QUANTUM PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 43
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

DILATION THEOREMS FOR COMPLETELY POSITIVE MAPS AND MAP-VALUED MEASURES

EWA HENSZ-CHĄDZYŃSKA, RYSZARD JAJTE and ADAM PASZKIEWICZ

Faculty of Mathematics, Łódź University
ul. Banacha 22, 90-238 Łódź, Poland
E-mail: ewahensz@math.uni.lodz.pl, rjajte@math.uni.lodz.pl, adampasz@math.uni.lodz.pl

Abstract. The Stinespring theorem is reformulated in terms of conditional expectations in a von Neumann algebra. A generalisation for map-valued measures is obtained.

1. Introduction. Traditionally, each dilation theorem is obtained by a construction of a 'huge' (Hilbert) space $\mathcal{H}$ containing a given space H in the following manner. A system $\psi(\cdot)$ of operators in H or transformations of an algebra acting in H can be represented in the form

$$\psi(\cdot) = P_H \Phi(\cdot) P_{H|H} \tag{1.1}$$

where $\Phi(\cdot)$ is more regular than $\psi(\cdot)$. Throughout, P_H denotes the orthogonal projection of $\mathcal{H}$ onto H.

The most impressive results in this theory are effects of sophisticated indexing of linear bases of $\mathcal{H}$ and a 'magic touch' of scalar product. Theorems of B. Sz.-Nagy [9] and K.R. Parthasarathy [5] are excellent examples of such approach.

Dealing with operator algebras it seems to be most natural and physically meaningful to use the conditional expectation $\mathbb{E}$ [7, p.116] instead of $P_H(\cdot)P_H$ (cf L. Accardi, M. Ohya [1]).

In the paper we follow both ideas. Roughly speaking we represent a completely positive map-valued measure via the following dilation. Namely, any completely positive map turns into multiplication by a projection in such a way that the map-valued measure is 'dilated' to a spectral measure (Section 2).

1991 *Mathematics Subject Classification*: 46L50, 28B05.

Key words and phrases: dilation, von Neumann algebra, completely positive map, map-valued measure.

Research supported by KBN grant 2 P 03A 048 08.

The paper is in final form and no version of it will be published elsewhere.

The outstanding theorem of Stinespring [6] gives the dilation of a completely positive map ψ in a C^*-algebra to its $*$-representation Φ via formula (1.1). Passing to a W^*-algebra $\mathcal{M}$ Stinespring's theorem can be formulated using a normal conditional expectation $\mathbb{E}$ from a 'huge' algebra $\mathcal{N}$ onto $\mathcal{M}$ instead of $P_H(\cdot)P_H$. Such a new version of Stinespring's result will be proved in Section 3 together with a dilation theorem for positive map-valued measures.

Section 4 is devoted to a short comparison of the results just mentioned with the previous ones concerning commutative W^*-algebras.

2. Dilation of completely positive map-valuded measure. Let $\mathcal{M}$ be a von Neumann algebra of operators acting in a Hilbert space H. By $CP(\mathcal{M})$ we shall denote the set of completely positive linear maps in $\mathcal{M}$. Let (X,Σ) be a measurable space and $Q:\Sigma\to CP(\mathcal{M})$ be a σ-additive operator-valued measure (i.e. $\Sigma\ni\Delta\mapsto Q(\Delta)x$ is σ-additive in the ultra weak topology in $\mathcal{M}$ for each $x\in\mathcal{M}$) with $Q(X)1=1$.

THEOREM 2.1. *There exist a Hilbert space $\mathcal{H}$, a natural linear injection $V:H\to\mathcal{H}$, a $*$-representation Φ of the algebra $\mathcal{M}$ in $\mathcal{H}$, a σ-additive vector measure $e:\Sigma\to\operatorname{Proj}\mathcal{H}$, such that*

$$Q(\Delta)x=V^*e(\Delta)\Phi(x)V,\quad x\in\mathcal{M},\quad \Delta\in\Sigma. \tag{2.1}$$

Moreover, $e(\Delta)$ *is a central projection in* $(\Phi(\mathcal{M})\cup e(\Sigma))''$.

Proof. Let us consider the algebraic tensor product of vector spaces

$$\mathcal{H}_0=\mathcal{M}\otimes H\otimes SF(X,\Sigma)$$

where $SF(X,\Sigma)$ denotes the vector space of simple functions on (X,Σ).

