CHAOTIC DECOMPOSITIONS IN
$\mathbf{Z}_2$-GRADED QUANTUM STOCHASTIC CALCULUS

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Abstract. A brief introduction to $\mathbf{Z}_2$-graded quantum stochastic calculus is given. By inducing a superalgebraic structure on the space of iterated integrals and using the heuristic classical relation $df(\Lambda) = f(\Lambda + d\Lambda) - f(\Lambda)$ we provide an explicit formula for chaotic expansions of polynomials of the integrator processes of $\mathbf{Z}_2$-graded quantum stochastic calculus.

1. Introduction. A theory of $\mathbf{Z}_2$-graded quantum stochastic calculus was was introduced in [EH] as a generalisation of the one-dimensional Boson-Fermion unification result of quantum stochastic calculus given in [HP2]. Of particular interest in $\mathbf{Z}_2$-graded quantum stochastic calculus is the result that the integrators of the theory provide a time-indexed family of representations of a broad class of Lie superalgebras. The notion of a Lie superalgebra was introduced in [K] and has received considerable attention since. It is essentially a $\mathbf{Z}_2$-graded analogue of the notion of a Lie algebra with a bracket that is, in a certain sense, partly a commutator and partly an anticommutator. Another work on Lie superalgebras is [S] and general superalgebras are treated in [C,S].

The Lie algebra representation properties of ungraded quantum stochastic calculus enabled an explicit formula for the chaotic expansion of elements of an associated universal enveloping algebra to be developed in [HPu]. The Lie superalgebra representation properties of $\mathbf{Z}_2$-graded quantum stochastic calculus enabled an analogous theory for the graded case to be developed in [E1]. The work of [E1] is presented in this expository paper in a simplified, shortened and less technical form. The work is treated with maximal detail in [E2].

Section 2 of this paper gives a brief description of the integrators of $\mathbf{Z}_2$-graded quantum stochastic calculus. Section 3 describes the Ito superalgebra of $\mathbf{Z}_2$-graded quantum stochastic differentials. Section 4 describes the Ito tensor superalgebra and how this en-

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ables the space of iterated quantum stochastic integrals to be treated algebraically in a fully rigorous manner. Section 5 establishes the existence of a chaos map, an explicit formula for which is provided in section 6.

2. The integrators of $\mathbf{Z}_2$-graded quantum stochastic calculus. In this section we see how the integrators of N-dimensional $\mathbf{Z}_2$-graded quantum stochastic calculus are indexed by elements of the superalgebra $M_0(N, r)$ of complex $(N+1) \times (N+1)$ matrices. Here r denotes a fixed integer with $0 \leq r < N$. An element A of $M_0(N, r)$ may be decomposed uniquely into two matrices as follows:

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0' \end{pmatrix} + \begin{pmatrix} 0 & A_1 \\ A_1' & 0 \end{pmatrix}.$$

Here A_0 denotes an $(r+1) \times (r+1)$ matrix, A_0' denotes an $(N-r) \times (N-r)$ matrix, A_1 denotes an $(r+1) \times (N-r)$ matrix and A_1' denotes an $(N-r) \times (r+1)$ matrix. If A is such that $A_1, A_1' = 0$ then A is said to be *even*. The subspace $M_0(N, r)_0 \subset M_0(N, r)$ consists of all such matrices. Similarly, if A is such that $A_0, A_0' = 0$ then A is said to be *odd*. The subspace $M_0(N, r)_1 \subset M_0(N, r)$ consists of all such matrices and it is clear that $M_0(N, r) = M_0(N, r)_0 + M_0(N, r)_1$. The elements of $M_0(N, r)_0 \cup M_0(N, r)_1$ are said to be the *homogeneous* elements of $M_0(N, r)$.

