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L^p-SPACES AND QUANTUM DYNAMICAL SEMIGROUPS

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Introduction. The purpose of the present note is to show the role played by noncommutative L^p-spaces in the theory of quantum dynamical semigroups. We consider both the C^*-algebra and von Neumann algebra case, concentrating on the latter. The two cases are very different, a phenomenon easy to explain on the grounds of noncommutative measure theory. If we take a locally compact space with a Radon measure, then the isomorphism class of the corresponding L^p-spaces ($p \neq 2$) depends crucially on the choice of the measure. It is therefore only natural to expect the isomorphism class of L^p-spaces associated with a noncommutative C^*-algebra to depend on the choice of a weight (or state) on the algebra. This is further supported by the results of the final section of the paper where the natural definition of L^p-spaces for UHF algebras leads to such a dependence. On the other hand, two isomorphic von Neumann algebras lead to linearly isometric L^p-spaces and that does not depend on the choice of (faithful) weights on the algebras. This corresponds to the classical fact that two equivalent measures on a measurable space give rise to isomorphic L^p-spaces. Note that a commutative von Neumann algebra corresponds to a quasi-measure space, i.e. a measure space with an associated

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class of equivalent measures. Thus, although we build different theories for von Neumann algebras with traces, states and weights, what really matters is whether the algebras are respectively semifinite, σ-finite or arbitrary. The functional is needed for reference purposes mainly. We can always choose a weight as our reference functional and, if the type of the algebra allows it, we can use a trace or a state. The importance of building the theory of quantum dynamical semigroups on non-semifinite (i.e. type III) von Neumann algebras is evident – these algebras arise frequently in physics. The usefulness of non-σ-finite algebras is not so evident. Nevertheless, many well-known constructions lead to such algebras (for example, the universal enveloping von Neumann algebra of a C^*-algebra). Also, a specific weight might be most natural in some situations, even if a state exists. That is why we insist on considering the most general set-up.

We seek to exploit as far as possible the idea of establishing a link between semigroups on a von Neumann algebra A, and semigroups on the Hilbert space $L^2(A)$. The aims are twofold. First, we can use Hille-Yosida theory and Dirichlet form techniques for the analysis of quantum dynamical semigroups on the algebra. Second, we may obtain new and interesting classes of semigroups on the algebra by suitable choices of generator (or form generator) on the Hilbert space $L^2(A)$.

Here we concentrate on semigroups of positive contractions on the algebra and try to associate with them corresponding semigroups of self-adjoint contractions on $L^2(A)$. The link is provided by the most natural, positivity preserving embedding of A into $L^2(A)$, the so called symmetric embedding. To guarantee both self-adjointness and contractivity of the L^2-semigroup, we impose an appropriate symmetry condition on the semigroup on the algebra, namely *KMS-symmetry.* The selfadjont contraction semigroup on $L^2(A)$ thus obtained is not arbitrary—it satisfies an interval preservation condition, explained below. We are currently also considering non-symmetric semigroups on A. In this case, the symmetry condition is replaced by an integrability condition, guaranteeing contractivity of the corresponding L^2-semigroup. In any case, to move from the algebra to the Hilbert space setting we use interpolation (see [GL 3]), and here the usefulness of the $L^p(A)$-spaces manifests itself. While it is possible to avoid L^p-spaces (at least for $p \neq 1, 2, \infty$), using Haagerup's spaces has the particular advantage of putting the whole machinery of *measurable operators* at one's disposal.

Many authors contributed to the theory of quantum Markov semigroups and noncommutative Dirichlet forms—Albeverio, Høegh-Krohn, Davies, Sauvageot, Cipriani, Fagnola, Guido, Isola, Scarlatti, Matsui and the authors ([AH-K], [DL 1], [Dav], [Sau], [Cip], [CFL], [GIS], [Mat], [GL 1], [GL 2]). Haagerup's L^p-spaces were introduced by Haagerup [Haa] and investigated by Terp [Te 1]. The interpolation of noncommutative L^p-spaces is described, among others, in [Te 2] and [GL 3]. L^p-spaces for C^*-algebras were constructed by Majewski and Zegarliński [MZ 1], [MZ 2]. The proofs of the results given here will appear in [GL 2] and [GPh].