Let us extend the measure Q from Σ to a linear mapping on $SF(X,\Sigma)$ putting

$$Q(f)=\sum_{\kappa=1}^{k}c_\kappa Q(\Delta_\kappa)\qquad\text{for}\qquad f=\sum_{\kappa=1}^{k}c_\kappa 1_{\Delta_\kappa}$$

where $\Delta_\kappa\in\Sigma$, $\kappa=1,\dots,k$.

In the sequel we shall briefly write Δ instead of 1_Δ, $\Delta\in\Sigma$. Notice that $\mathcal{H}_0$ is formed by elements of the form

$$\xi=\sum_{i=1}^{n}x_i\otimes h_i\otimes\Delta_i \tag{2.2}$$

where $x_i\in\mathcal{M}$, $h_i\in H$, $\Delta_i\in\Sigma$, $i=1,\dots,n$, $n=1,2,\dots$.

In the space $\mathcal{H}_0$ we can define a sesquilinear form $\langle\cdot,\cdot\rangle$ by

$$\langle\xi,\eta\rangle=\sum_{i=1}^{n}\sum_{j=1}^{m}(Q(\Delta_i\cap\Gamma_j)(y_j^*x_i)h_i,g_j)$$

for

$$\xi=\sum_{i=1}^{n}x_i\otimes h_i\otimes\Delta_i\qquad\text{and}\qquad \eta=\sum_{j=1}^{m}y_j\otimes g_j\otimes\Gamma_j.$$

The symbol $(\cdot,\cdot)$ denotes here the inner product in H. We shall show that $\langle\cdot,\cdot\rangle$ is positive. Indeed, for ξ of form (2.2) we consider the partition $\{\sigma_1,\dots,\sigma_k\}$ of $\bigcup_{i=1}^n\Delta_i$ given by

$\Delta_i, \ldots, \Delta_n$. Putting $\varepsilon_s^i = 1$ when $\sigma_s \subset \Delta_i$ and $\varepsilon_s^i = 0$ when $\sigma_s \cap \Delta_i = \emptyset$ we can write

$$\begin{aligned}\langle \xi, \xi \rangle &= \sum_{i,j=1}^{n} (Q(\Delta_i \cap \Delta_j)(x_j^* x_i) h_i, h_j) = \sum_{i,j=1}^{n} \Big(\Big(\sum_{s=1}^{k} \varepsilon_s^i \varepsilon_s^j Q(\sigma_s)\Big)(x_j^* x_i) h_i, h_j\Big) \\ &= \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(\sigma_s)(x_j^* x_i) h_i^s, h_j^s)\end{aligned}$$

where $h_i^s = \varepsilon_s^i h_i$, $i = 1, \ldots, n$.

The complete positivity of $Q(\sigma_s)$ gives the inequality

$$\sum_{i,j=1}^{n} (Q(\sigma_s)(x_j^* x_i) h_i^s, h_j^s) \geq 0, \qquad s = 1, \ldots, k,$$

thus $\langle \xi, \xi \rangle \geq 0$. Let us denote $\|\xi\|_0 = \sqrt{\langle \xi, \xi \rangle}$ and put $\mathcal{H}_1 = \mathcal{H}_0 \,/\, N$ where $N = \{\xi \in \mathcal{H}_0 : \|\xi\|_0 = 0\}$. Finally, let us set $\mathcal{H} = \overline{\mathcal{H}_1}^{\langle \cdot, \cdot \rangle}$.

We define $V : H \to \mathcal{H}$ by putting $Vh = [1 \otimes h \otimes X]$ for $h \in H$. Then

$$\langle Vh, Vh \rangle = ((Q(X)1)h, h) = (h, h)$$

so V is an isometry.

Now let us construct a $*$-representation Φ of the algebra $\mathcal{M}$ in $\mathcal{H}$. Namely, for $x \in \mathcal{M}$ let us set

$$\Phi(x) : [y \otimes h \otimes \Delta] \mapsto [xy \otimes h \otimes \Delta]$$

where $y \in \mathcal{M}$, $h \in H$, $\Delta \in \Sigma$. $\Phi(x)$ is well defined. Indeed, we prove the following inequality

$$\Big\| \sum_{i=1}^{n} x y_i \otimes h_i \otimes \Delta_i \Big\|_0 \leq \|x\| \cdot \Big\| \sum_{i=1}^{n} y_i \otimes h_i \otimes \Delta_i \Big\|_0 \tag{2.3}$$

for $y_i \in \mathcal{M}$, $h_i \in H$, $\Delta_i \in \Sigma$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$. As above, we can write

$$\Big\| \sum_{i=1}^{n} y_i \otimes h_i \otimes \Delta_i \Big\|_0^2 = \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(\sigma_s)(y_j^* y_i) h_i^s, h_j^s),$$

$$\Big\| \sum_{i=1}^{n} x y_i \otimes h_i \otimes \Delta_i \Big\|_0^2 = \sum_{s=1}^{k} \sum_{i,j=1}^{n} (Q(\sigma_s)(y_j^* x^* x y_i) h_i^s, h_j^s). \tag{2.4}$$