Define E_β^α to be the $(N+1) \times (N+1)$ matrix that is zero everywhere except for the value 1 in the αth column at the βth row. We can see that for all $0 \leq \alpha, \beta \leq N$, E_β^α will be homogeneous. We define the value σ_β^α for $0 \leq \alpha, \beta \leq N$ by

$$\sigma_\beta^\alpha = \begin{cases} 0 & \text{if } E_\beta^\alpha \text{ even}; \\ 1 & \text{if } E_\beta^\alpha \text{ odd}. \end{cases}$$

For an arbitrary homogeneous element A of $M_0(N, r)$ we define $\sigma(A) = \gamma$ where $A \in M_0(N, r)_\gamma$.

Define $\Delta \in M_0(N, r)$ by $\Delta := \sum_{\alpha=1}^N E_\alpha^\alpha$. The multiplication . in $M_0(N, r)$ is defined for arbitrary $A, B \in M_0(N, r)$ by

$$A.B = A\Delta B.$$

We now define the quantum stochastic process G on boson Fock space, $\Gamma(L^2(\mathbf{R}_+; \mathbf{C}^N))$. The totality of the exponential vectors means that it suffices to define G at an arbitrary time t on an arbitrary exponential vector $e(f)$ as follows:

$$G(t)e((f^1, \ldots, f^N)) = e(\chi_{[0,t]}(f^1, \ldots, f^r, -f^{r+1}, \ldots, -f^N) + \chi_{(s,\infty)}f).$$

Note that $G(t)$ is a self-adjoint unitary and therefore is defined on the whole of the Fock space. The process G is known as the *grading process*.

The quantum stochastic process Ξ_β^α is defined as follows:

$$\Xi_\beta^\alpha = \begin{cases} \int_0^{\cdot} d\Lambda_\beta^\alpha(s) & \text{if } \sigma_\beta^\alpha = 0; \\ \int_0^{\cdot} G(s) \, d\Lambda_\beta^\alpha(s) & \text{if } \sigma_\beta^\alpha = 1. \end{cases}$$

An arbitrary element A of $M_0(N, r)$ may be expressed (using Einstein summation) as $\lambda_\alpha^\beta E_\beta^\alpha$ with each $\lambda_\alpha^\beta \in \mathbf{C}$. The corresponding quantum stochastic integrator process Ξ_A is defined to be $\lambda_\alpha^\beta \Xi_\beta^\alpha$. If A is even then $\Xi_A = \Lambda_A$.

3. The Ito superalgebra. The integrator processes Ξ_A yield the quantum stochastic differentials $d\Xi_A$. Ito multiplication of these differentials yields the equality

$$(1) \qquad\qquad d\Xi_A d\Xi_B = d\Xi_{A.B}.$$

In the one-dimensional case where $N = 1$ and $r = 0$, (1) yields the Fermionic Ito table [HP2]. The results of [HP2] show that Ξ_1^0 is the Fermionic creation process and Ξ_0^1 is the Fermionic annihilation process. Indeed, $\mathbf{Z}_2$-graded quantum stochastic calculus is a generalisation of this result.

The Ito multiplication is associative and so the vector space $\{d\Xi_A : A \in M_0(N, r)\}$ forms an associative algebra which we denote $\mathcal{I}$. It is clear that $\mathcal{I}$ may be $\mathbf{Z}_2$-graded by means of the decomposition $\mathcal{I} = \mathcal{I}_0 + \mathcal{I}_1$ where $\mathcal{I}_0 := \{d\Xi_A : A \in M_0(N, r)_0\}$ and $\mathcal{I}_1 := \{d\Xi_A : A \in M_0(N, r)_1\}$. Given arbitrary $\alpha, \beta \in \{0, 1\}$ it is easy to show that $\mathcal{I}_\alpha \mathcal{I}_\beta \subset \mathcal{I}_{\alpha +_2 \beta}$ where $+_2$ denotes addition modulo 2. It follows that $\mathcal{I}$ is an associative superalgebra and hence the Lie superalgebra $\mathcal{I}_{SLie}$ may be formed [K,S] with superbracket $\{.,.\}$ defined by linear extension of the following rule for arbitrary homogeneous $A, B \in M_0(N, r)$:

$$\{d\Xi_A, d\Xi_B\} = d\Xi_A d\Xi_B - (-1)^{\sigma(A)\sigma(B)} d\Xi_B d\Xi_A = d\Xi_{A.B} - (-1)^{\sigma(A)\sigma(B)} d\Xi_{B.A}.$$

