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QUANTUM DYNAMICAL ENTROPY REVISITED

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Abstract. We define a new quantum dynamical entropy for a C*-algebra automorphism with an invariant state (and for an appropriate 'approximating' subalgebra), which entropy is a 'hybrid' of the two alternative definitions by Connes, Narnhofer and Thirring resp. by Alicki and Fannes (and earlier, Lindblad). We report on this entropy's properties and on three examples.

1. Introduction. The quantum dynamical entropy in the sense of Connes, Narnhofer and Thirring [8], originating from the first, still more restricted definition by Connes and Størmer [9], has been studied intensively during the past decade (see in particular [6] for a comparison with the popular notion of 'quantum chaos'). More recently, Alicki and Fannes [1] proposed a more direct definition of quantum dynamical entropy, related to earlier work of G. Lindblad [15]. This latter entropy definition has been applied and studied in considerable detail in [1, 5, 4, 2, 3, 21] and may also be the subject of other contributions in this present Volume. Yet another, promising approach is due to Voiculescu [22].

In our contribution here, we report on our recent definition of a new quantum dynamical entropy [11], which is a 'hybrid' of the above two earlier definitions and has partly 'merged' properties, resp. values in the examples. Note that an earlier version of the forthcoming preprint [11] was circulated under the same title of this contribution.

Our standing notation will be the following: $\mathcal{A}$ is a general unital C*-algebra, with unit $\mathbb{1} \in \mathcal{A}$. All subalgebras of $\mathcal{A}$ are unital $*$-subalgebras and all maps are unital (i.e. unit-preserving), linear maps; in particular, $\theta : \mathcal{A} \to \mathcal{A}$ is a $*$-endomorphism (or $*$-automorphism), and $\varphi = \varphi \circ \theta$ is a θ-invariant state on $\mathcal{A}$. The basic quantum entropy functional will be denoted by the letter S (and 'log' will denote natural logarithms):

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Definition 1 (von Neumann entropy). For a state ψ on the $(n \times n)$-matrix algebra $M_n(\mathbb{C})$, its *von Neumann quantum entropy* is defined by $S(\psi) = -\operatorname{Tr}_n(\rho_\psi \log \rho_\psi)$, where ρ_ψ is the density matrix for the state ψ, and Tr_n is the $(n \times n)$-matrix trace.

2. Quick review of the two alternative definitions, to be partly merged

Definition 2 (Connes–Narnhofer–Thirring entropy). The basic entropy functional in the definition by Connes–Narnhofer–Thirring [8] of the quantum dynamical entropy $h_\varphi(\theta)$ is a more 'sophisticated' functional than the von Neumann entropy Def. 1 above, defined in a first step as follows:

(i) For a *single* completely positive, unit–preserving, linear map $\gamma : M_n(\mathbb{C}) \to \mathcal{A}$, again from the C*-algebra of complex $(n \times n)$-matrices into the fixed C*-algebra $\mathcal{A}$, its *CNT entropy w.r.t. the state* φ *on* $\mathcal{A}$ is

$$H_\varphi(\gamma) = S(\varphi \circ \gamma) - \inf_{\{\varphi = \sum_i \varphi_i\}} \sum_i \varphi_i(\mathbb{1}) S(\hat{\varphi}_i \circ \gamma)\,,$$

where $\varphi = \sum_i \varphi_i$ is any (finite) decomposition of the state φ into positive linear functionals φ_i on $\mathcal{A}$, and $\hat{\varphi}_i = (\varphi_i(\mathbb{1}))^{-1} \cdot \varphi_i$ is the respectively corresponding normalized state on $\mathcal{A}$.

(ii) The second step of the definition in [8] is to generalize the entropy functional (i) above to more than one argument, extending the original definition by Connes and Størmer [9] (there for trace states φ on $\mathcal{A}$) to $\ell \in \mathbb{N}$ completely positive maps $\gamma_k : M_{n_k}(\mathbb{C}) \to \mathcal{A}$ for $k = 1, \ldots, \ell$, from respective $(n_k \times n_k)$-matrix algebras (but not necessarily subalgebras of $\mathcal{A}$, in contrast to [9]) into $\mathcal{A}$. As we will not need this multi-argument entropy functional of [8] in our approach to quantum dynamical entropy, we do not recall its definition here, but we just emphasize that it is a 'well-developed' generalization of (i) above, denoted by $H_\varphi(\gamma_1, \ldots, \gamma_\ell)$.

(iii) This then leads to the definition of the *CNT entropy of* θ *w.r.t.* γ (*as in* (*i*) *above*) *given* φ, which uses (ii) above in the same multi–argument way as already in [9]:

$$h_\varphi(\theta, \gamma) = \lim_{m \to \infty} \frac{1}{m} H_\varphi(\gamma, \theta \circ \gamma, \ldots, \theta^{m-1} \circ \gamma)\,,$$

where the limit exists due to a multi–argument subadditivity (and invariance) property of the permutation symmetric functional (ii), see [8, 9].

(iv) The final step is the definition of the *CNT dynamical entropy of* θ *given* φ, first as an abstract supremum

$$h_\varphi(\theta) = \sup_\gamma h_\varphi(\theta, \gamma)\,,$$

where the supremum is taken over all completely positive maps γ from finite–dimensional matrix algebras into $\mathcal{A}$ (or equivalently [8], from finite–dimensional C*-algebras into $\mathcal{A}$, with (i) and (ii) above slightly generalized).