For a linear map $\alpha : \mathcal{M} \to \mathcal{M}$ let us denote by $\alpha^{(n)}$ the map $\alpha^{(n)} : \mathrm{Mat}_n(\mathcal{M}) \to \mathrm{Mat}_n(\mathcal{M})$ given by the formula

$$\alpha^{(n)}([z_{i,j}]) = [\alpha(z_{i,j})]$$

where $[z_{i,j}]_{i,j \leq n} \in \mathrm{Mat}_n(\mathcal{M})$. $\mathrm{Mat}_n(\mathcal{M})$ denotes here the C^*-algebra of all $n \times n$ matrices $[z_{i,j}]_{i,j \leq n}$ with entries $z_{i,j}$ in $\mathcal{M}$.

Now, we follow Takesaki [10, p. 196]. The Schwarz inequality for operators, by the complete positivity of $Q(\sigma_s)$, gives

$$Q(\sigma_s)^{(n)}(\widetilde{y}^* \widetilde{x}^* \widetilde{x} \widetilde{y}) \leq \|\widetilde{x}\|^2 Q(\sigma_s)^{(n)}(\widetilde{y}^* \widetilde{y}) \tag{2.5}$$

for each $\widetilde{x},\widetilde{y} \in \mathrm{Mat}_n(\mathcal{M})$. Setting $\widetilde{x} = [\delta_{i,j}x]$, $\widetilde{y} = [\delta_{1,i}y_i]$ we get $\widetilde{y}^*\widetilde{x}^*\widetilde{x}\widetilde{y} = [y_i^*x^*xy_j]$, $\widetilde{y}^*\widetilde{y} = [y_i^*y_j]$. Thus, by (2.5) and $\|\widetilde{x}\| = \|x\|$, we have

$$[Q(\sigma_s)(y_i^*x^*xy_j)] \le \|x\|^2[Q(\sigma_s)(y_i^*y_j)].$$

Hence

$$\sum_{i,j=1}^{n}(Q(\sigma_s)(y_i^*x^*xy_j)h_j^s, h_i^s) \le \|x\|^2\sum_{i,j=1}^{n}(Q(\sigma_s)(y_i^*y_j)h_j^s, h_i^s).$$

Finally, by (2.4), we get (2.3). Then $\|\sum_{i=1}^n y_i\otimes h_i\otimes\Delta_i\|_0 = 0$ implies $\|\sum_{i=1}^n xy_i\otimes h_i\otimes\Delta_i\|_0 = 0$ and $\Phi(x)$ is well defined. Obviously by (2.3), $\Phi(x) : \mathcal{H}\to\mathcal{H}$ is a linear bounded operator in $B(\mathcal{H})$. It is easy to check that $\Phi : \mathcal{M}\to B(\mathcal{H})$ is a $*$-representation $\mathcal{M}$ in $\mathcal{H}$.

Now for $\Delta\in\Sigma$ we define $e(\Delta) : \mathcal{H}\to\mathcal{H}$ putting

$$e(\Delta) : [y\otimes h\otimes\Delta'] \mapsto [y\otimes h\otimes(\Delta\cap\Delta')]$$

where $y\in\mathcal{M}$, $h\in H$, $\Delta'\in\Sigma$. The operator $e(\Delta)$ is well defined because $\|\sum_{i=1}^n y_i\otimes h_i\otimes\Delta_i\|_0 = 0$ implies $\|\sum_{i=1}^n y_i\otimes h_i\otimes(\Delta\cap\Delta_i)\|_0 = 0$. Indeed, let $\{\sigma_1,\ldots,\sigma_k\}$ be a partition of $\bigcup_{i=1}^n\Delta_i$ given by $\Delta,\Delta_1,\ldots,\Delta_k$. Let us put $\varepsilon_s^i = 1$ when $\sigma_s\subset\Delta_i$ and $\varepsilon_s^i = 0$ when $\sigma_s\cap\Delta_i = \emptyset$. Similarly, let $\varepsilon_s = 1$ when $\sigma_s\subset\Delta$ and $\varepsilon_s = 0$ when $\sigma_s\cap\Delta = \emptyset$. Then