A superbracket $\{.,.\}$ may be defined on $M_0(N, r)$ in a similar way by linear extension of the following rule for arbitrary homogeneous $A, B \in M_0(N, r)$:

$$\{A, B\} = A.B - (-1)^{\sigma(A)\sigma(B)} B.A.$$

Equipped with this bracket, $M_0(N, r)$ becomes a Lie superalgebra which we denote $gl_0(N, r)$.

More surprisingly, the processes Ξ_A themselves form a time-indexed family of representations of $gl_0(N, r)$. We have the relation

$$\{\Xi_A, \Xi_B\} = \Xi_{\{A,B\}}$$

where $\{\Xi_A, \Xi_B\}$ must be interpreted in terms of adjoints and the inner product. If $e(f), e(g)$ are arbitrary exponential vectors and A, B are arbitrary homogeneous elements of $gl_0(N, r)$ we have that for all $t \geq 0$

$$\langle e(f), \{\Xi_A, \Xi_B\}(t)e(g)\rangle =$$

$$\langle \Xi_A(t)^\dagger e(f), \Xi_B(t)e(g)\rangle - (-1)^{\sigma(A)\sigma(B)} \langle \Xi_B(t)^\dagger e(f), \Xi_A(t)e(g)\rangle.$$

Full details and a proof of this result may be found in [EH,E2].

4. The Ito tensor superalgebra. To any Lie superalgebra there corresponds a universal enveloping superalgebra with properties analogous to those of the universal enveloping algebra of a Lie algebra. In this instance we choose to take the universal enveloping superalgebra of the matrix superalgebra $gl_0(N, r)$ and denote it by $\mathcal{U}$. We have that, for any associative superalgebra A, a Lie superalgebra morphism $\phi : gl_0(N, r) \to A_{SLie}$ will extend uniquely to a superalgebra morphism $\tilde{\phi} : \mathcal{U} \to A$. In this section we construct the associative superalgebra $(\mathcal{P}, \odot)$ of iterated quantum stochastic integrals which will form the target superalgebra for the universal extension yielding chaotic expansions that is to take place in section 5.

Consider first the weak tensor superalgebra $\mathcal{T}(\mathcal{I})$ consisting of all finite sums of tensors in $\mathcal{I}$:

$$\mathcal{T}(\mathcal{I}) = \mathbf{C} + \mathcal{I} + \mathcal{I} \otimes \mathcal{I} + \mathcal{I} \otimes \mathcal{I} \otimes \mathcal{I} + \cdots.$$

Now define the integrator map I on $\mathcal{T}(\mathcal{I})$ by linear extension of the following rule for an n-fold product tensor:

$$I : d\Xi_{A_1} \otimes \cdots \otimes d\Xi_{A_n} \mapsto \int\limits_{0 < t_1 < \cdots < t_n <\cdot} d\Xi_{A_1}(t_1) \ldots d\Xi_{A_n}(t_n).$$

Naturally, for $z \in \mathbf{C}$ we define $I(z) = zId$, where Id is the identity process. Thus, for example, $I(z + d\Xi_{A_1} \otimes d\Xi_{A_2} + d\Xi_{B_1} \otimes d\Xi_{B_2} \otimes d\Xi_{B_3}) = zId + \iint d\Xi_{A_1} d\Xi_{A_2} + \iiint d\Xi_{B_1} d\Xi_{B_2} d\Xi_{B_3}$.