Remark 1. The generally abstract supremum in the final step (iv) of Definition 2 may be exactly computed for a (separable) *nuclear* C*-algebra, for which there exists a sequence $\sigma_n : \mathcal{A} \to \mathcal{A}_n$ resp. $\tau_n : \mathcal{A}_n \to \mathcal{A}$ of completely positive, unital maps with finite–dimensional C*-algebras $\mathcal{A}_n$ such that $\tau_n \circ \sigma_n \to \mathrm{Id}_\mathcal{A}$ in pointwise norm: see [8] (and references there), where it is shown that in this case $h_\varphi(\theta) = \lim_{n\to\infty} h_\varphi(\theta, \tau_n)$ holds, due to a certain joint norm continuity of the multi–argument entropy functional (ii) in Definition 2 above.

Remark 2. In particular, for an AF (approximately finite–dimensional) algebra $\mathcal{A} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{A}_n}$ (the norm inductive limit of an increasing sequence of finite–dimensional subalgebras $\mathcal{A}_n$), one may take the canonical inclusion $*$-homomorphisms $\tau_n := \imath_{\mathcal{A}_n} : \mathcal{A}_n \hookrightarrow \mathcal{A}$ and then obtains the CNT dynamical entropy as the increasing limit $h_\varphi(\theta) = \lim_{n\to\infty} h_\varphi(\theta, \mathcal{A}_n)$, where we use the short notation for subalgebras: e.g. $H_\varphi(\mathcal{A}_n) \equiv H_\varphi(\imath_{\mathcal{A}_n})$ in (i) of Def. 2.

Remark 3. While the AF algebras of the previous Remark are the natural non-Abelian generalization of (the C*-algebras of continuous functions on) totally disconnected, compact metric spaces, the natural generalization of connected, compact metric spaces are the projectionless (unital, separable, in particular nuclear) non-Abelian C*-algebras. For such a C*-algebra, there do not exist any finite-dimensional $*$-subalgebras as used in Remark 2, but still there exist plenty of completely positive unital maps $\gamma : M_n(\mathbb{C}) \to \mathcal{A}$ for any $n \in \mathbb{N}$: As pointed out in [8] (see references there), such a map, defined by linearly extending $\gamma(E_{ij}) = A_{ij} \in \mathcal{A}$ with the matrix units $E_{ij} \in M_n(\mathbb{C})$ $(i, j = 1, \ldots, n)$, is completely positive *iff* the matrix $A = [A_{ij}]_{ij}$ in the C*-algebra $M_n(\mathcal{A})$ is *positive*.

Definition 3 (Alicki–Fannes / Lindblad entropy). The definition of quantum dynamical entropy $h^{\mathrm{AF}}_{\varphi,\mathcal{B}}(\theta)$ by Alicki and Fannes [1] uses the original von Neumann quantum entropy of Def. 1 in a completely different way than in Def. 2 above, related to earlier work of G. Lindblad [15], compare also [16, p. 121] and the reference to the work of Lindblad given there. Following the suggestion of R. Alicki during his talk at the Quantum Probability 1997 meeting, we will now use the short–cut: ALF entropy (despite the non-alphabetic order of the authors' initials, but in a sense 'time-ordered' from the middle initial) in contrast to the CNT entropy of Def. 2 above; but we will retain the notation h^{AF} (resp. H^{AF}) as above, hoping that no confusion will be possible with the notion of AF algebras as in Remark 2 above.

An *operational partition of unity* in $\mathcal{A}$ is an m-tuple (for $m \in \mathbb{N}$ arbitrary) $\alpha = (A_i)_{i=1,\ldots,m}$ of elements $A_i \in \mathcal{A}$ such that $\sum_{i=1}^m A_i^* A_i = \mathbb{1}$.

For any $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, we denote by $\mathcal{O}_1(\mathcal{B})$ the set of all its operational partitions of unity $\beta = (B_i)_i$ with $B_i \in \mathcal{B}$; and by $\mathcal{O}_1(\mathcal{B}, n)$ we denote the subset of all those β with exactly $n \in \mathbb{N}$ elements.

(i) For $\alpha \in \mathcal{O}_1(\mathcal{A}, n)$, $\alpha = (A_i)_{i=1,\ldots,n}$, its *ALF entropy w.r.t. the state* φ on $\mathcal{A}$ is defined by the von Neumann entropy of the state $\psi_\varphi[\alpha]$ on $M_n(\mathbb{C})$ with density matrix $\rho_\varphi[\alpha]_{ij} := \varphi(A_j^* A_i)$ $(i, j = 1, \ldots, n)$:

$$H^{\mathrm{AF}}_\varphi(\alpha) := S(\psi_\varphi[\alpha]), \quad \text{where } \psi_\varphi[\alpha](m) = \mathrm{Tr}_n(m \cdot \rho_\varphi[\alpha]) \ \forall m \in M_n(\mathbb{C}).$$

(ii) For two operational partitions of unity $\alpha, \beta \in \mathcal{O}_1(\mathcal{B})$, where $\alpha = (A_i)_{i=1,\ldots,m} \in \mathcal{O}_1(\mathcal{B}, m)$ and $\beta = (B_j)_{j=1,\ldots,n} \in \mathcal{O}_1(\mathcal{B}, n)$ within a $*$-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, we define their *ordered refinement* by

$$\alpha\vec{\nabla}\beta := (B_1A_1, B_2A_1, \ldots, B_nA_1, B_1A_2, \ldots, B_nA_2, \ldots, B_nA_m)$$

in the order indicated with the resulting (mn)-tuple, such that obviously again $\alpha\vec{\nabla}\beta \in \mathcal{O}_1(\mathcal{B})$, and in particular $\alpha\vec{\nabla}\beta \in \mathcal{O}_1(\mathcal{B}, mn)$, as we have to count all the zero components in the resulting tuple.

