DILATION THEOREMS FOR COMPLETELY POSITIVE MAPS AND MAP-VALUED MEASURES

EWA HENSSZ-CHĄDZIŃSKA, RYSZARD JAJTE and ADAM PASZKIEWICZ

Faculty of Mathematics, Łódź University
ul. Banacha 22, 90-238 Łódź, Poland
E-mail: ewahensz@math.uni.lodz.pl, rjajte@math.uni.lodz.pl, adampasz@math.uni.lodz.pl

Abstract. The Stinespring theorem is reformulated in terms of conditional expectations in a von Neumann algebra. A generalisation for map-valued measures is obtained.

1. Introduction. Traditionally, each dilation theorem is obtained by a construction of a ‘huge’ (Hilbert) space \( \mathcal{H} \) containing a given space \( H \) in the following manner. A system \( \psi(\cdot) \) of operators in \( H \) or transformations of an algebra acting in \( H \) can be represented in the form

\[
\psi(\cdot) = P_H \Phi(\cdot) P_H |_H \quad (1.1)
\]

where \( \Phi(\cdot) \) is more regular than \( \psi(\cdot) \). Throughout, \( P_H \) denotes the orthogonal projection of \( \mathcal{H} \) onto \( H \).

The most impressive results in this theory are effects of sophisticated indexing of linear bases of \( \mathcal{H} \) and a ‘magic touch’ of scalar product. Theorems of B. Sz.-Nagy [9] and K.R. Parthasarathy [5] are excellent examples of such approach.

Dealing with operator algebras it seems to be most natural and physically meaningful to use the conditional expectation \( E \) [7, p.116] instead of \( P_H(\cdot)P_H \) (cf L. Accardi, M. Ohya [1]).

In the paper we follow both ideas. Roughly speaking we represent a completely positive map-valued measure via the following dilation. Namely, any completely positive map turns into multiplication by a projection in such a way that the map-valued measure is ‘dilated’ to a spectral measure (Section 2).

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The outstanding theorem of Stinespring [6] gives the dilation of a completely positive map \( \psi \) in a \( C^* \)-algebra to its \( * \)-representation \( \Phi \) via formula (1.1). Passing to a \( W^* \)-algebra Stinespring’s theorem can be formulated using a normal conditional expectation \( E \) from a ‘huge’ algebra \( N \) onto \( M \) instead of \( P_H(\cdot)P_H \). Such a new version of Stinespring’s result will be proved in Section 3 together with a dilation theorem for positive map-valued measures.

Section 4 is devoted to a short comparison of the results just mentioned with the previous ones concerning commutative \( W^* \)-algebras.

2. Dilation of completely positive map-valued measure. Let \( M \) be a von Neumann algebra of operators acting in a Hilbert space \( H \). By \( CP(M) \) we shall denote the set of completely positive linear maps in \( M \). Let \( (X, \Sigma) \) be a measurable space and \( Q : \Sigma \to CP(M) \) be a \( \sigma \)-additive operator-valued measure (i.e. \( \Sigma \ni \Delta \mapsto Q(\Delta)x \) is \( \sigma \)-additive in the ultra weak topology in \( M \) for each \( x \in M \) with \( Q(X)1 = 1 \).

**Theorem 2.1.** There exist a Hilbert space \( H \), a natural linear injection \( V : H \to H \), a \( \sigma \)-additive vector measure \( e : \Sigma \to \text{Proj} H \), such that
\[
Q(\Delta)x = V^* e(\Delta)\Phi(x)V, \quad x \in M, \quad \Delta \in \Sigma.
\] (2.1)

Moreover, \( e(\Delta) \) is a central projection in \( (\Phi(M) \cup e(\Sigma))'' \).

**Proof.** Let us consider the algebraic tensor product of vector spaces
\[
H_0 = M \otimes H \otimes SF(X, \Sigma)
\]
where \( SF(X, \Sigma) \) denotes the vector space of simple functions on \( (X, \Sigma) \).