In [E1,E2] may be found the definition of the $\star$ product in $\mathcal{T}(\mathcal{I})$, the technical details of which are omitted here. This product has the property that, given an arbitrary time $t \geq 0$, arbitrary exponential vectors $e(f), e(g)$ and arbitrary elements a, b of $\mathcal{T}(\mathcal{I})$ we have

$$\langle I(a)(t)^{\dagger} e(f), I(b)(t)e(g)\rangle = \langle e(f), I(a \star b)e(g)\rangle.$$

It follows that $\star$ provides a good notion of multiplying iterated $\mathbf{Z}_2$-graded quantum stochastic integrals even though they are, in general, unbounded.

We are now in a position to define the superalgebra $\mathcal{P}$. The underlying vector space of $\mathcal{P}$ is the space $\{I(a) : a \in \mathcal{T}(\mathcal{I})\}$ of all linear combinations of iterated quantum stochastic integrals. The product $\odot$ in $\mathcal{P}$ is defined for $I(a), I(b) \in \mathcal{P}$ by

$$I(a) \odot I(b) := I(a \star b).$$

As $\mathcal{T}(\mathcal{I})$ is closed under $\star$ we have that $\mathcal{P}$ is closed under $\odot$. Furthermore, it is tedious rather than difficult to show from the definition of $\star$ that $(\mathcal{P}, \odot)$ has a natural superalgebra structure ultimately deriving from that of $M_0(N, r)$. It is also true that $\odot$ is associative. Full details of these results may be found in [E1,E2]. Thus $(\mathcal{P}, \odot)$ is an associative superalgebra and as such we may form $\mathcal{P}_{SLie}$ with bracket $\{.,.\}$ defined by linear extension of the following rule for homogeneous $I(a), I(b)$ in $\mathcal{P}$:

$$(2) \qquad \{I(a), I(b)\} = I(a) \odot I(b) - (-1)^{\sigma(I(a))\sigma(I(b))} I(b) \odot I(a).$$

In (2), we denote by $\sigma(I(c))$ the grade of $I(c)$. This may be determined via $M_0(N, r)$, $\mathcal{I}$, $\mathcal{T}(\mathcal{I})$ and I or by the fact that $GI(c)G = (-1)^{\sigma(I(c))} I(c)$.

5. Existence of a chaos map. Let j be the map defined as follows:

$$j : gl_0(N, r) \to \mathcal{P}_{SLie}$$
$$j : A \mapsto \Xi_A.$$

PROPOSITION 1. *The map j is a Lie superalgebra morphism.*

Proof. It suffices to show that the result holds for arbitrary homogeneous A, B in $gl_0(N, r)$. Thus we have

$$\{j(A), j(B)\} = \{\Xi_A, \Xi_B\}$$
$$(3) \qquad\qquad = I(d\Xi_A) \odot I(d\Xi_B) - (-1)^{\sigma(A)\sigma(B)} I(d\Xi_B) \odot I(d\Xi_A)$$

Using the definition of $\star$ given in [E1] we calculate that (3) is equal to

$$I((0, d\Xi_{A.B} - (-1)^{\sigma(A)\sigma(B)}d\Xi_{B.A}, (d\Xi_A \otimes d\Xi_B + (-1)^{\sigma(A)\sigma(B)}d\Xi_B \otimes d\Xi_A)$$
$$- (-1)^{\sigma(A)\sigma(B)}(d\Xi_B \otimes d\Xi_A + (-1)^{\sigma(B)\sigma(A)}d\Xi_A \otimes d\Xi_B), 0, \ldots))$$

$$= I((0, d\Xi_{\{A,B\}}, 0, \ldots)) = \Xi_{\{A,B\}}.$$

Thus j is a Lie superalgebra morphism. ∎

It follows from universality and Proposition 1 that j may be extended uniquely to a map J from $\mathcal{U}$ to $\mathcal{P}$. This map J is effectively the chaotic decomposition of a formal polynomial in the integrator processes of $\mathbf{Z}_2$-graded quantum stochastic calculus. It will become clear why we use the elements of $gl_0(N, r)$ rather than $\mathcal{P}$ for these polynomials in the course of the main proof of this paper contained in section 6.