(iii) The *ALF entropy of* θ *w.r.t.* α *given* φ is defined as

$$h_\varphi^{\mathrm{AF}}(\theta, \alpha) = \limsup_{k\to\infty} \frac{1}{k} H_\varphi^{\mathrm{AF}}(\alpha\vec{\nabla}\theta(\alpha)\vec{\nabla}\ldots\vec{\nabla}\theta^{k-1}(\alpha)).$$

(iv) Let θ and φ be as before, and $\mathcal{B} \subset \mathcal{A}$ be any (typically, norm–dense) unital $*$-subalgebra of $\mathcal{A}$. The *ALF entropy of* θ *given* φ *and* $\mathcal{B}$ is defined as

$$h_{\varphi,\mathcal{B}}^{\mathrm{AF}}(\theta) = \sup_{\alpha\in\mathcal{O}_1(\mathcal{B})} h_\varphi^{\mathrm{AF}}(\theta, \alpha).$$

R e m a r k 4. In contrast to the CNT entropy for AF algebras as in Remark 2, where the norm–dense, algebraic inductive limit, $*$-algebra $\mathcal{B} := \bigcup_{n\in\mathbb{N}} \mathcal{A}_n \subset \mathcal{A}$ already gives $h_\varphi(\theta)$ as in Def. 2,(iv) by taking the supremum there over all completely positive maps from finite–dimensional matrix algebras *only* into $\mathcal{B} \subset \mathcal{A}$ (this is an obvious consequence of Remark 2), the ALF entropy has the following general continuity *problem*: For a norm–dense $*$-subalgebra $\mathcal{B} \subset \mathcal{A}$ (as in the AF algebra case), there is no general norm continuity property of the ALF entropy available which would again lead to the desired computation of the abstract full supremum for all of $\mathcal{A}$, in the form $h_{\varphi,\mathcal{B}}^{\mathrm{AF}}(\theta) = h_{\varphi,\mathcal{A}}^{\mathrm{AF}}(\theta)$.

R e m a r k 5. This latter problem was solved for the Abelian case $\mathcal{A} = L^\infty(X, \mu)$, of the von Neumann algebra $\mathcal{A}$ of L^∞-functions on a probability space (X, μ), in [2] where the natural notion of a *H-dense* (here, the precise notation should be 'H^{AF}-dense') subalgebra $\mathcal{B} \subset \mathcal{A}$ was introduced, for which the above equality of suprema holds. See also the corresponding Chapter 5 in the Thesis of Tuyls [21], in particular Section 5.2.

3. The new 'hybrid' quantum dynamical entropy and its properties

DEFINITION 4 (The map of a partition). For an operational partition of unity $\alpha = (A_i \in \mathcal{A})_{i=1,\ldots,n}$, with $\alpha \in \mathcal{O}_1(\mathcal{A}, n)$, we define a completely positive map denoted by $\gamma[\alpha] : M_n(\mathbb{C}) \to \mathcal{A}$ by linearly extending

$$\gamma[\alpha](E_{ij}) := A_i^* A_j, \;\; i, j = 1, \ldots, n,$$

from the matrix units $E_{ij} \in M_n(\mathbb{C})$, numbered in the canonical order.

That $\gamma[\alpha]$ is a *completely* positive map is obvious from the result quoted in Remark 3 above: The matrix $A = [A_i^* A_j]_{ij} \in M_n(\mathcal{A})$ is evidently positive, as it may be expressed as $A = B^*B$ with the matrix $B \in M_n(\mathcal{A})$ having the top row equal to $(A_1, A_2, \ldots, A_n)$ and all zero entries on the lower rows.

DEFINITION 5 (The hybrid entropy). We have to modify only the first step (i) of Def. 3 above, and leave the other three steps completely analogous:

(i) For $\alpha \in \mathcal{O}_1(\mathcal{A})$, the *hybrid entropy of* α *given the state* φ *on* $\mathcal{A}$ is defined by

$$HH_\varphi(\alpha) := H_\varphi(\gamma[\alpha]),$$

with the single–argument CNT entropy functional, of Def. 2,(i) before, on the right.

(ii) The ordered refinement $\vec{\vee}$ is defined in Def. 3,(ii) above.

(iii) The *hybrid entropy of* θ *w.r.t.* α *given* φ is defined by

$$hh_\varphi(\theta,\alpha) = \limsup_{k\to\infty} \frac{1}{k} HH_\varphi(\alpha\vec{\vee}\theta(\alpha)\vec{\vee}\dots\vec{\vee}\theta^{k-1}(\alpha)).$$

(iv) The *hybrid entropy of* θ *w.r.t.* φ *and* $\mathcal{B}$ (as in Def. 3,(iv) above) is defined by

$$hh_{\varphi,\mathcal{B}}(\theta) = \sup_{\alpha\in\mathcal{O}_1(\mathcal{B})} hh_\varphi(\theta,\alpha).$$

PROPOSITION 1. *The hybrid entropy Def. 5,(i) of a partition has the following general algebraic properties*:

(i) *For* $\alpha \in \mathcal{O}_1(\mathcal{A})$, *the entropy has the general upper bound*:

$$HH_\varphi(\alpha) \le S(\varphi\circ\gamma[\alpha]) = H_\varphi^{\mathrm{AF}}(\alpha),$$

where the r.h.s. equation states an equivalent reformulation of the ALF entropy Def. 3,(i) in terms of the S-entropy Def. 1 together with the new Def. 4 of the map of a partition: it is easy to see that the two states $\psi_\varphi[\alpha]$ *and* $\varphi\circ\gamma[\alpha]$ *on* $M_n(\mathbb{C})$ *coincide. Further, both sides of the inequality are* independent *of the order of the tuple* α.

(ii) *For the 'trivial' partition* $\nu = (\mu_1 1\!\!1, \mu_2 1\!\!1, \dots, \mu_n 1\!\!1) \in \mathcal{O}_1(\mathcal{A},n)$, *with* $\mu_i \in \mathbb{C}$ $\forall i = 1,\dots,n \in \mathbb{N}$ *such that* $\sum_{i=1}^n |\mu_i|^2 = 1$, *the entropy vanishes*: $HH_\varphi(\nu) = 0$.

