,} \end{cases}$$

and $\mu_x^{"}(d\varphi) = \frac{1}{2\pi}(1 - \frac{|f(x)|}{x^N})d\varphi$. Now, let $\mu(dz) = \mu_x(d\varphi)\nu(dx)$. ∎

Application. With the notation of Corollary 1 let us put $\nu_1 = \overline{\nu}^a(x^2)$, $\nu_2(x) = \overline{\nu}^b(x^2)$, $a \neq b$,

$$f(x) = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}x^2 \quad \text{and} \quad \nu(x) = \frac{\nu_1(x) + \nu_2(x)}{2}.$$

Then $|f(x)| \leq 2x^2$ and $\int_0^\infty x^{2n}\nu(dx) = [n]_q!$, $\int_0^\infty f(x)x^{2n}\nu(dx) = 0$ for every $n \in N$.

Now we can apply Theorem 2 to obtain a non-radial measure which gives Bargmann's representation for $q > 1$.

Remark 5. From the given solutions we may obtain new ones: by rotations of the underlying plane, by convex linear combinations of already obtained measures and as weak limits of them.

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