$$\begin{aligned}\Big\|\sum_{i=1}^{n}(y_i\otimes h_i\otimes(\Delta\cap\Delta_i)\Big\|_0^2 &= \sum_{s=1}^{k}\varepsilon_s\sum_{i,j=1}^{n}\varepsilon_s^i\varepsilon_s^jQ(\sigma_s)(y_j^*y_i)h_i,h_j) \\ &\le \sum_{s=1}^{k}\sum_{i,j=1}^{n}\varepsilon_s^i\varepsilon_s^j(Q(\sigma_s)(y_j^*y_i)h_i,h_j) = \Big\|\sum_{i=1}^{n}y_i\otimes h_i\otimes\Delta_i\Big\|_0^2\end{aligned}$$

because, by the complete positivity of $Q(\sigma_s)$, we have

$$\sum_{i,j=1}^{n}\varepsilon_s^i\varepsilon_s^j(Q(\sigma_s)(y_j^*y_i)h_i,h_j) \ge 0.$$

Obviously, $e(\Delta)$ is an orthogonal projection in $\mathcal{H}$. Moreover, for $x\in\mathcal{M}$ and $\Delta\in\Sigma$ we have

$$\Phi(x)e(\Delta)[y\otimes h\otimes\Delta'] = e(\Delta)\Phi(x)[y\otimes h\otimes\Delta']$$

where $y\in\mathcal{M}$, $h\in H$, $\Delta'\in\Sigma$, so $e(\Delta)$ is a central projection in the algebra $(\Phi(\mathcal{M}\cup e(\Sigma))''$.

Finally, for all $h,g\in H$, $x\in\mathcal{M}$ and $\Delta\in\Sigma$

$$(V^*e(\Delta)\Phi(x)Vh,g) = \langle e(\Delta)\Phi(x)Vh,Vg\rangle = \langle x\otimes h\otimes\Delta,\,1\otimes g\otimes X\rangle = ((Q(\Delta)x)h,g),$$

so formula (2.1) holds. ∎

3. Dilations via conditional expectations. At the very beginning dilation theory was motivated by physical applications. In particular, the classical Naimark theorem gives a construction of a good self-adjoint quantum observable expressed by its spectral measure beyond the Hilbert space H in which acts a 'candidate' for physical observable being only an unbounded symmetric operator (see [9] for precise explanation). On the other hand, passing from a given operator algebra to a bigger one, physically means passing from a given system to a bigger one. That is why general ideas of dilation theory can be interpreted as follows. Enlarging a Hilbert space we usually pass to a new (better) model of the same physical system whereas the construction of a dilation in a bigger algebra means passing to a bigger system enjoying more regular evolution ([3], [2]).

In particular, the physical meaning of Stinespring's theorem can be enriched if we express the dilation in terms of the conditional expectation in the enlarged algebra. Such a construction, with consequences for map-valued measures, will be done in this section.

It turns out that some natural properties of an equivalence relation in the lattice of projections are crucial.

A basic tool is *the comparison theorem* for projections ([8], Thm. 4.6).

THEOREM 3.1. *For any* $p, q \in \operatorname{Proj}\mathcal{N}$, *there exists a projection* $e \in \mathcal{N} \cap \mathcal{N}'$ *such that* $p\,e \succcurlyeq q\,e$ *and* $p(1-e) \preccurlyeq q(1-e)$.

Clearly, $p \preccurlyeq q$ means $uu^* = p$, $u^*u \le q$ for some partial isometry $u \in \mathcal{N}$.

The following consequence of the above theorem will be used.

PROPOSITION 3.2. *Let* $\mathcal{N}$ *be a von Neumann algebra and let* p *be a projection in* $\mathcal{N}$ *with the central support* $z(p) = \mathbf{1}$. *There exists a system of mutually orthogonal projections* $(p_i;\ i < k_0)$ *in* $\operatorname{Proj}\mathcal{N}$, k_0 *being an ordinal number*, *such that* $p_i \preccurlyeq p$, $\sum_{i<k_0} p_i = 1$, *and* $p_1 = p$.

Proof. We use the *transfinite* induction, treating $1, 2, \dots$ as ordinals. Denote $e_1 = 0$, $p_1 = p$. Assume that, for some ordinals k and for any $i < k$, projections $e_i, p_i \in \operatorname{Proj}\mathcal{N}$ satisfying the conditions

$$\begin{aligned} & e_i \in \mathcal{N}', \\ & (e_i;\ i<k) \quad \text{are mutually orthogonal}, \\ & (p_i;\ i<k) \quad \text{are mutually orthogonal}, \\ & \sum_{j\le i} p_j \ge \sum_{j \le i} e_j, \\ & p_i \preccurlyeq p \end{aligned} \tag{3.1}$$

have already been defined. If $\sum_{i<k} p_i = 1$, the construction is complete with $k_0 = k$.