Let us extend the measure \( Q \) from \( \Sigma \) to a linear mapping on \( SF(X, \Sigma) \) putting
\[
Q(f) = \sum_{\kappa=1}^k c_\kappa Q(\Delta_\kappa) \quad \text{for} \quad f = \sum_{\kappa=1}^k c_\kappa 1_{\Delta_\kappa},
\]
where \( \Delta_\kappa \in \Sigma, \kappa = 1, \ldots, k \).

In the sequel we shall briefly write \( \Delta \) instead of \( 1_{\Delta} \). Notice that \( H_0 \) is formed by elements of the form
\[
\xi = \sum_{i=1}^n x_i \otimes h_i \otimes \Delta_i
\] (2.2)
where \( x_i \in M, h_i \in H, \Delta_i \in \Sigma, i = 1, \ldots, n, n = 1, 2, \ldots \).

In the space \( H_0 \) we can define a sesquilinear form \( \langle \cdot, \cdot \rangle \) by
\[
\langle \xi, \eta \rangle = \sum_{i=1}^n \sum_{j=1}^m (Q(\Delta_i \cap \Gamma_j)(y_j x_i) h_i, g_j)
\]
for
\[
\xi = \sum_{i=1}^n x_i \otimes h_i \otimes \Delta_i \quad \text{and} \quad \eta = \sum_{j=1}^m y_j \otimes g_j \otimes \Gamma_j.
\]
The symbol \( \langle \cdot, \cdot \rangle \) denotes here the inner product in \( H \). We shall show that \( \langle \cdot, \cdot \rangle \) is positive.

Indeed, for \( \xi \) of form (2.2) we consider the partition \( \{\sigma_1, \ldots, \sigma_k\} \) of \( \bigcup_{i=1}^n \Delta_i \) given by
\[ \langle \xi, \xi \rangle = \sum_{i,j=1}^{n} (Q(\Delta_i \cap \Delta_j)(x^*_i x_i)h_i, h_j) = \sum_{i,j=1}^{n} \left( \sum_{s=1}^{k} e^*_s e^i_s Q(x_s)(x^*_j x_i)h_i, h_j \right) \]

\[ = \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(x_s)(x^*_j x_i)h^*_i, h^*_j) \]

where \( h^*_i = e^i_s h_i, i = 1, \ldots, n. \)

The complete positivity of \( Q(\sigma_s) \) gives the inequality

\[ \sum_{i,j=1}^{n} (Q(\sigma_s)(x^*_j x_i)h^*_i, h^*_j) \geq 0, \quad s = 1, \ldots, k, \]

thus \( \langle \xi, \xi \rangle \geq 0. \) Let us denote \( \|\xi\|_0 = \sqrt{\langle \xi, \xi \rangle} \) and put \( H_1 = H_0 / N \) where \( N = \{ \xi \in H_0 : \|\xi\|_0 = 0 \}. \) Finally, let us set \( H = \overline{H_1}. \)

We define \( V : H \rightarrow H \) by putting \( Vh = [1 \otimes h \otimes X] \) for \( h \in H. \) Then

\[ \langle Vh, Vh \rangle = ((Q(X)1)h, h) = (h, h) \]

so \( V \) is an isometry.

Now let us construct a \( * \)-representation \( \Phi \) of the algebra \( M \) in \( H. \) Namely, for \( x \in M \) let us set

\[ \Phi(x) : [y \otimes h \otimes \Delta] \mapsto [xy \otimes h \otimes \Delta] \]

where \( y \in M, h \in H, \Delta \in \Sigma. \) \( \Phi(x) \) is well defined. Indeed, we prove the following inequality

\[ \left\| \sum_{i=1}^{n} xy_i \otimes h_i \otimes \Delta_i \right\|_0 \leq \|x\| \cdot \left\| \sum_{i=1}^{n} y_i \otimes h_i \otimes \Delta_i \right\|_0 \]  \hspace{1cm} (2.3)

for \( y_i \in M, h_i \in H, \Delta_i \in \Sigma, i = 1, \ldots, n, n = 1, 2, \ldots. \) As above, we can write

\[ \left\| \sum_{i=1}^{n} y_i \otimes h_i \otimes \Delta_i \right\|_0^2 = \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(\sigma_s)(y^*_j y_i)h^*_i, h^*_j), \]

\[ \left\| \sum_{i=1}^{n} xy_i \otimes h_i \otimes \Delta_i \right\|_0^2 = \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(\sigma_s)(y^*_j x_i y_i)h^*_i, h^*_j). \]  \hspace{1cm} (2.4)

