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HALL'S TRANSFORMATION VIA QUANTUM STOCHASTIC CALCULUS

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Abstract. It is well known that Hall's transformation factorizes into a composition of two isometric maps to and from a certain completion of the dual of the universal enveloping algebra of the Lie algebra of the initial Lie group. In this paper this fact will be demonstrated by exhibiting each of the maps in turn as the composition of two isometries. For the first map we use classical stochastic calculus, and in particular a stochastic analogue of the Dyson perturbation expansion. For the second map we make use of quantum stochastic calculus, in which the circumambient space is the complexification of the Lie algebra equipped with the *ad*-invariant inner product.

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1. Introduction. Hall's transformation [Hall] is a generalization of the unitary transformation which intertwines the Schrödinger representation, realised in the Hilbert space got by equipping the configuration space with a Gaussian rather than Lebesgue measure, with the Bargmann-Fock [Barg] representation of the canonical commutation relations, and which maps one to the other the respective functions which are identically one (the harmonic oscillator ground state vectors). In the generalization the configuration space is replaced by a Lie group G whose Lie algebra $\mathcal{L}$ carries an *ad*-invariant inner product, and the Gaussian measure becomes a heat-kernel measure $d\gamma_t$, where t is a positive real number, for the Laplacian determined canonically by the inner product. The complexification of the Lie algebra generates a Lie group with complex structure, into which G is embedded, and which also possesses a (differently normalized) Laplacian. Hall's transformation is an isometry $\mathcal{H}_t$ from the L^2 space of the original heat kernel measure $d\gamma_t$ onto the holomorphic subspace $HL^2(d\tilde{\gamma}_t)$ of the L^2 space of a heat kernel measure $d\tilde{\gamma}_t$ for the complexified Laplacian. It is given explicitly by the convolution

$$\mathcal{H}_t f(x) = f * d\gamma_t(x) \tag{1.1}$$

for x belonging to the embedding of G in its complexification.

Following Hall's original paper, there is now an extensive literature (see [Driv, DrGr, Gros, GrMa] and references contained in them) in which, in particular, the Hall isomorphism $\mathcal{H}_t$ is related to maps $\mathcal{D}_t$ and $\tilde{\mathcal{D}}_t$ from the respective L^2 spaces to the completion in a certain norm $\|\,.\,\|_t$ of the dual $\mathcal{U}^*$ of the universal enveloping algebra $\mathcal{U}$ of the Lie algebra $\mathcal{L}$ of the initial Lie group G by

$$\mathcal{D}_t = \tilde{\mathcal{D}}_t \circ \mathcal{H}_t. \tag{1.2}$$

The maps $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are given formally by

$$\mathcal{D}_t f(U) = U f(e) \tag{1.3}$$

$$\tilde{\mathcal{D}}_t \tilde{f}(U) = \tilde{U} \tilde{f}(e) \tag{1.4}$$

where e is the neutral element of $G \subseteq \tilde{G}$ and the action of $U \in \mathcal{U}$ on f is the extension of that of $\mathcal{L}$ by left-invariant vector fields on G, and $\tilde{U}$ refers to the corresponding action on $\tilde{G}$ generated by the action of $\mathcal{L}$ by holomorphic vector fields. The norm $\|\,.\,\|_t$ which makes these maps isometric can be given an intrinsic characterization. The resulting isometries are bijective if G is simply connected.

In this paper we use quantum stochastic calculus to construct the Hall transformation $\mathcal{H}_t$ and to interpret the maps $\mathcal{D}_t$ and $\tilde{\mathcal{D}}_t$ and the norms $\|\,.\,\|_t$. In Section 2 we consider the Bargmann transformation which is related to anti-normal-ordered quantization. In Section 3 we generalize this transformation to the Lie group context as a relation between two stochastic flows of which one is classical and the other quantum in character. The isometry property and the formula (1.1) are deduced from this flow description in Section 4. Finally in Section 5 we give some indications of extensions of this work.