(iii) *For a faithful state* φ *and any* $B \in \mathcal{A}$, *we define the linear functional* $[\varphi^{1/2}B\varphi^{1/2}]$ *on* $\mathcal{A}$ *as also before Prop. (VIII.3) in [8]: we identify* $\mathcal{A}$ *with its isomorphic image in the GNS representation constructed with* φ, *and on the generated von Neumann algebra in this representation we use the modular automorphism group* $\sigma_t^\varphi (t \in \mathbb{R} \subset \mathbb{C})$ *of* φ *to define this linear functional by*:

$$[\varphi^{1/2}B\varphi^{1/2}](A) := \varphi(A\sigma^\varphi_{-i/2}(B)),\ \ \forall A \in \mathcal{A}.$$

Then, for any operational partition β *of the unit by mutually orthogonal projections, i.e.* $\beta = (p_i \in \mathcal{A})_{i=1,\dots,n} \in \mathcal{O}_1(\mathcal{A},n)$ *such that* $p_i = p_i^* = p_i^2$, $p_ip_j = 0$ $\forall i \neq j$, *we have the inequalities*

$$HH_\varphi(\beta) \le \sum_{i=1}^n \eta(\varphi(p_i)) = S(\varphi\circ\gamma[\beta])$$

and

$$HH_\varphi(\beta) \ge 2S(\varphi\circ\gamma[\beta]) - \sum_{i=1}^n\sum_{j=1}^n \eta\Big([\varphi^{1/2}p_i\varphi^{1/2}](p_j)\Big) \ge 0,$$

where as usual, $\eta(x) \equiv -x\log x$ $\forall x \in [0,1]$.

If in addition $\beta \in \mathcal{O}_1(\mathcal{A}_\varphi)$, *where the* φ*-centralizer* $\mathcal{A}_\varphi := \{B \in \mathcal{A} | \varphi(AB) = \varphi(BA)\ \forall A \in \mathcal{A}\}$ *coincides with the fixed point algebra of* σ_t^φ *restricted to* $\mathcal{A}$, *then* $HH_\varphi(\beta) = -\sum_{i=1}^n \varphi(p_i)\log\varphi(p_i)$.

More generally, for not necessarily faithful φ *but with* $\mathcal{A}_\varphi$ *still defined as previously above, if* $\alpha = (A_i)_i \in \mathcal{O}_1(\mathcal{A}_\varphi)$ *is an 'anti–orthogonal' partition in the sense that* $A_iA_j^* = 0$ $\forall i \neq j$, *then always* $HH_\varphi(\alpha) = H_\varphi^{\mathrm{AF}}(\alpha)$ *holds (i.e., the inequality (i) is saturated). Note that for such a partition also* $\varphi(A_i^*A_j) = 0$ $\forall i \neq j$ *holds.*

(iv) *$HH_\varphi(\alpha\vec{\nabla}\beta) \geq HH_\varphi(\alpha)$ for $\alpha, \beta \in \mathcal{O}_1(\mathcal{A})$. Note that $HH_\varphi(\alpha\vec{\nabla}\beta) \neq HH_\varphi(\beta\vec{\nabla}\alpha)$, i.g.*

(v) *If $\theta : \mathcal{A} \to \mathcal{A}$ is a unital $*$-endomorphism with $\varphi \circ \theta = \varphi$, then $HH_\varphi(\theta(\alpha)) \leq HH_\varphi(\alpha)$ for $\alpha \in \mathcal{O}_1(\mathcal{A})$. In particular, equality holds for a $*$-automorphism θ.*

(vi) *For $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ of tensor product form, and with the canonical tensor product $\alpha_1 \otimes \alpha_2 \in \mathcal{O}_1(\mathcal{A})$ of two respective partitions $\alpha_1 = (A_i)_i \in \mathcal{O}_1(\mathcal{A}_1)$ and $\alpha_2 = (B_j)_j \in \mathcal{O}_1(\mathcal{A}_2)$ defined by $\alpha_1 \otimes \alpha_2 := (\alpha_1 \otimes \mathbb{1}_2)\vec{\nabla}(\mathbb{1}_1 \otimes \alpha_2) = (A_i \otimes B_j)_{(i,j)}$, the hybrid entropy with a product state $\varphi = \varphi_1 \otimes \varphi_2$ on $\mathcal{A}$ is superadditive:*

$$HH_\varphi(\alpha_1 \otimes \alpha_2) \geq HH_{\varphi_1}(\alpha_1) + HH_{\varphi_2}(\alpha_2).$$

Proof. We give a brief sketch of the methods of proof for these properties:

(i) follows directly from Def. 2,(i) and from Def. 1, which together with (i) also implies (ii), observing that $\varphi \circ \gamma[\nu]$ in (ii) is a *pure* state on $M_n(\mathbb{C})$.

The first inequality of (iii) is just a repetition of (i) using Def. 1 in this special case, and the second inequality is an immediate consequence of Prop. (VIII.4) and Lemma (VIII.5) in [8], observing that with the Abelian n-dimensional subalgebra $\mathcal{B} \subset \mathcal{A}$ generated by β, we have by Proposition (III.6,a) in [8]: $HH_\varphi(\beta) = H_\varphi(\mathcal{B})$, where on the r.h.s. we use the notation from the end of Remark 2. The third statement of (iii) follows from the first two inequalities and the definition of $[\varphi^{1/2}p_i\varphi^{1/2}]$. Note that generally, $\varphi = \sum_{i=1}^{n}[\varphi^{1/2}p_i\varphi^{1/2}]$ is a decomposition of φ. The more general final claim follows directly from Def. 2.