6. An explicit formula for the chaotic decomposition map J. In this section we state and prove an explicit formula for the chaotic decomposition map whose existence was established in the previous section.

We begin by considering a classical heuristic formula concerning the differential of a polynomial f of processes Λ:

(4) $$df(\Lambda) = f(\Lambda + d\Lambda) - f(\Lambda).$$

For example, we have formally by (4) and the one dimensional quantum Ito formula:

$$d\Lambda^3 = (\Lambda + d\Lambda)^3 - \Lambda^3 = \Lambda^3 + 3\Lambda^2 d\Lambda + 3\Lambda(d\Lambda)^2 + (d\Lambda)^3 - \Lambda^3$$

(5) $$= 3\Lambda^2 d\Lambda + 3\Lambda d\Lambda + d\Lambda.$$

By applying (4) to the first term of (5) we obtain

$$d\Lambda^3 = 6\int \Lambda d\Lambda d\Lambda + 6\Lambda d\Lambda + d\Lambda$$

which is the same as the chaotic decomposition of Λ^3 as obtained by the second fundamental formula [HP1,P].

Drawing on this relation we now introduce a collection of co-(super)algebraic maps which will enable (4) to be applied rigorously to the elements of $\mathcal{U}$.

The *co-unit* $\eta : \mathcal{U} \to \mathbf{C}$ is the unique extension of the zero map

$$0 : gl_0(N, r) \to \mathbf{C}_{SLie} \quad 0 : A \mapsto 0$$

to $\mathcal{U}$. The kernel of this map is $\mathcal{U}M_0(N, r)\mathcal{U}$ and we denote this kernel by $\mathcal{K}$. Roughly speaking, $\mathcal{K}$ consists of all those elements of $\mathcal{U}$ that have a zero 'constant' term.

The *co-product* map $\gamma : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ is the unique extension of the map

$$g : gl_0(N, r) \to (\mathcal{U} \otimes \mathcal{U})_{SLie} \quad g : A \mapsto A \otimes 1 + 1 \otimes A$$

to $\mathcal{U}$ where 1 denotes the unit element of $\mathcal{U}$. By means of γ we define the family $(\kappa_i)_{i \geq 0}$ of *difference maps* by

$$\kappa_1 : \mathcal{U} \to \mathcal{U} \otimes \mathcal{K} \quad \kappa_1 : U \mapsto \gamma(U) - U \otimes 1$$
$$\kappa_i := (\kappa_1 \otimes id \otimes \cdots \otimes id) \circ \kappa_{i-1} \quad \text{for } i \geq 2.$$

Note that κ_1 maps $\mathcal{U}$ into $\mathcal{U} \otimes \mathcal{K}$. It can also be shown [E1,E2] that, if $\kappa_1(U) = \sum_i V_i \otimes K_i$ then the 'degree' of each of the 'polynomials' V_i is strictly less than the

degree of U. This notion of degree is treated with full rigour in [E1,E2] via the 'super' analogue of the Poincaré-Birkhoff-Witt theorem [S]. The finite degree of each element of $\mathcal{U}$ ensures that for each $U \in \mathcal{U}$ there exists an $n \in N$ such that $\kappa_n(U) = 0$.

We define the map $\xi : \mathcal{U} \to \mathcal{I}$ to be the unique extension to $\mathcal{U}$ of the Lie superalgebra morphism

$$d : gl_0(N,r) \to \mathcal{I}_{SLie} \quad d : A \mapsto d\Xi_A.$$

We may now give an explicit formula for the chaotic expansion of an element of $\mathcal{U}$.