L^p-spaces and embeddings. We consider here the whole scale of Haagerup's L^p-spaces, $L^p(A)$, $p \in [1, \infty]$, for an arbitrary von Neumann algebra A. We choose a reference weight φ which is assumed faithful normal and semifinite. We denote by $\mathbb{A}$ the crossed product of A by the modular automorphism group associated with φ and by τ the canonical trace on $\mathbb{A}$. With each normal weight ω on A one can associate the dual weight $\widetilde{\omega}$ on $\mathbb{A}$ and then the Radon-Nikodym derivative $h_\omega = \frac{d\widetilde{\omega}}{d\tau}$, a generalized positive operator affiliated with $\mathbb{A}$. It turns out that h_ω is measurable if and only if ω is finite, and the map $\omega \mapsto h_\omega$ can be extended linearly to the whole predual A_* of A. On the other hand, the operator $h := h_\varphi$, which plays a crucial role in the theory, is in general nonmeasurable.

Measurable operators stand out among other closed, densely defined operators affiliated with a von Neumann algebra as much as analytic functions among infinitely differentiable ones. It is their rigidity which makes them so interesting and useful — if two measurable operators coincide on a dense subspace of their domains, they must be equal.

The nonmeasurability of h is the feature that makes the theory more difficult, but also more interesting. The operator h should be regarded as a 'unit' in $L^1(A)$, note however that it does not belong to the space. Similarly, $h^{1/2}$ plays a role of a unit in $L^2(A)$.

The space $L^1(A)$ consists of operators h_ω, $\omega \in A_*$, and the norm of $L^1(A)$ is defined in such a way that $\|h_\omega\|_1 = \|\omega\|$. $L^\infty(A)$ is simply A (or, more precisely, its isomorphic image in $\mathbb{A}$) and $L^p(A)$ consists of operators with polar decomposition $x = u|x|$ in which $u \in A$ and $|x|^p \in L^1(A)$, with $\|x\|_p = \||x|^p\|_1^{1/p}$. A *trace-like functional* is defined on $L^1(A)$ by $\mathrm{tr}(h_\omega) = \omega(1)$, and the norm on $L^2(A)$ is induced by the inner product $(x, y) \mapsto \mathrm{tr}(x^* y)$.

For purposes of application, we need to single out that part of A which can be naturally embedded into $L^p(A)$. It should be noted that there is no canonical way of doing it. We are interested in the noncommutative counterpart of $L^p \cap L^\infty$. The proper, set-theoretic intersection of $L^p(A)$ and $A = L^\infty(A)$ consists of the zero operator alone. We define a family of ideals, $\mathfrak{n}^{(q)}$, $q \in [2, \infty]$, and a family of subalgebras $\mathfrak{m}^{(p)}$, $p \in [1, \infty]$, by

$$\begin{aligned} \mathfrak{n}^{(q)} &:= \{a \in A : ah^{1/q} \text{ is closable and } [ah^{1/q}] \in L^q(A)\}, \\ \mathfrak{m}^{(p)} &:= \mathrm{lin}\{b^* c : b, c \in \mathfrak{n}^{(2p)}\}. \end{aligned}$$

Both families turn out to be increasing, $\mathfrak{n}^{(\infty)} = \mathfrak{m}^{(\infty)} = A$, $\mathfrak{n}^{(2)} = \mathfrak{n}_\varphi := \{a \in A\colon \varphi(a^* a) < \infty\}$ and $\mathfrak{m}^{(1)} = \mathfrak{m}_\varphi := \{a \in A\colon \varphi(|a|) < \infty\}$.

Now we can construct a *left embedding* $\mathfrak{j}^{(q)}$ of $\mathfrak{n}^{(q)}$ into $L^q(A)$ and a *symmetric embedding* $\mathfrak{i}^{(p)}$ of $\mathfrak{m}^{(p)}$ into $L^p(A)$. They are given by

$$\mathfrak{j}^{(q)}\colon a \mapsto [ah^{1/q}],$$

$$\mathfrak{i}^{(p)} \text{ is a linear extension of } \mathfrak{m}_+^{(p)} \ni a \mapsto \mathfrak{j}^{(2p)}(a^{1/2})^* \mathfrak{j}^{(2p)}(a^{1/2}) \in L^p(A).$$

The symmetric embeddings are manifestly positivity preserving; the ideals $\mathfrak{n}^{(q)}$ and subalgebras $\mathfrak{m}^{(p)}$ are all σ-weakly dense in A and their images $\mathfrak{j}^{(q)}(\mathfrak{n}^{(q)})$ and $\mathfrak{i}^{(p)}(\mathfrak{m}^{(p)})$ are norm dense in $L^q(A)$ and $L^p(A)$, respectively.

To sum up, the 'intersections' are closely related to the embeddings and one can treat them as subspaces of A or $L^p(A)$, as the needs arise.