For a linear map \( \alpha : M \rightarrow M \) let us denote by \( \alpha^{(n)} \) the map \( \alpha^{(n)} : \text{Mat}_n(M) \rightarrow \text{Mat}_n(M) \) given by the formula

\[ \alpha^{(n)}([z_{i,j}]) = [\alpha(z_{i,j})] \]

where \([z_{i,j}]_{i,j \leq n} \in \text{Mat}_n(M). \) \( \text{Mat}_n(M) \) denotes here the \( C^* \)-algebra of all \( n \times n \) matrices \([z_{i,j}]_{i,j \leq n} \) with entries \( z_{i,j} \) in \( M. \)

Now, we follow Takesaki [10, p. 196]. The Schwarz inequality for operators, by the complete positivity of \( Q(\sigma_s), \) gives

\[ Q(\sigma_s)^{(n)}(\bar{y}^* \tilde{x}^* \tilde{x} y) \leq \|\tilde{x}\|^2 Q(\sigma_s)^{(n)}(\bar{y}^* \tilde{y}) \]  \hspace{1cm} (2.5)
for each $\tilde{x}, \tilde{y} \in \text{Mat}_n(\mathcal{M})$. Setting $\tilde{x} = [\delta_{i,j}]$, $\tilde{y} = [\delta_{i,j}]$, we get $\tilde{y}^* \tilde{x}^* \tilde{y} = [y_i^* x_j y_j]$, $\tilde{y}^* \tilde{x} = [y_i^* y_j]$. Thus, by (2.5) and \( ||x|| = ||x|| \), we have

$$\|Q(\sigma_s)(y_i^* x_j y_j)\| \leq \|x\|^2 \|Q(\sigma_s)(y_i^* y_j)\|.$$

Hence

$$\sum_{i,j=1}^n (Q(\sigma_s)(y_i^* x_j y_j) h_i^* h_j^*) \leq \|x\|^2 \sum_{i,j=1}^n (Q(\sigma_s)(y_i^* y_j) h_i^* h_j^*).$$

Finally, by (2.4), we get (2.3). Then \( \sum_{i=1}^n y_i \otimes h_i \otimes \Delta_i ||0 = 0 \) implies \( \sum_{i=1}^n y_i \otimes h_i \otimes \Delta_i ||0 = 0 \) and \( \Phi(x) \) is well defined. Obviously by (2.3), \( \Phi(x) : \mathcal{H} \rightarrow \mathcal{H} \) is a linear bounded operator in \( B(\mathcal{H}) \). It is easy to check that \( \Phi : \mathcal{M} \rightarrow B(\mathcal{H}) \) is a \(*\)-representation \( \mathcal{M} \in \mathcal{H} \).

Now for \( \Delta \in \Sigma \) we define \( e(\Delta) : \mathcal{H} \rightarrow \mathcal{H} \) putting

$$e(\Delta) : [y \otimes h \otimes \Delta'] \mapsto [y \otimes h \otimes (\Delta \cap \Delta')]$$

where \( y \in \mathcal{M}, h \in \mathcal{H}, \Delta' \in \Sigma \). The operator \( e(\Delta) \) is well defined because \( \|\sum_{i=1}^n y_i \otimes h_i \otimes \Delta_i ||0 = 0 \) implies \( \|\sum_{i=1}^n y_i \otimes h_i \otimes (\Delta \cap \Delta_i) ||0 = 0 \). Indeed, let \( \{\sigma_1, \ldots, \sigma_k\} \) be a partition of \( \bigcup_{i=1}^n \Delta_i \) given by \( \Delta, \Delta_1, \ldots, \Delta_k \). Let us put \( \varepsilon_1 = 1 \) when \( \sigma_s \subset \Delta_i \) and \( \varepsilon_2 = 0 \) when \( \sigma_s \cap \Delta_i = \emptyset \). Similarly, let \( \varepsilon_1 = 1 \) when \( \sigma_s \subset \Delta \) and \( \varepsilon_2 = 0 \) when \( \sigma_s \cap \Delta = \emptyset \). Then