If not, we consider separately the following two cases.

Case 1^0. Assume that

$$\Big(\sum_{j<k} e_j\Big)^{\perp} p \preccurlyeq \Big(\sum_{j<k} e_j\Big)^{\perp}\Big(\sum_{j<k} p_j\Big)^{\perp}. \tag{$*$}$$

Then it is enough to put $e_k = 0$, p_k an arbitrary projection in $\mathcal{N}$ satisfying

$$p_k \sim \Big(\sum_{j<k} e_j\Big)^{\perp} p, \qquad p_k \le \Big(\sum_{j<k} e_j\Big)^{\perp}\Big(\sum_{j<k} p_j\Big)^{\perp}$$

(clearly, $p \sim q$ means $p = u^*u$, $q = uu^*$, for some $u \in \mathcal{N}$).

Case 2^0. Assume that $(*)$ does not hold. Then we consider the algebra

$$\mathcal{M} = \Big(\sum_{j<k} e_j\Big)^{\perp} \mathcal{N} \Big(\sum_{j<k} e_j\Big)^{\perp} = \mathcal{N}\Big(\sum_{j<k} e_j\Big)^{\perp}. \tag{3.2}$$

Restricting operators to a subspace $(\sum_{j<k} e_j)^{\perp}(H)$, one can treat $\mathcal{M}$ as a von Neumann algebra with the projections $\widetilde{p} = p(\sum_{j<k} e_j)^{\perp}$, $\widetilde{p}_i = p_i(\sum_{j<k} e_j)^{\perp}$. By the comparison

theorem there exists a central projection in $\mathcal{M}$, say e_k, satisfying the conditions

$$\widetilde{p}e_k \succcurlyeq \Big(\sum_{j<k} \widetilde{p}_j\Big)^{\perp} e_k \quad \text{and} \quad \widetilde{p}(1_{\mathcal{M}} - e_k) \preccurlyeq \Big(\sum_{j<k} \widetilde{p}_j\Big)^{\perp} (1_{\mathcal{M}} - e_k).$$

Since the reduction of $\mathcal{N}$ to $\mathcal{M}$ is done by the central projection $(\sum_{j<k} e_j)^{\perp}$, e_k can be obviously treated as a central projection in $\mathcal{N}$ as well.

Let $\widetilde{p}_k$ be an arbitrary projection in $\mathcal{M}$ satisfying

$$\widetilde{p}_k \le \Big(\sum_{j<k} p_j\Big)^{\perp} (1_{\mathcal{M}} - e_k), \qquad \widetilde{p}_k \sim p(1_{\mathcal{M}} - e_k).$$

We put

$$p_k = \widetilde{p}_k + \Big(\sum_{j<k} p_j\Big)^{\perp} e_k.$$

Obviously, we can treat p_k as a projection in $\mathcal{N}$. All conditions (3.1) are now satisfied for $k+1$ (instead for k).

Clearly, $\sum_{i<k} p_i = 1$ necessarily for some ordinal k (since $\dim H$ is a fixed cardinal). ∎

We shall need the following consequences of Proposition 3.2.

LEMMA 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras acting in Hilbert spaces H and $\mathcal{H}$, respectively, with $H \subset \mathcal{H}$. Denote by P_H the orthogonal projection from $\mathcal{H}$ onto H. Assume that*

$$P_H^* \,\mathcal{M}\, P_H \subset \mathcal{N}, \quad \text{the central support} \ \ z(P_H) = 1.$$

Then there exists an isometric injection $v : \mathcal{H} \to H \otimes K$, for some Hilbert space K such that

$$v\mathcal{N}v^* \subset \mathcal{M} \otimes B(K), \tag{3.3}$$

$$v\zeta = \zeta \otimes \eta_1, \quad \zeta \in H, \quad \text{for some } \eta_1 \in K. \tag{3.4}$$

Proof. Keeping the notation of Proposition 3.2, with $p = P_H \subset \mathcal{N}$, let us fix a Hilbert space K with an orthogonal basis $(\eta_j, j < k_0)$. As $p_i \preccurlyeq p$, we can use projections $r_i \le p$ satisfying $p_i = w_i^* w_i$, $r_i = w_i w_i^*$ for some partial isometries $w_i \in \mathcal{N}$, $i < k_0$. Obviously, we can assume that $w_1 = p$.