Markov operators and Markov semigroups. We are now in a position to introduce the notions of Markov (on $L^p(A)$, w.r.t. $h^{1/p}$) and KMS-symmetry (on A) for both operators and semigroups of operators. An operator S on $L^p(A)$ is *Markov with respect to* $h^{1/p}$ (L^p-Markov for short) if (i) $\mathrm{Dom}(S) \supset [0, h^{1/p}]_{L^p}$ and (ii) $S([0, h^{1/p}]_{L^p}) \subset [0, h^{1/p}]_{L^p}$;

an operator T on A is *KMS-symmetric with respect to* φ if (i) $\mathrm{Dom}(T) \supset \mathfrak{m}$; (ii) $\mathrm{tr}\,(Ta \cdot \mathfrak{i}(b)) = \mathrm{tr}(\mathfrak{i}(a) \cdot Tb)$ $\forall\ a,\, b \in \mathfrak{m}$. We are mainly interested in establishing a link between operators on A and operators on $L^2(A)$. It is given by the following intertwining formula:

$$\mathfrak{i}^{(2)} \circ T = S \circ \mathfrak{i}^{(2)}. \tag{$*$}$$

Now let T be a positivity preserving contraction on A. It is easy to check that the operator S defined on $\mathfrak{i}^{(2)}(\mathfrak{m}^{(2)})$ by (*) is necessarily positivity preserving and L^2-Markov. The hard question is to decide whether S is bounded. When T is KMS-symmetric, the boundedness of S follows by interpolation. The following interpolation inequality turns out to be crucial:

$$\|\mathfrak{i}^{(2)}(a)\| \le \|a\|^{1/2}\|\mathfrak{i}^{(1)}(a)\|^{1/2} \quad \forall a \in \mathfrak{m}.$$

Using the KMS-symmetry we first show that T can be extended to a bounded operator on $L^1(A)$. The above inequality then gives the boundedness of S.

Going in the other direction, from Hilbert space to algebra, requires first establishing $\mathfrak{i}^{(2)}$ as an order isomorphism of $[0,1]_{\mathfrak{m}^{(2)}}$ and $[0,h^{1/p}]_{L^p}$. This is a consequence of the following result.

Theorem 1 (Radon-Nikodym Type Theorem). *Let $x \in L^p(A)$ satisfy $0 \le x \le h^{1/p}$, where $p \in [1,\, \infty[$. Then there is $b \in \mathfrak{n}^{(2p)}$ such that*

$$x^{1/2} = h^{1/2p}b^* = [bh^{1/2p}]\,;\ ||\,b\,|| \le 1.$$

Self-adjointness of S guarantees that the operator T obtained from (*) is, in fact, KMS-symmetric. This implies that lifting T to $L^2(A)$ leads us back to S. Here is the statement of the precise result.

Theorem 2. *The identity (*) establishes a bijective correspondence between positive normal contractions T on A which are KMS-symmetric with respect to φ, and positive selfadjoint contractions S on $L^2(A)$ which are Markov with respect to $h^{1/2}$.*

Similar results hold also for semigroups of operators. A *Markov semigroup on* (A,φ) is a semigroup of positive normal contractions (T_t) on A such that $t \mapsto T_t a$ is σ-weakly continuous for all $a \in \mathfrak{m}_\varphi$, and a *Markov semigroup on* $(L^2(A), h^{1/2})$ is a strongly continuous (i.e. pointwise norm continuous) semigroup of Markov operators. Here is the proper statement.

Theorem 3. *Let (T_t) be a KMS-symmetric Markov semigroup on (A,φ), then (S_t), where S_t correspond to T_t as in Theorem 2, is an L^2-Markov semigroup. Conversely, if (S_t) is a symmetric Markov semigroup on $(L^2(A), h^{1/2})$, then there is a KMS-symmetric Markov semigroup (T_t) on A such that S_t correspond to T_t as in Theorem 2—in particular (S_t) is a self-adjoint contraction semigroup.*

The results extend to all values $p \in [1,\infty[$, by interpolation ([GL3]). We have therefore unified the results obtained in [AH-K], [DaL] for traces and in [Cip], [GL1] for states. Moreover the form generators of symmetric L^2-Markov semigroups are again characterised allowing the application of Dirichlet form techniques in the present general context ([GL2]).

L^p-spaces for C^*-algebras. The aim of the section is to persuade the reader that L^p-spaces may be 'canonically' associated with a pair consisting of a UHF algebra A and a state φ on the algebra. Obviously, no sensible definition would give any dependence on φ in case A is finite-dimensional—for any two states φ_1, φ_2 on A the spaces $L^p(A, \varphi_1)$ and $L^p(A, \varphi_2)$ should be isometric Banach spaces. For UHF algebras, it is most natural to define the L^p-norm as a limit of L^p-norms of an approximating sequence of finite-dmensional factors. It turns out that for an important class of states the isometry class of $L^p(A, \varphi)$, $p \neq 2$, depends on the isomorphic class of the von Neumann algebra $\pi_\varphi(A)''$.