$$\left\| \sum_{i=1}^n (y_i \otimes h_i \otimes (\Delta \cap \Delta_i)) \right\|_0^2 = \sum_{s=1}^k \varepsilon_1 \sum_{i,j=1}^n \varepsilon_2 Q(\sigma_s)(y_i^* y_j) h_i, h_j$$

$$\leq \sum_{s=1}^k \sum_{i,j=1}^n \varepsilon_1^2 \varepsilon_2 Q(\sigma_s)(y_i^* y_j) h_i, h_j = \left\| \sum_{i=1}^n y_i \otimes h_i \otimes \Delta_i \right\|_0^2$$

because, by the complete positivity of \( Q(\sigma_s) \), we have

$$\sum_{i,j=1}^n \varepsilon_1 \varepsilon_2 Q(\sigma_s)(y_i^* y_j) h_i, h_j \geq 0.$$
of the same physical system whereas the construction of a dilation in a bigger algebra means passing to a bigger system enjoying more regular evolution ([3], [2]).

In particular, the physical meaning of Stinespring’s theorem can be enriched if we express the dilation in terms of the conditional expectation in the enlarged algebra. Such a construction, with consequences for map-valued measures, will be done in this section.

It turns out that some natural properties of an equivalence relation in the lattice of projections are crucial.

A basic tool is the comparison theorem for projections ([8], Thm. 4.6).

**Theorem 3.1.** For any \( p, q \in \text{Proj} \mathcal{N} \), there exists a projection \( e \in \mathcal{N} \cap \mathcal{N}' \) such that \( pe \succ qe \) and \( p(1-e) \prec q(1-e) \).

Clearly, \( p \preceq q \) means \( uu^* = p, u^*u \leq q \) for some partial isometry \( u \in \mathcal{N} \).

The following consequence of the above theorem will be used.

**Proposition 3.2.** Let \( \mathcal{N} \) be a von Neumann algebra and let \( p \) be a projection in \( \mathcal{N} \) with the central support \( z(p) = 1 \). There exists a system of mutually orthogonal projections \( (p_i; i < k_0) \) in \( \text{Proj} \mathcal{N} \), \( k_0 \) being an ordinal number, such that \( p_i \preceq p, \sum_{i < k_0} p_i = 1 \), and \( p_1 = p \).

**Proof.** We use the transfinite induction, treating \( 1, 2, \ldots \) as ordinals. Denote \( e_1 = 0, p_1 = p \). Assume that, for some ordinals \( k \) and for any \( i < k \), projections \( e_i, p_i \in \text{Proj} \mathcal{N} \) satisfying the conditions

\[
\begin{align*}
& e_i \in \mathcal{N}', \\
& (e_i; i < k) \text{ are mutually orthogonal,} \\
& (p_i; i < k) \text{ are mutually orthogonal,} \\
& \sum_{j \leq i} p_j \geq \sum_{j \leq i} e_j, \\
& p_i \preceq p \\
\end{align*}
\]  

(3.1)

have already been defined. If \( \sum_{i < k} p_i = 1 \), the construction is complete with \( k_0 = k \).

If not, we consider separately the following two cases.

**Case 1.** Assume that \( \left( \sum_{j < k} e_j \right)^\perp p \preceq \left( \sum_{j < k} e_j \right)^\perp \left( \sum_{j < k} p_j \right)^\perp \). (*)

Then it is enough to put \( e_k = 0, p_k \) an arbitrary projection in \( \mathcal{N} \) satisfying

\[
\sum_{j < k} e_j \perp p_k \sim \left( \sum_{j < k} e_j \right)^\perp p_k \leq \left( \sum_{j < k} e_j \right)^\perp \left( \sum_{j < k} p_j \right)^\perp
\]

(clearly, \( p \sim q \) means \( p = u^*u, q = uu^* \), for some \( u \in \mathcal{N} \)).