Let us take $v_i\zeta = w_i\zeta \otimes \eta_i$, $i < k_0$, for $\zeta \in H$. Then we get an isometry

$$v = \sum_{i<k_0} v_i, \qquad v : \mathcal{H} \to H \otimes K.$$

Formula (3.4) is obvious. It remains to show (3.3) or, equivalently, $\mathcal{N} \subset v^*\mathcal{M} \otimes B(K)v$. This can be checked by the commutant technique as follows.

We have

$$p_i\mathcal{N}p_i \cup \{w_i^*, w_i\} \subset v^*\mathcal{M} \otimes B(K)v, \quad i < k_0. \tag{3.5}$$

Indeed,

$$\begin{aligned} p_i\mathcal{N}p_i &= v^*(r_i\mathcal{M}r_i \otimes \langle \cdot, \eta_i\rangle \eta_i)v, \\ w_i &= v^*(r_i \otimes \langle \cdot, \eta_i\rangle \eta_1)v, \\ w_i^* &= v^*(r_i \otimes \langle \cdot, \eta_1\rangle \eta_i)v. \end{aligned}$$

For example we check the first equality. Obviously $p_i\mathcal{N}p_i = w_i^*\mathcal{N}w_i = w_i^*\mathcal{M}w_i$, and for any $x \in \mathcal{M}$, $\zeta \in \mathcal{H}$, denoting $\zeta_j = p_j\zeta$, $j < k_0$, we have

$$v\zeta = \sum_{j<k_0} w_j\zeta_j \otimes \eta_j$$

and

$$\begin{aligned} v^*(r_i\, x\, r_i \otimes \langle\cdot,\eta_i\rangle\eta_i)v\zeta &= v^*(r_i\, x\, w_i\zeta_i \otimes \eta_i) = (w_i\,\cdot\,\otimes\eta)i)^*(r_i\, x\, w_i\zeta_i \otimes \eta_i) \\ &= w_i^*\, x\, w_i\zeta_i = w_i^*\, x\, w_i\zeta. \end{aligned}$$

On the other hand, we have

$$\Big(\bigcup_{i<k_0} p_i\mathcal{N}p_i \cup \{w_i, w_i^*\}\Big)' = \mathcal{N}'. \tag{3.6}$$

The inclusion "$\supset$" is obvious. Conversely, let y commute with all $p_i\mathcal{N}p_i$, w_i, w_i^*. An arbitrary $z \in \mathcal{N}$ can be represented as $z = \sum_{i,j<k_0} p_i z p_j$. Take $x \in \mathcal{N}$ of the form $x = p_i\, z\, p_j$. We have, since $w_i z w_j^* \in p_1\mathcal{N}p_1$,

$$yx = yw_i^*w_izw_j^*w_j = w_i^*y(w_izw_j^*)w_j = w_i^*(w_izw_j^*)yw_j = xy.$$

Taking commutants on both sides of (3.6) and taking into account (3.5), we get (3.3). ∎

PROPOSITION 3.4. *For any completely positive map α in a von Neumann algebra $\mathcal{M}$ acting in a Hilbert space H there exists a Hilbert space K and a $*$-representation $\Phi : \mathcal{M} \to \mathcal{M} \otimes B(K)$ satisfying*

$$\alpha x = \Pi^*\Phi(x)\Pi$$

where, for $\xi \in H$, $\Pi\xi = \xi \otimes \eta_1$ for a fixed vector $\eta_1 \in K$, $\|\eta_1\| = 1$.

Proof. Take any Stinespring triple: $(\mathcal{H}, P_H, \Psi)$ where $\mathcal{H} \supset H$, P_H is an orthogonal projection of $\mathcal{H}$ onto H, and $\Psi : \mathcal{M} \to B(\mathcal{H})$ is a $*$-representation satisfying

$$\alpha x = P_H\Psi(x)P_H|_H.$$

Denote $\mathcal{N} = (\mathcal{M} \cup \Psi(\mathcal{M}))''$ (obviously, we identify $\mathcal{M} \ni x \equiv xP_H \in B(\mathcal{H})$). According to the Stinespring's construction [6], [10, p. 195] the projection P_H has in $\mathcal{N}$ the central support $z(P_H) = 1_{\mathcal{N}}$. By Lemma 3.3, there exists a Hilbert space K, an isometry $v : \mathcal{H} \to H \otimes K$ and a vector $\eta_1 \in K$ satisfying (3.3) and (3.4). We set

$$\Phi(x) = v\Psi(x)v^*, \qquad x \in \mathcal{M}.$$

Then Φ is a $*$-representation of $\mathcal{M}$ into $\mathcal{M} \otimes B(K)$. Moreover, as $\Pi\xi = \xi \otimes \eta_1$ for $\xi \in H$, we have, for any $x \in \mathcal{M}$,

$$\begin{aligned} (\Pi^*\Phi(x)\Pi)\xi &= (\Pi^*v\Psi(x)v^*)(\xi \otimes \eta_1) = \Pi^*v\Psi(x)\xi \\ &= \Pi^*v\Psi(x)P_H\xi = P_H\Psi(x)P_H\xi = \alpha(x)\xi \end{aligned}$$