(iv) resp. (v) follow from Proposition (III.6), (a) resp. (b), in [8]. Finally, (vi) is a consequence of the superadditivity of the single–argument CNT entropy Def. 2,(i) w.r.t. tensor product maps, which is part of the argument in Lemma (3.4) of [20] (and using the order independence (i) in the proof of (vi)). ■

PROPOSITION 2. *The hybrid entropy Def. 5,(iii) of an endomorphism w.r.t. a partition has the following general algebraic properties:*

(i) *For $\alpha \in \mathcal{O}_1(\mathcal{A}, n)$, we have the general upper bound $hh_\varphi(\theta, \alpha) \leq \log n$.*

(ii) *$hh_\varphi(\theta^N, \alpha\vec{\nabla}\theta(\alpha)\vec{\nabla}\dots\vec{\nabla}\theta^{N-1}(\alpha)) \geq N \cdot hh_\varphi(\theta, \alpha)$, $\forall N \in \mathbb{N}$. If in Def. 5,(iii) of the r.h.s., even the limit exists (not only the limit superior), then equality holds.*

(iii) *For all $*$-isomorphisms $\sigma : \mathcal{A} \to \sigma(\mathcal{A})$, $hh_{\varphi\circ\sigma^{-1}}(\sigma \circ \theta \circ \sigma^{-1}, \sigma(\alpha)) = hh_\varphi(\theta, \alpha)$.*

(iv) *$hh_\varphi(\theta, \alpha) \leq h_\varphi^{\mathrm{AF}}(\theta, \alpha)$, where the r.h.s. is the ALF entropy of Def. 3,(iii).*

(v) *For $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\varphi = \varphi_1 \otimes \varphi_2$ as in Prop. 1,(vi), with a product $*$-endomorphism $\theta = \theta_1 \otimes \theta_2$ of two endomorphisms θ_i of $\mathcal{A}_i$ with $\varphi_i \circ \theta_i = \varphi_i$ $(i = 1, 2)$, the entropy is superadditive: $hh_\varphi(\theta, \alpha_1 \otimes \alpha_2) \geq hh_{\varphi_1}(\theta_1, \alpha_1) + hh_{\varphi_2}(\theta_2, \alpha_2)$ for $\alpha_i \in \mathcal{O}_1(\mathcal{A}_i)$.*

Proof. (i) is obvious from Prop. 1,(i) and the definitions. (ii) and also (iii) follow directly from Def. 5,(iii) itself, and (iv) is a more refined consequence of Prop. 1,(i). Finally, (v) follows from Prop. 1,(vi) and the definitions, after the observation that the operation $\otimes$ on $\mathcal{O}_1(A_1) \times \mathcal{O}_1(\mathcal{A}_2)$ is distributive w.r.t. the (multiple) operation $\vec{\nabla}$ on $\mathcal{O}_1(\mathcal{A}_1)$ resp. on $\mathcal{O}_1(\mathcal{A}_2)$. ■

COROLLARY 3. *The hybrid entropy Def. 5,(iv) of an endomorphism given a $*$-subalgebra has the following general properties:*

(i) *$hh_{\varphi,\mathcal{B}}(\theta^N) \geq N \cdot hh_{\varphi,\mathcal{B}}(\theta)$, $\forall N \in \mathbb{N}$. This inequality shows that the identity* *automorphism* $\theta = \mathrm{Id}_{\mathcal{A}}$ *has either* $hh_{\varphi,\mathcal{B}}(Id_{\mathcal{A}}) = 0$, *or in principle also* $hh_{\varphi,\mathcal{B}}(\mathrm{Id}_{\mathcal{A}}) = \infty$ (*but no example is known for the latter* ∞; *compare Prop. 6 below*, *however*).

(ii) $hh_{\varphi\circ\sigma^{-1},\sigma(\mathcal{B})}(\sigma\circ\theta\circ\sigma^{-1}) = hh_{\varphi,\mathcal{B}}(\theta)$, *for all* $*$*-isomorphisms* $\sigma : \mathcal{A} \to \sigma(\mathcal{A})$.

(iii) $hh_{\varphi,\mathcal{B}}(\theta) \le h^{\mathrm{AF}}_{\varphi,\mathcal{B}}(\theta)$, *where the r.h.s. is the ALF entropy of Def. 3,(iv).*

(iv) *In the situation of Prop. 2,(v), for* $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ *with* $*$*-subalgebras* $\mathcal{B}_i \subset \mathcal{A}_i$ $(i = 1, 2)$, *the entropy is superadditive*: $hh_{\varphi,\mathcal{B}}(\theta) \ge h_{\varphi_1,\mathcal{B}_1}(\theta_1) + h_{\varphi_2,\mathcal{B}_2}(\theta_2)$.

Proof. (i), (ii), resp. (iii) follow straightforwardly from Prop. 2, (ii), (iii) resp. (iv). Finally, (iv) here follows from Prop. 2,(v) together with the observation that $\{\alpha_1\otimes\alpha_2|\alpha_i \in \mathcal{O}_1(\mathcal{B}_i),\ \ i = 1, 2\} \subset \mathcal{O}_1(\mathcal{B}_1 \otimes \mathcal{B}_2)$ is a (strict) set inclusion of these classes. ∎

COROLLARY 4. *Let* X *be a compact Hausdorff space*, $T : X \to X$ *a homeomorphism and* $\mu = \mu \circ T^{-1}$ *a* T*-invariant Borel probability measure on* X. *By* $h^{\mathrm{KS}}_{\mu}(T)$ *we denote the Kolmogorov–Sinai entropy of* T *w.r.t.* μ *[12]. Let* θ_T *be the induced* $*$*-automorphism of* $\mathcal{A} = C(X)$, *via* $\theta_T(A) = A \circ T\ \ \forall A \in \mathcal{A}$, *and* $\omega_\mu = \int_X d\mu$ *be the corresponding state on* $\mathcal{A}$. *Then we have*:

$$hh_{\omega_\mu,\mathcal{A}}(\theta_T) \le h^{\mathrm{KS}}_{\mu}(T).$$

Proof. In [2] it is shown that $h^{\mathrm{AF}}_{\omega_\mu,\mathcal{A}}(\theta_T) = h^{\mathrm{KS}}_{\mu}(T)$, see also the corresponding Chapter 5 in the Thesis of Tuyls [21] (Section 5.2). Applying Cor. 3,(iii) above gives the result. ∎

THEOREM 5. *In the situation of the above Corollary 4, the final inequality is in fact a general equality* (*as for the ALF entropy*, *also quoted in the proof of that latter Corollary*).