THEOREM 2. *For an arbitrary element U of $\mathcal{U}$ we have that $J(U) = I(\chi(U))$ where*

$$\chi(U) = \eta(U) + \eta \otimes \xi \circ \kappa_1(U) + \eta \otimes \xi \otimes \xi \circ \kappa_2(U) + \cdots$$
$$+ \eta \otimes \underbrace{\xi \otimes \cdots \otimes \xi}_{n \text{ times}} \circ \kappa_n(U) + \cdots.$$

P r o o f. By the uniqueness of the universal extension of j it suffices to show that the map $I \circ \chi$ satisfies the following relations:

(i) $\forall A \in M_0(N,r), \quad I(\chi(A)) = \Xi_A;$
(ii) $\forall U, V \in \mathcal{U}, \ I(\chi(U)) \odot I(\chi(V)) = I(\chi(UV)).$

To show (i) is straightforward. As $A \in M_0(N,r) \subset \mathcal{K}$ we have $\eta(A) = 0$. Furthermore, the degree-reducing property of the maps κ_i ensures that for all $i \geq 2$ we have $\kappa_i(A) = 0$. For $i = 1$ we have $I \circ (\eta \otimes \xi) \circ \kappa_1(A) = I \circ (\eta \otimes \xi)(A \otimes 1 + 1 \otimes A - A \otimes 1) = I \circ (\eta \otimes \xi)(1 \otimes A) = I(d\Xi_A) = \Xi_A$. Thus relation (i) holds.

We now establish (ii) through a rigorous analogue of the classical relation (4). Using the notation just established, the integral form of (4) may be expressed as

$$(6) \qquad\qquad J(U) = \eta(U)Id + \int J \otimes \xi \circ \kappa_1(U).$$

Establishing that (6) holds with J replaced by $I \circ \chi$, that is to say, establishing the equality

$$(7) \qquad\qquad I \circ \chi(U) = \eta(U)Id + \int (I \circ \chi) \otimes \xi \circ \kappa_1(U)$$

enables (ii) to be established by means of the quantum Ito formula and induction. This is preferable to the combinatorial approach used for the ungraded case in [HPu].

It follows from the independence of quantum stochastic integrators [L] that the map $I : \mathcal{T}(\mathcal{I}) \to \mathcal{P}$ is injective [E1,E2]. Therefore, in order to establish (7), it suffices to show that the equality

$$(8) \qquad\qquad \chi(U) = \eta(U) + \chi \otimes \xi \circ \kappa_1(U)$$

holds. The zeroth order component of each side of (8) is $\eta(U)$. For $n \geq 1$, the nth order component of $\chi(U)$ is

$$(9) \qquad\qquad (\eta \otimes \underbrace{\xi \otimes \cdots \otimes \xi}_{n-1 \text{ times}} \circ \kappa_{n-1}) \otimes \xi \circ \kappa_1(U).$$

The rightmost ξ in this expression operates on the second entry of the tensor $\kappa_1(U)$ whereas the map $\eta \otimes \xi \otimes \cdots \otimes \xi \circ \kappa_{n-1}$ operates on the first entry of this tensor. Thus

we may delay the operation of ξ by one composition and include it in the leftmost map. This re-writes (9) as

$$(10) \qquad (\eta \otimes \xi \otimes \cdots \otimes \xi) \circ \kappa_{n-1} \otimes id \circ \kappa_1(U).$$

It is straightforward to show that for all $n \geq 1$ we have $\kappa_n \otimes id \circ \kappa_1 = \kappa_{n+1}$. This identity enables (10) to be re-written as

$$(\eta \otimes \xi \otimes \cdots \otimes \xi) \circ \kappa_n(U),$$

this being the nth order component of $\chi(U)$ on the left hand side of (8). Having established (8) we conclude that (7) indeed holds.