(since $\langle v^*(\xi\otimes\eta_1),\zeta\rangle = \langle\xi,\zeta\rangle$, $\langle\Pi^*v\rho,\zeta\rangle = \langle v(P_H\rho+P_H^{\perp}\rho),\zeta\otimes\eta_1\rangle = \langle(P_H\rho)\otimes\eta_1,\zeta\otimes\eta_1\rangle = \langle P_H\rho,\zeta\rangle$ for $\zeta \in H$, $\rho \in \mathcal{H}$, the orthogonality $vP_H^{\perp}\rho \perp \zeta \otimes \eta_1$ is a consequence of (3.4)). ∎

Now we are in a position to prove dilation theorems in the language of conditional expectations in W^*-algebras (see [7], Chapter 2 for basic facts).

THEOREM 3.5. *For any W^*-algebra $\mathcal{M}$ and any completely positive map α in $\mathcal{M}$ there exist a W^*-algebra $\mathcal{N}$, $\mathcal{N} \supset \mathcal{M}$ (i.e. $\mathcal{M}$ is a W^*-subalgebra of $\mathcal{N}$) and a $*$-representation*

$\Phi : \mathcal{M} \to \mathcal{N}$ such that

$$\alpha x = \mathbb{E}^{\mathcal{M}} \Phi(x), \qquad x \in M, \tag{3.7}$$

where $\mathbb{E}^{\mathcal{M}}$ is a normal conditional expectation of $\mathcal{N}$ onto $\mathcal{M}$.

Proof. We keep the notation of Proposition 3.4. We identify $\mathcal{M}$ with $\mathcal{M} \otimes 1_K$ by a natural isomorphism $x \equiv x \otimes 1_K$. We define a conditional expectation $\mathbb{E}^{\mathcal{M} \otimes 1_K}$ by putting, for $y \in \mathcal{N} = \mathcal{M} \otimes B(K)$

$$\mathbb{E}^{\mathcal{M} \otimes 1_K}(y) = (\Pi^* y \Pi) \otimes 1_K,$$

where $\Pi\xi = \xi \otimes \eta_1$, $\xi \in H$. It is easy to check that $\mathbb{E}^{\mathcal{M} \otimes 1_K}$ is a projection of norm one, so conditional expectation [7, p. 116]. Taking Φ as in Proposition 3.4, we have $\alpha x = \Pi^* \Phi(x) \Pi$, so

$$\alpha x \otimes 1_K = (\Pi^* \Phi(x) \Pi) \otimes 1_K = \mathbb{E}^{\mathcal{M} \otimes 1_K} \Phi(x),$$

which is equivalent to (3.7). ∎

Now, keeping notation as in Section 2, our Theorem 2.1 can be rewritten in the following way:

THEOREM 3.6. *For a W^*-algebra $\mathcal{M}$ and for a measure $Q : \Sigma \to CP(\mathcal{M})$, there exists a W^*-algebra $\mathcal{N}$, $\mathcal{N} \supset \mathcal{M}$ (i.e. $\mathcal{M}$ is a W^*-subalgebra of $\mathcal{N}$) and a spectral measure $e : \Sigma \to \mathrm{Proj}\,\mathcal{N}$ such that*

$$Q(\Delta)x = \mathbb{E}^{\mathcal{M}}(e(\Delta)\Phi(x))$$

for some $*$-representation Φ of $\mathcal{M}$ in $\mathcal{N}$ and a conditional expectation $\mathbb{E}^{\mathcal{M}}$ of $\mathcal{N}$ onto $\mathcal{M}$.

4. Dilations in conditional expectations scheme. In this section we compare our results of Sections 2 and 3 with theorems concerning measures with values being positive operators in L_1. It turns out that these results can be reformulated to the case of the algebra L_∞ and then treated as theorems on commutative W^*-algebras.

In this context, constructiong a dilation, we shall try to use most natural transformations (projections) appearing in the L_1-space theory, like conditional expectation, indicator multiplication operator etc.