For a finite-dimensional factor A, we put $L^p(A, \varphi) := A$ and the norm in $L^p(A, \varphi)$, $p \in [1, \infty[$, is given by

$$\|a\|_p = \tau(|h^{1/2p} a h^{1/2p}|^p)^{1/p}, \text{ where } h := \frac{d\varphi}{d\tau}.$$

Now let A be a UHF algebra and φ a faithful product state on A for an approximating sequence (A_n) of finite dimensional subfactors. Let $\|\cdot\|_p^{(n)}$ denote the norm of $L^p(A_n, \varphi|A_n)$. We define the norm $\|a\|_p$ of an element of $\bigcup A_n$ by

$$\|a\|_p = \|a\|_p^{(n)} \text{ when } a \in A_n.$$

The norm $\|\cdot\|_p$ turns $\bigcup A_n$ into a normed space. We denote by $L^p(A, \varphi)$ the completion of the space.

The main result can be stated as follows.

THEOREM 4. *Let A be a UHF C^*-algebra, φ a faithful product state on A for an approximating sequence A_n of finite-dimensional subfactors of A. Then $L^p(A, \varphi)$ and $L^p(\pi_\varphi(A)'')$ are isomorphic as Banach spaces. In particular, the space $L^p(A, \varphi)$ does not depend on the choice of the approximating sequence for which φ is product.*

This result can be generalized to a larger class of states and a larger class of C^*-algebras.

References

[AH-K] S. Albeverio and R. Høegh-Krohn, *Dirichlet forms and Markov semigroups on C^*-algebras*, Comm. Math. Phys. 56 (1977), 173–187.

[Cip] F. Cipriani, *Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras*, J. Funct. Anal. 147 (1977), 259–300.

[CFL] F. Cipriani, F. Fagnola and J. M. Lindsay, *Spectral analysis and Feller property for the quantum Ornstein-Uhlenbeck semigroup*, preprint, 1997.

[Dav] E. B. Davies, *Analysis on graphs and noncommutative geometry*, J. Funct. Anal. 111 (1993), 398–430.

[DL 1] E. B. Davies and J. M. Lindsay, *Noncommutative symmetric Markov semigroups*, Math. Z. 210 (1992), 379–411.

[DL 2] E. B. Davies and J. M. Lindsay, *Superderivations and symmetric Markov semigroups*, Comm. Math. Phys. 157 (1993), 359–370.

[GL 1] S. Goldstein and J. M. Lindsay, *KMS-symmetric Markov semigroups*, Math. Z. 219 (1995), 591–608.

[GL 2] S. Goldstein and J. M. Lindsay, *Markov semigroups KMS-symmetric for a weight*, preprint, 1997.

[GL 3] S. Goldstein and J. M. Lindsay, *Noncommutative interpolation and Markov semigroups*, in preparation.

[GPh] S. Goldstein and V. T. Phan, *L^p-spaces for UHF algebras*, to appear in Internat. J. Theoret. Phys., 1998.

[GIS] R. Guido, T. Isola and S. Scarlatti, *Non-symmetric Dirichlet forms on semifinite von Neumann algebras*, J. Funct. Anal. 135 (1996), 50–75.

[Haa] U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, in: Algèbres d'Opérateurs et leur Application en Physique Mathématique. Colloques Internationaux du CNRS, no. 274, Marseille, 20-24 juin 1977, Éditions du CNRS, Paris, 1979, 175–184.

[MZ 1] A. W. Majewski and B. Zegarliński, *Quantum stochastic dynamics I*: *Spin Systems on a lattice*, Math. Phys. Electron. J. 1 (1995), no. 1.

[MZ 2] A. W. Majewski and B. Zegarliński, *On quantum stochastic dynamics. Disordered systems and statistical physics*: *rigorous results* (*Budapest*, *1995*), Markov Process. Related Fields 2 (1996), 87–116.

[Mat] T. Matsui, *Markov semigroups on UHF algebras*, Rev. Math. Phys. 5 (1993), 587–600.

[Sau] J.-L. Sauvageot, *Semigroupe de la chaleur transverse sur la C^*-algèbre d'un feuilletage riemannien*, C. R. Acad. Sci. Paris Sér. I Math. 310 (1990), 531–536.

[Te 1] M. Terp, *L^p-spaces associated with von Neumann algebras*, notes, Report No. 3a + 3b, Københavns Universitets Matematiske Institut, 1981.

[Te 2] M. Terp, *Interpolation spaces between a von Neumann algebra and its predual*, J. Operator Theory 8 (1982), 327–360.