**Case 2.** Assume that (*) does not hold. Then we consider the algebra

\[
\mathcal{M} = \left( \sum_{j < k} e_j \right)^\perp \mathcal{N} \left( \sum_{j < k} e_j \right)^\perp = \mathcal{N} \left( \sum_{j < k} e_j \right)^\perp.
\]

(3.2)

Restricting operators to a subspace \( \left( \sum_{j < k} e_j \right)^\perp (H) \), one can treat \( \mathcal{M} \) as a von Neumann algebra with the projections \( \tilde{p} = p \left( \sum_{j < k} e_j \right)^\perp, \tilde{p}_i = p_i \left( \sum_{j < k} e_j \right)^\perp \). By the comparison
theorem there exists a central projection in \( \mathcal{M} \), say \( e_k \), satisfying the conditions
\[
\tilde{p}_k \geq \left( \sum_{j < k} \tilde{p}_j \right)^{\perp} e_k \quad \text{and} \quad \tilde{p}(1_{\mathcal{M}} - e_k) \leq \left( \sum_{j < k} \tilde{p}_j \right)^{\perp} (1_{\mathcal{M}} - e_k).
\]

Since the reduction of \( \mathcal{N} \) to \( \mathcal{M} \) is done by the central projection \( \left( \sum_{j < k} e_j \right)^{\perp} \), \( e_k \) can be obviously treated as a central projection in \( \mathcal{N} \) as well.

Let \( \tilde{p}_k \) be an arbitrary projection in \( \mathcal{M} \) satisfying \( \tilde{p}_k \leq \left( \sum_{j < k} p_j \right)^{\perp} (1_{\mathcal{M}} - e_k) \), \( \tilde{p}_k \sim p(1_{\mathcal{M}} - e_k) \).

We put
\[
p_k = \tilde{p}_k + \left( \sum_{j < k} p_j \right)^{\perp} e_k.
\]

Obviously, we can treat \( p_k \) as a projection in \( \mathcal{N} \). All conditions (3.1) are now satisfied for \( k + 1 \) (instead for \( k \)).

Clearly, \( \sum_{i < k} p_i = 1 \) necessarily for some ordinal \( k \) (since \( \dim H \) is a fixed cardinal).

We shall need the following consequences of Proposition 3.2.

**Lemma 3.3.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be von Neumann algebras acting in Hilbert spaces \( H \) and \( \mathcal{H} \), respectively, with \( H \subset \mathcal{H} \). Denote by \( P_H \) the orthogonal projection from \( \mathcal{H} \) onto \( H \).

Assume that \( P_H^* \mathcal{M} P_H \subset \mathcal{N} \), the central support \( z(P_H) = 1 \).

Then there exists an isometric injection \( v : \mathcal{H} \to H \otimes K \), for some Hilbert space \( K \) such that
\[
v \mathcal{N} v^* \subset \mathcal{M} \otimes B(K), \quad (3.3)
\]
\[
v \zeta = \zeta \otimes \eta_1, \quad \zeta \in \mathcal{H}, \quad \text{for some } \eta_1 \in K. \quad (3.4)
\]

**Proof.** Keeping the notation of Proposition 3.2, with \( p = P_H \subset \mathcal{N} \), let us fix a Hilbert space \( K \) with an orthogonal basis \( (\eta_j, j < k_0) \). As \( p_i \not\preceq p \), we can use projections \( r_i \leq p_i \) satisfying \( p_i = w_i^* w_i \), \( r_i = w_i w_i^* \) for some partial isometries \( w_i \in \mathcal{N}, i < k_0 \).

Obviously, we can assume that \( w_1 = p \).

Let us take \( v_i \zeta = w_i \zeta \otimes \eta_i, i < k_0, \) for \( \zeta \in \mathcal{H} \). Then we get an isometry
\[
v = \sum_{i < k_0} v_i, \quad v : \mathcal{H} \to H \otimes K.
\]

Formula (3.4) is obvious. It remains to show (3.3) or, equivalently, \( \mathcal{N} \subset v^* \mathcal{M} \otimes B(K)v \).
This can be checked by the commutant technique as follows.