Remark 5. For the less trivial proof of the converse inequality to Cor. 4, we have to refer the reader to [11].

The proof of the analog here for the purely measure–theoretic Theorem (3.3) in [2], however, can be made short enough to be postponed to this Remark: Let X be a finite (Lebesgue) measure space with a probability measure μ and with an invertible measure–preserving automorphism $T : X \to X$, i.e. such that $\mu \circ T = \mu \circ T^{-1} = \mu$. We use the analogous notations ω_μ resp. θ_T as in the topological case of Cor. 4 above, but now for the Abelian C*-algebra $\mathcal{A} = L^\infty(X, \mu)$.

Theorem (3.3) in [2] says that also here $h^{\mathrm{AF}}_{\omega_\mu,\mathcal{A}}(\theta_T) = h^{\mathrm{KS}}_{\mu}(T)$, and the general inequality Cor. 3,(iii) applies again in the form $hh_{\omega_\mu,\mathcal{A}}(\theta_T) \le h^{\mathrm{AF}}_{\omega_\mu,\mathcal{A}}(\theta_T)$. The converse inequality follows from Prop. 1,(iii) applied to any partition $\beta = (p_1, \ldots, p_n) \in \mathcal{O}_1(\mathcal{A}, n)$ with mutually orthogonal projections (i.e. characteristic functions of measurable sets) $p_i = p_i^* = p_i^2 \in \mathcal{A}$ $(\forall i = 1, \ldots, n)$, $p_ip_j = 0\ \ \forall i \ne j$, where in this Abelian case $HH_{\omega_\mu}(\beta)$ is equal to the classical KS entropy functional of the corresponding n-element measurable partition of X (as the centralizer $\mathcal{A}_{\omega_\mu} = \mathcal{A}$; the details are left to the reader). ∎

Now, we consider instead of the Abelian von Neumann algebra $\mathcal{A} = L^\infty(X, \mu)$ an extremely *non*–Abelian case of von Neumann algebras: Let $\mathcal{A}$ be an infinite factor (cf. e.g. [7]) with an automorphism θ of $\mathcal{A}$ and an invariant state $\varphi = \varphi \circ \theta$ on $\mathcal{A}$.

PROPOSITION 6. *There is a sequence of partitions* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathcal{O}_1(\mathcal{A})$ *such that* $\alpha_n \in \mathcal{O}_1(\mathcal{A}, n)$ *and* $h^{\mathrm{AF}}_{\varphi}(\theta, \alpha_n) = \log n$ *while always* $hh_{\varphi}(\theta, \alpha_n) = 0$, *independently of* θ *and* φ.

Note that this implies that here $h^{\mathrm{AF}}_{\varphi,\mathcal{A}}(\mathrm{Id}_{\mathcal{A}}) = \infty$ *for the identity automorphism* $\mathrm{Id}_{\mathcal{A}}$ *on* $\mathcal{A}$, *see also Cor. 3,(i) above.*

Remark 6. For the proof of the strict positivity of the ALF entropy, we can refer the reader to Theorem 4.3 of [21] (Section 4.6), while for the proof of the hybrid entropy vanishing for this sequence of partitions, we have to refer to [11]. We still have to leave it as an open problem to decide if the second alternative of the dichotomy in Cor. 3,(i) is realized for the hybrid entropy, or if the latter is always equal to zero for the identity automorphism on any C*-algebra. Also, Remark 4 has to be reconsidered here.

4. Three familiar examples for the calculation of the 'hybrid' entropy

EXAMPLE 1 (The shift on the quantum spin chain). Let $\mathcal{A} = \bigotimes_{k\in\mathbb{Z}}(M_n(\mathbb{C}))_k$ be the n^∞-UHF algebra ('quantum spin chain') as the bilaterally infinite tensor product of copies of the $(n \times n)$-matrix algebra $M_n(\mathbb{C})$. We choose the norm–dense $*$-subalgebra $\mathcal{A}_\infty \subset \mathcal{A}$ using the natural AF structure $\mathcal{A}_\infty = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ (algebraic inductive limit), where we put $\mathcal{A}_k = \mathbb{1}_n^{\otimes\infty} \otimes \bigotimes_{j=-k}^{k}(M_n)_j \otimes \mathbb{1}_n^{\otimes\infty}$ $\forall k \in \mathbb{N} \cup \{0\}$.

The unit shift on $\mathbb{Z}$ determines a $*$-automorphism θ of $\mathcal{A}$ by $\theta(\mathbb{1}_n^{\otimes\infty} \otimes (M_n)_k \otimes \mathbb{1}_n^{\otimes\infty}) = \mathbb{1}_n^{\otimes(\infty+1)} \otimes (M_n)_{k+1} \otimes \mathbb{1}_n^{\otimes(\infty-1)}$, and we let φ be any translation–invariant state on $\mathcal{A}$: $\varphi\circ\theta = \varphi$. We use the additional notation $\mathcal{B}_{[\ell,m]} = \mathbb{1}_n^{\otimes\infty} \otimes \bigotimes_{j=\ell}^{m}(M_n)_j \otimes \mathbb{1}_n^{\otimes\infty}$ $\forall \ell < m \in \mathbb{Z}$, such that $\mathcal{A}_k = \mathcal{B}_{[-k,k]}$.