Moreover, we use a conditional expectation $E_P^{\mathcal{A}}$ for some probability measure P (and σ-field $\mathcal{A}$) instead of a projection $P_H : \mathcal{H} \to H$ (from beyond the Hilbert space H).

Using here the space L_1 instead of L_∞ seems to be a better idea.

Let (X, Σ) be a topological Borel measurable space. Let $(M, \mathfrak{M}, \mu)$ be a probability space. A map $Q : \Sigma \to B(L_1(M, \mathfrak{M}, \mu))$ is said to be a regular positive operator measure (shortly PO-measure) if the following conditions are satisfied:

1. $Q(\Delta)f \geq 0$ for $0 \leq f \in L_1$;
2. $Q\Big(\bigcup_{s=1}^{\infty} \Delta_s\Big)\, f = \sum_{s=1}^{\infty} Q(\Delta_s)f$, for $f \in L_1$, and pairwise disjoint Δ_i's, the series being convergent in $L_1(M, \mathfrak{M}, \mu)$;
3. Q is regular in the sense that for each $\varepsilon > 0$ and each $\Delta \in \Sigma$ there exist in X a compact set Z and an open set V such that

$$\int_M Q(V - Z) 1_M \, d\mu < \varepsilon, \qquad Z \subset \Delta \subset V;$$

4. $Q(X)1_M \le 1_M$;
5. $\int_M Q(X)f\,d\mu \le \int_M f\,d\mu, \quad 0 \le f \in L_1$.

We have the following

THEOREM 4.1 [4]. *Let Q be a regular positive operator measure. Then there exist a 'huge' measure space $(\Omega, \mathcal{F}, P)$, a σ-field $\mathcal{A} \subset \mathcal{F}$, a σ-lattice homomorphism $e : \Sigma \to \mathcal{F}$ and two measurable maps $i : \Omega \stackrel{\text{onto}}{\longrightarrow} M$, $j : \Omega \stackrel{\text{onto}}{\longrightarrow} M$ such that*

$$(Q(\Delta)f) \circ j = \mathbb{E}_P^{\mathcal{A}} 1_{e(\Delta)}(f \circ i), \qquad \Delta \in \Sigma, \quad f \in L_1(M).$$

THEOREM 4.2 [4]. *There exist a measurable space $(\Omega, \mathcal{F})$, a measurable map $i : \Omega \to M$ (onto), σ-fields $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$, a σ-lattice homomorphism $e : \Sigma \to \mathcal{F}$, a set $\Omega_0 \in \mathcal{F}$ such that, for every PO-measure $Q : \Sigma \to B(L_1(M, \mathfrak{M}, \mu))$, there exists a probability measure P on $(\Omega, \mathcal{F})$, for which the following formula holds*:

$$(Q(\Delta)f) \circ i = 4\mathbb{E}_P^{\mathcal{A}} 1_{e(\Delta)} \mathbb{E}_P^{\mathcal{B}} 1_{\Omega_0}(f \circ i), \qquad \Delta \in \Sigma, \quad f \in L_1(M).$$

For other similar results we refer to [4].

References

[1] L. Accardi and M. Ohya, *Compound channels, transition expectations and liftings*, preprint.

[2] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics, I, II* New York-Heidelberg-Berlin, Springer, (1979).

[3] D. E. Evans and J. T. Lewis, *Dilation of irreversible evolutions in algebraic quantum theory*, Communications of the Dublin Institute for Advanced Studies, Series A (Theoretical Physics) 24 (1977).

[4] E. Hensz-Chądzyńska, R. Jajte and A. Paszkiewicz, *Dilation theorems for positive operator-valued measures*, Probab. Math. Statist. 17 (1997), 365–375.

[5] K. R. Parthasarathy, *A continuous time version of Stinespring's theorem on completely positive maps*, Quantum probability and Applications V, Proceedings, Heidelberg 1988, L. Accardi, W. von Waldenfels (eds.), Lecture Notes Math., Springer-Verlag (1988).

[6] W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. 6 (1965), 211–216.

[7] S. Strătilă, *Modular theory in operator algebras*, Editura Academiei, Bucuresti, Abacus Press (1981).

[8] S. Strătilă and L. Zsidó, *Lectures on von Neumann algebras*, Editura Academiei, Bucuresti, (1979).

[9] B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space which extend beyond this space*, Appendix to: F. Riesz and B. Sz.-Nagy, Functional Analysis, Frederick Ungar Publishing Co.

[10] M. Takesaki, *Theory of operator algebras, I*, Springer, Berlin-Heidelberg-New York (1979).