We have
\[
p_i \mathcal{N} p_i \cup \{ w_i^*, w_i \} \subset v^* \mathcal{M} \otimes B(K)v, \quad i < k_0. \quad (3.5)
\]

Indeed,
\[
p_i \mathcal{N} p_i = v^*(r_i M r_i \otimes \langle \cdot , \eta_i \rangle \eta_i) v,
\]
\[
w_i = v^*(r_i \otimes \langle \cdot , \eta_i \rangle \eta_i) v,
\]
\[
w_i^* = v^*(r_i \otimes \langle \cdot , \eta_i \rangle \eta_i) v.
\]
For example we check the first equality. Obviously \( p_k \mathcal{N} p_k = w_k^* \mathcal{N} w_k = w_k^* \mathcal{M} w_k \), and for any \( x \in \mathcal{M}, \zeta \in \mathcal{H} \), denoting \( \zeta_j = p_j \zeta, j < k_0 \), we have
\[
v \zeta = \sum_{j < k_0} w_j \zeta_j \otimes \eta_j
\]
and
\[
v^*(r_i x r_i \otimes \zeta, \eta) v \zeta = v^*(r_i x w_i \zeta \otimes \eta) = (w_i \cdot \zeta, \eta) i^*(r_i x w_i \zeta, \eta) = w_i^* x w_i \zeta.
\]
On the other hand, we have
\[
\left( \bigcup_{i < k_0} p_i \mathcal{N} p_i \cup \{ w_i, w_i^* \} \right)' = \mathcal{N}'.
\] (3.6)
The inclusion "\( \supset \)" is obvious. Conversely, let \( y \) commute with all \( p_k \mathcal{N} p_k, w_i, w_i^* \). Any arbitrary \( z \in \mathcal{N} \) can be represented as
\[
z = \sum_{i, j < k_0} p_i z p_j.
\]
We have, since \( w_i z w_j^* \in p_i \mathcal{N} p_i \),
\[
xy = yw_i^* w_i z w_j^* w_j = w_i^* y (w_i z w_j^*) w_j = w_i^* (w_i z w_j^*) y w_j = xy.
\]
Taking commutants on both sides of (3.6) and taking into account (3.5), we get (3.3). \( \blacksquare \)

**Proposition 3.4.** For any completely positive map \( \alpha \) in a von Neumann algebra \( \mathcal{M} \) acting in a Hilbert space \( \mathcal{H} \) there exists a Hilbert space \( \mathcal{K} \) and a \( * \)-representation \( \Phi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B} (\mathcal{K}) \) satisfying
\[
\alpha x = \Pi^* \Phi(x) \Pi
\]
where, for \( \xi \in \mathcal{H}, \Pi \xi = \xi \otimes \eta_1 \) for a fixed vector \( \eta_1 \in \mathcal{K}, \| \eta_1 \| = 1 \).