PROPOSITION 7. $hh_{\varphi,\mathcal{A}_\infty}(\theta) = \limsup_{k\to\infty} \frac{1}{k} H_\varphi(\mathcal{B}_{[0,k]}) \le s(\varphi)$, *where the* H_φ *is the CNT entropy functional of Def. 2,(i) with the notation from the end of Remark 2; and* $s(\varphi)$ *is the* quantum entropy density *of* $\varphi = \varphi \circ \theta$ *(cf. [7]). If the state* φ *satisfies the so–called 'cluster condition' as defined in equations (11.18), (11.19) of [16], then* $hh_{\varphi,\mathcal{A}_\infty}(\theta) = s(\varphi)$.

Remark 7. We have to refer to [11] for the proof.

It is shown in [8] that generally the CNT entropy $h_\varphi(\theta) \le \limsup_{k\to\infty} \frac{1}{k} H_\varphi(\mathcal{B}_{[0,k]})$, where equality holds only if a possibly stronger clustering condition holds for φ (cf. pp. 203/204 in [16]).

It is shown in [1] that the Alicki–Fannes entropy is *generally* (i.e. without any additional clustering condition on φ) equal to $h^{\mathrm{AF}}_{\varphi,\mathcal{A}_\infty}(\theta) = s(\varphi) + \log n$, with n from the notation of Ex. 1 above; cf. also Theorem 6.2 (Sect. 6.2) in the Thesis of Tuyls [21], where it is also shown (Theorem 6.3 in Sect. 6.3) that this result may be extended to a so–called algebra of 'smooth' elements $\mathcal{A}_q^\infty \subset \mathcal{A}$ (any $q > 0$), defined via so–called tensorable oscillation norms (see Sect. 6.1 in [21] and the reference there). We refer the reader to [11] as to analogous results for the hybrid entropy, extending Prop. 7 above.

EXAMPLE 2 (The Powers–Price shift). If $X \subseteq \mathbb{N}$, we denote by $\mathcal{A}(X)$ the (universal) C*-algebra generated by a two–sided sequence $(s_n)_{n\in\mathbb{Z}}$ of self–adjoint unitaries s_n with the commutation relation $s_i s_j = (-1)^{g(|i-j|)} s_j s_i$, $\forall i \ne j \in \mathbb{Z}$, where g is the characteristic function of X.

The canonical trace τ on $\mathcal{A}(X)$ is defined by $\tau(\mathbb{1}) = 1$ and $\tau(s_{i_1} s_{i_2} \cdots s_{i_k}) = 0$ $\forall i_1 < i_2 < \ldots < i_k$, $\forall k \in \mathbb{N}$. We denote by θ_X the shift automorphism on $\mathcal{A}(X)$: $\theta_X(s_i) = s_{i+1}$, $\forall i \in \mathbb{Z}$, such that obviously $\tau = \tau \circ \theta_X$ is invariant. For the entropy definition

Def. 5,(iv) we will use the canonical AF structure for $\mathcal{A}(X)$, i.e. with $A_N := \mathrm{C}^*(\{s_i | i = -N,\dots,N\})$, let $\mathcal{A}_\infty(X) = \bigcup_{N\in\mathbb{N}} \mathcal{A}_N$, so that $\mathcal{A}(X) = \overline{\mathcal{A}_\infty(X)}$ (norm completion).

In addition, we use again the notation $\mathcal{B}_{[k,\ell]} := \mathrm{C}^*(\{s_k, s_{k+1},\dots,s_\ell | k < \ell \in \mathbb{Z}\})$, so that $\mathcal{B}_{[-N,N]} = \mathcal{A}_N$. Recall that by Lemma (3.5) of [17], $\mathcal{B}_{[0,n]}$ is always isomorphic to the direct sum of $2^{p(n)}$ many $2^{k(n)} \times 2^{k(n)}$-matrix algebras, where $2k(n) + p(n) = n + 1$, $\forall n \in \mathbb{N}$ (correcting an obvious misprint in [17], where the factor 2 of $k(n)$ is missing, with the same notation). In particular, the linear dimension $\dim \mathcal{B}_{[0,n]} = 2^{(n+1)}$.

By Theorem (4.6) in [17], the sequence $(p(n))_{n\in\mathbb{N}}$ consists of a concatenation of finite 'strings' of the form $(0, 1, 2, 3,\dots,(m-1), m,(m-1),\dots, 3, 2, 1)$ with $m \in \mathbb{N}$, where the value of m may vary in the sequence (in particular, $p(n)$ may be unbounded in n). In turn, for any such sequence $(p(n))$ there is $X \subseteq \mathbb{N}$ with $\mathcal{A}_\infty(X)$ leading to this $(p(n))$.

PROPOSITION 8. (i) *Generally*, $hh_{\tau,\mathcal{A}_\infty(X)}(\theta_X) \le \limsup_{n\to\infty} \frac{1}{n} H_\tau(\mathcal{B}_{[0,n]}) \le \log 2$, *where* H_τ *is the entropy functional of Def. 2,(i) with the notation from the end of Remark 2, or equivalently, as already in [9].*

(ii) *If* X *is 'nonperiodic' (in the sense of [10], i.e.* $-X \cup \{0\} \cup X$ *is nonperiodic)* and *the sequence* $(p(n))_{n\in\mathbb{N}}$ *as above is* bounded, *then* $hh_{\tau,\mathcal{A}_\infty(X)}(\theta_X) \le \frac{1}{2}\log 2$.
If for nonperiodic X, *the sequence* $(p(n))_{n\in\mathbb{N}}$ *is* not *bounded, clearly at least the following sequence of ratios* $r(n) := (p(n) + k(n))(p(n) + 2k(n))^{-1}$ *is bounded:* $\frac{1}{2} \le r(n) \le 1$. *Then we have that* $hh_{\tau,\mathcal{A}_\infty(X)}(\theta_X) \le \limsup_{n\to\infty} r(n) \cdot \log 2$.