We now proceed to prove (ii) by means of induction on the degree of $UV = \deg U + \deg V$. If at least one of U, V is of degree zero then the result is immediate. If $\deg U = \deg V = 1$ then $U = \lambda 1 + L$, $V = \mu 1 + M$ with $\lambda, \mu \in \mathbf{C}$ and $L, M \in M_0(N, r)$. By linearity we may assume that L and M are of definite parity. By (6) we have $I(\chi(U)) = \lambda Id + \Xi_L$, $I(\chi(V)) = \mu Id + \Xi_M$ so we may write

$$
\begin{aligned}
I(\chi(U)) \odot I(\chi(V)) &= \lambda\mu Id + \lambda\Xi_M + \mu\Xi_L + \Xi_L \odot \Xi_M \\
&= I(\lambda\mu + \lambda d\Xi_M + \mu d\Xi_L + d(\Xi_L \otimes \Xi_M)) \\
&= I(\lambda\mu + \lambda d\Xi_M + \mu d\Xi_L + d\Xi_L d\Xi_M + d\Xi_L \otimes d\Xi_M \\
&\qquad + (-1)^{\sigma(L)\sigma(M)} d\Xi_M \otimes d\Xi_L) \\
&= I(\eta(UV) + \eta \otimes \xi \circ \kappa_1(UV) + \eta \otimes \xi \otimes \xi \circ \kappa_2(UV)) \\
&= I(\chi(UV)).
\end{aligned}
$$

Therefore (ii) holds for all $U, V \in \mathcal{U}$ with $\deg U + \deg V \leq 2$.

Now assume, by way of induction, that (ii) holds for all U, V with $\deg U + \deg V < k$ for some positive integer k. Suppose we have $U, V \in \mathcal{U}$ such that $\deg U + \deg V = k$. We must show that $I(\chi(U)) \odot I(\chi(V)) = I(\chi(UV))$. We take a differential approach to this and so begin by showing that the equality holds at time $t = 0$. From (7) we have $I(\chi(U))(0) = \eta(U)Id$ and $I(\chi(V))(0) = \eta(V)Id$. Therefore $(I(\chi(U)) \odot I(\chi(V)))(0) = (\eta(U)Id \odot \eta(V)Id)(0) = \eta(U)\eta(V)Id = \eta(UV)Id$ by the definition of $\odot$ and the morphism property of η. We also have that $I(\chi(UV))(0) = \eta(UV)Id$ so that the equality holds at time zero. It now suffices to show that

$$(11) \qquad d(I(\chi(U)) \odot I(\chi(V))) = dI(\chi(UV)).$$

Application of the quantum Ito formula to the left hand side of (11) yields

$$(12) \qquad I(\chi(U))\,d(I(\chi(V))) + d(I(\chi(U)))\,I(\chi(V)) + d(I(\chi(U)))\,d(I(\chi(V))).$$

The differential form of (7) gives $d(I(\chi(W))) = (I \circ \chi) \otimes \xi \circ \kappa_1(W)$ for arbitrary $W \in \mathcal{U}$. Thus (12) may be re-written as

$$
\begin{aligned}
(13) \qquad & I(\chi(U))\,((I \circ \chi) \otimes \xi \circ \kappa_1(V)) + ((I \circ \chi) \otimes \xi \circ \kappa_1(U))\,I(\chi(V)) \\
& \qquad + (I \circ \chi) \otimes \xi \circ \kappa_1(U)\,(I \circ \chi) \otimes \xi \circ \kappa_1(V).
\end{aligned}
$$

The degree-reducing property of κ_1 allows us to invoke the inductive hypothesis and re-write (13) as

$$(I \circ \chi) \otimes \xi \left(U \otimes 1 \kappa_1 (V) + \kappa_1 (U) V \otimes 1 + \kappa_1 (U) \kappa_1 (V) \right)$$
$$= (I \circ \chi) \otimes \xi \big(U \otimes 1 \left(\gamma(V) - V \otimes 1 \right) + \left(\gamma(U) - U \otimes 1 \right) V \otimes 1$$
$$+ \left(\gamma(U) - U \otimes 1 \right) \left(\gamma(V) - V \otimes 1 \right) \big)$$
$$= (I \circ \chi) \otimes \xi \left(\gamma(UV) - UV \otimes 1 \right)$$
$$(14) \qquad = (I \circ \chi) \otimes \xi \circ \kappa_1(UV).$$

We have from (7) that (14) is equal to $d(I(\chi(UV)))$ as required. Thus (ii) holds by induction and the theorem is established. ∎

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