**Proof.** Take any Stinespring triple: \( (\mathcal{H}, P_H, \Psi) \) where \( \mathcal{H} \supset \mathcal{H}, P_H \) is an orthogonal projection of \( \mathcal{H} \) onto \( \mathcal{H} \), and \( \Psi : \mathcal{M} \to \mathcal{B} (\mathcal{H}) \) is a \( * \)-representation satisfying
\[
\alpha x = P_H \Psi(x) P_H |_{\mathcal{H}}.
\]
Denote \( \mathcal{N} = (\mathcal{M} \cup \Psi(\mathcal{M}))'' \) (obviously, we identify \( \mathcal{M} \ni x \equiv x P_H \in \mathcal{B} (\mathcal{H}) \)). According to the Stinespring's construction [6], [10, p. 195] the projection \( P_H \) has in \( \mathcal{N} \) the central support \( z (P_H) = 1_N \). By Lemma 3.3, there exists a Hilbert space \( \mathcal{K} \), an isometry \( v : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K} \) and a vector \( \eta_1 \in \mathcal{K} \) satisfying (3.3) and (3.4). We set
\[
\Phi(x) = v \Psi(x) v^*, \quad x \in \mathcal{M}.
\]
Then \( \Phi \) is a \( * \)-representation of \( \mathcal{M} \) into \( \mathcal{M} \otimes \mathcal{B} (\mathcal{K}) \). Moreover, as \( \Pi \xi = \xi \otimes \eta_1 \) for \( \xi \in \mathcal{H} \), we have, for any \( x \in \mathcal{M} \),
\[
(\Pi^* \Phi(x) \Pi) \xi = (\Pi^* v \Psi(x) v^*) (\xi \otimes \eta_1) = \Pi^* v \Psi(x) \xi
\]
\[
= \Pi^* v \Psi(x) P_H \xi = P_H \Psi(x) P_H \xi = \alpha(x) \xi
\]
(since \( v^* (\xi \otimes \eta_1), \zeta = \langle \xi, \zeta \rangle, (\Pi^* v \rho, \zeta) = \langle v (P_H \rho + P_H^\perp \rho), \zeta, \eta_1 \rangle = \langle (P_H \rho) \otimes \eta_1, \zeta \otimes \eta_1 \rangle = \langle P_H \rho, \zeta \rangle \) for \( \zeta \in \mathcal{H}, \rho \in \mathcal{H} \), the orthogonality \( v P_H^\perp \rho \perp \zeta \otimes \eta_1 \) is a consequence of (3.4)). \( \blacksquare \)

Now we are in a position to prove dilation theorems in the language of conditional expectations in \( W^* \)-algebras (see [7], Chapter 2 for basic facts).

**Theorem 3.5.** For any \( W^* \)-algebra \( \mathcal{M} \) and any completely positive map \( \alpha \) in \( \mathcal{M} \) there exist a \( W^* \)-algebra \( \mathcal{N} \), \( \mathcal{N} \ni \mathcal{M} \) (i.e. \( \mathcal{M} \) is a \( W^* \)-subalgebra of \( \mathcal{N} \)) and a \( * \)-representation...
\( \Phi : \mathcal{M} \rightarrow \mathcal{N} \) such that
\[
\alpha x = \mathbb{E}^{\mathcal{M}} \Phi(x), \quad x \in \mathcal{M},
\]
where \( \mathbb{E}^{\mathcal{M}} \) is a normal conditional expectation of \( \mathcal{N} \) onto \( \mathcal{M} \).

**Proof.** We keep the notation of Proposition 3.4. We identify \( \mathcal{M} \) with \( \mathcal{M} \otimes 1_K \) by a natural isomorphism \( x \equiv x \otimes 1 \).

We define a conditional expectation \( \mathbb{E}^{\mathcal{M} \otimes 1_K} \) by putting, for \( y \in \mathcal{N} = \mathcal{M} \otimes B(K) \)
\[
\mathbb{E}^{\mathcal{M} \otimes 1_K}(y) = (\Pi \ast y \Pi) \otimes 1_K,
\]
where \( \Pi \xi = \xi \otimes \eta \), \( \xi \in H \). It is easy to check that \( \mathbb{E}^{\mathcal{M} \otimes 1_K} \) is a projection of norm one, so conditional expectation [7, p. 116]. Taking \( \Phi \) as in Proposition 3.4, we have
\[
\alpha x = (\Pi \ast \Phi(x) \Pi) \otimes 1_K = \mathbb{E}^{\mathcal{M} \otimes 1_K} \Phi(x),
\]
which is equivalent to (3.7).

Now, keeping notation as in Section 2, our Theorem 2.1 can be rewritten in the following way:

**Theorem 3.6.** For a \( W^* \)-algebra \( \mathcal{M} \) and for a measure \( Q : \Sigma \rightarrow CP(\mathcal{M}) \), there exists a \( W^* \)-algebra \( \mathcal{N} \), \( \mathcal{N} \supset \mathcal{M} \) (i.e. \( \mathcal{M} \) is a \( W^* \)-subalgebra of \( \mathcal{N} \)) and a spectral measure \( e : \Sigma \rightarrow \text{Proj}\mathcal{N} \) such that
\[
Q(\Delta) = \mathbb{E}^{\mathcal{M}}(e(\Delta) \Phi(x))
\]
for some \( * \)-representation \( \Phi \) of \( \mathcal{M} \) in \( \mathcal{N} \) and a conditional expectation \( \mathbb{E}^{\mathcal{M}} \) of \( \mathcal{N} \) onto \( \mathcal{M} \).