(iii) *In particular, at least if* $X = \mathbb{N}$, *in fact equality holds:* $hh_{\tau,\mathcal{A}_\infty(X)}(\theta_X) = \frac{1}{2}\log 2$. *On the other hand, if* X *is periodic in the sense of [10] (i.e. 'mirror–periodic' in the sense of [17]), we get generally the 'classical' value:* $hh_{\tau,\mathcal{A}_\infty(X)}(\theta_X) = \log 2$.

Remark 8. See [11] for the proof. It is shown in Theorem (1) of [4] that the ALF entropy Def. 3 is identically $h^{\mathrm{AF}}_{\tau,\mathcal{A}_\infty(X)}(\theta_X) \equiv \log 2$, independently of X. Compare also e.g. [10, 13, 14, 19] for the CNT resp. Connes–Størmer entropy Def. 2 in this example.

EXAMPLE 3 (The quantum Arnold map). Using the same notation as [5], we define the $SL(2,\mathbb{Z})$-action on the 'irrational rotation' C*-algebra as follows: Let $\mathcal{B}_q$ be the (universal) $*$-algebra generated by unitaries $W(\chi)$, $\chi \in \mathbb{Z}^2$, satisfying the commutation relations $W(\chi_1)W(\chi_2) = e^{iq\sigma(\chi_1,\chi_2)/2}W(\chi_1 + \chi_2)$, $\forall \chi_1, \chi_2 \in \mathbb{Z}^2$, where $q \in [0, 2\pi)$ is 'preferably' an *irrational* multiple of 2π, and where $\sigma(\chi_1, \chi_2) = \chi_1(1)\chi_2(2) - \chi_1(2)\chi_2(1)$ for $\chi_i = (\chi_i(1), \chi_i(2)) \in \mathbb{Z}^2, i = 1, 2$.

We denote by $\mathcal{A}_q = \overline{\mathcal{B}_q}$ the generated C*-algebra ('norm completion'), and on $\mathcal{A}_q$ we define a tracial state τ by $\tau(W(\chi)) = \delta_{\chi 0}$ $\forall \chi \in \mathbb{Z}^2$, and a $*$-automorphism of $\mathcal{A}_q$ by $\theta_T(W(\chi)) = W(T^t\chi)$ for any $T \in SL(2,\mathbb{Z})$, $\mathrm{Tr}_2(T) > 2$.

Then it follows trivially from the main Theorem in [5] together with Prop. 3,(iv) here that for any $q \in [0, 2\pi)$, $hh_{\tau,\mathcal{B}_q}(\theta_T) \le \log\lambda$, where λ is the *larger* eigenvalue of T.

LEMMA 9. *Assume that for each* $m \in \mathbb{N}$, *we can find a partition* $\beta \in \mathcal{O}_1(\mathcal{B}_q)$ *of the form* $\beta = (e^{i\beta_1}W(\chi_1)/\sqrt{\ell},\dots, e^{i\beta_\ell}W(\chi_\ell)/\sqrt{\ell})$, *with* $\chi_1,\dots,\chi_\ell \in \mathbb{Z}^2$ *and* $\beta_1,\dots,\beta_\ell \in \mathbb{R}$, *such that* $hh_\tau(\theta_T^m, \beta) \ge c \cdot \log[\lambda^m]$, *where we assume that such a* $c \in (0, 1]$ *exists* independently *of* m, *and where* $[\nu]$ *is the integer part of* $\nu \in \mathbb{R}$. *Then* $hh_{\tau,\mathcal{B}_q}(\theta_T) \ge c\cdot\log\lambda$.

See [11] for the proof. In order to prove the *assumption* of Lemma 9, along the lines of [5] (in the remainder of the Proof of Prop. (1) there), we would need a *lower* bound for the entropy of a partition $\beta \in \mathcal{O}_1(\mathcal{B}_q, \ell)$ of the general form as in Lemma 9 above but in particular with $\chi_i \neq \chi_j \ \forall i \neq j \in \{1, \ldots, \ell\}$, namely: $HH_\tau(\beta) \geq c \cdot \log \ell$, for some 'universal' $c \in (0, 1]$ *independent* of ℓ (and of $\beta \in \mathcal{O}_1(\mathcal{B}_q, \ell)$). In this direction, we have:

PROPOSITION 10. *Let $\beta \in \mathcal{O}_1(\mathcal{B}_q, \ell)$ be of the form as in Lemma 9, but in particular with $\chi_i \neq \chi_j$ for $i \neq j \in \{1, \ldots, \ell\}$, and for the sake of simplicity with $\beta_1 = \ldots = \beta_\ell = 0$. Then $HH_\tau(\beta) > 0$ (strictly positive), with a lower bound depending on ℓ (but with this proof, not of the form $c \cdot \log \ell$). In particular, for $\ell = 2$ we even have that $HH_\tau(\beta) = \log 2$,* independently *of $\beta \in \mathcal{O}_1(\mathcal{B}_q, 2)$.*

Remark 9. See [11] for the proof of this partial result, to be extended for $\ell > 2$. In the Thesis of Tuyls [21], the 'classical' value of the ALF entropy in this example is again extended even for the so-called 'smooth' subalgebra $\mathcal{S}_q \supset \mathcal{B}_q$ of $\mathcal{A}_q$: $h^{\mathrm{AF}}_{\tau,\mathcal{S}_q}(\theta_T) = \log \lambda$.

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