4. Dilations in conditional expectations scheme. In this section we compare our results of Sections 2 and 3 with theorems concerning measures with values being positive operators in \( L_1 \). It turns out that these results can be reformulated to the case of the algebra \( L_\infty \) and then treated as theorems on commutative \( W^* \)-algebras.

In this context, constructing a dilation, we shall try to use most natural transformations (projections) appearing in the \( L_1 \)-space theory, like conditional expectation, indicator multiplication operator etc.

Moreover, we use a conditional expectation \( \mathbb{E}^{\mathcal{M}}_P \) for some probability measure \( P \) (and \( \sigma \)-field \( \mathcal{A} \)) instead of a projection \( P_H : \mathcal{H} \rightarrow H \) (from beyond the Hilbert space \( H \)).

Using here the space \( L_1 \) instead of \( L_\infty \) seems to be a better idea.

Let \((X, \Sigma)\) be a topological Borel measurable space. Let \((M, \mathcal{M}, \mu)\) be a probability space. A map \( Q : \Sigma \rightarrow B(L_1(M, \mathcal{M}, \mu)) \) is said to be a regular positive operator measure (shortly PO-measure) if the following conditions are satisfied:

1. \( Q(\Delta) f \geq 0 \) for \( 0 \leq f \in L_1 ; \)
2. \( Q \left( \bigcup_{s=1}^{\infty} \Delta_s \right) f = \sum_{s=1}^{\infty} Q(\Delta_s) f , \) for \( f \in L_1 , \) and pairwise disjoint \( \Delta_i \)'s, the series being convergent in \( L_1(M, \mathcal{M}, \mu) \);
3. \( Q \) is regular in the sense that for each \( \varepsilon > 0 \) and each \( \Delta \in \Sigma \) there exist in \( X \) a compact set \( Z \) and an open set \( V \) such that
\[
\left\{ Q(V - Z) 1_M \, d\mu < \varepsilon , \quad Z \subset \Delta \subset V \right\}
\]


4. $Q(X)1_M \leq 1_M$;
5. $\int_M Q(X)f \, d\mu \leq \int_M f \, d\mu$, $0 \leq f \in L_1$.

We have the following

**Theorem 4.1** [4]. Let $Q$ be a regular positive operator measure. Then there exist a ‘huge’ measure space $(\Omega, \mathcal{F}, \mathcal{P})$, a $\sigma$-field $\mathcal{A} \subset \mathcal{F}$, a $\sigma$-lattice homomorphism $e : \Sigma \to \mathcal{F}$ and two measurable maps $i : \Omega \to M$, $j : \Omega \to M$ such that

$$(Q(\Delta)f) \circ j = \mathbb{E}^P \mathbb{E}_\mathcal{A} 1_{\mathcal{E}(\Delta)} (f \circ i), \quad \Delta \in \Sigma, \quad f \in L_1(M).$$

**Theorem 4.2** [4]. There exist a measurable space $(\Omega, \mathcal{F})$, a measurable map $i : \Omega \to M$ (onto), $\sigma$-fields $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$, a $\sigma$-lattice homomorphism $e : \Sigma \to \mathcal{F}$, a set $\Omega_0 \in \mathcal{F}$ such that, for every PO-measure $Q : \Sigma \to B(L_1(M, \mathcal{M}, \mu))$, there exists a probability measure $P$ on $(\Omega, \mathcal{F})$, for which the following formula holds:

$$(Q(\Delta)f) \circ i = 4\mathbb{E}^P \mathbb{E}_\mathcal{A} 1_{\mathcal{E}(\Delta)} \mathbb{E}_\mathcal{B} 1_{\Omega_0} (f \circ i), \quad \Delta \in \Sigma, \quad f \in L_1(M).$$

For other similar results we refer to [4].

**References**