2. Bargmann's transformation and deformation quantization. Let $\mathbb{C}\langle p, q\rangle$ denote the algebra of complex polynomials in two commuting indeterminates p and q, and let $\mathbb{C}\langle \mathbf{p}, \mathbf{q}\rangle$ denote the corresponding algebra in indeterminates satisfying the Heisenberg relation

$$\mathbf{p}\mathbf{q} - \mathbf{q}\mathbf{p} = -it$$

where t is a fixed positive number. By a *quantization* we shall mean a linear bijective unital map w from $\mathbb{C}\langle p, q\rangle$ to $\mathbb{C}\langle \mathbf{p}, \mathbf{q}\rangle$ which intertwines the *translation* actions of $\mathbb{R}^2$ which are given by the automorphisms $s_{x,y}$ and $\mathbf{s}_{x,y}$ of the respective algebras for which

$$s_{x,y}p = p + x, \quad s_{x,y}q = q + y, \quad \mathbf{s}_{x,y}\mathbf{p} = \mathbf{p} + x, \quad \mathbf{s}_{x,y}\mathbf{q} = \mathbf{q} + y$$

and which intertwines the involutions for which p, q, $\mathbf{p}$ and $\mathbf{q}$ are self-adjoint. The *Weyl* quantization w_0 is that which, for arbitrary $x, y \in \mathbb{R}^2$ and $n = 0, 1, \ldots$, maps $(xp + yq)^n$ to $(x\mathbf{p} + y\mathbf{q})^n$. Any other quantization is of the form $w_0 \circ F(\partial_1, \partial_2)$ [Huds] where F is a formal power series in the partial derivatives ∂_1 and ∂_2 with respect to p and q whose absolute term $F(0, 0) = 1$, so that F is invertible. In particular the *anti-normal ordered* quantization w_a is given by

$$w_a = w_0 \circ \exp\left[\frac{t}{2}\left(\partial_1^2 + \partial_2^2\right)\right]. \tag{2.1}$$

Equivalently, for $m, n = 0, 1, 2, \ldots$,

$$w_a\left((p - iq)^m (p + iq)^n\right) = (\mathbf{p} - i\mathbf{q})^m (\mathbf{p} + i\mathbf{q})^n. \tag{2.2}$$

A quantization w defines a deformed associative multiplication on $\mathbb{C}\langle p, q\rangle$ by

$$Y \circ_w Z = w^{-1}\left(w(Y)w(Z)\right). \tag{2.3}$$

The translation action continues to be by automorphisms and the original involution continues to be an involution for the deformed multiplication. Conversely [Huds] it can be shown that any associative multiplication on $\mathbb{C}\langle p, q\rangle$ with these properties is of the form (2.3).

In the case of the Weyl quantization the deformed product is given by

$$Y \circ Z(p, q) = \Upsilon(\partial_1^{(1)}, \partial_2^{(1)}, \partial_1^{(2)}, \partial_2^{(2)}) Y^{(1)} Z^{(2)} \mid_{p^{(1)} = p^{(2)} = p,\, q^{(1)} = q^{(2)} = q} \tag{2.4}$$

where $\Upsilon(x_1, x_2, y_1, y_2) = \exp\left(\frac{it}{2}(x_1 y_2 - y_1 x_2)\right)$. For the general quantization $w = w_0 \circ F(\partial_1, \partial_2)$ we replace Υ in (2.4) by

$$\Upsilon_{[F]}(x_1, x_2, y_1, y_2) = \Upsilon(x_1, x_2, y_1, y_2) \frac{F(x_1, x_2)F(y_1, y_2)}{F(x_1 + y_1, x_2 + y_2)}.$$

In the case of anti-normal ordered quantization this becomes

$$\Upsilon_a(x_1, x_2, y_1, y_2) = \exp\left[-t(x_1 + iy_1)(x_2 - iy_2)\right]. \tag{2.5}$$

THEOREM 2.1. *The anti-normal ordered quantization of a polynomial $f(p)$ in p alone is the polynomial $\tilde{f}(\mathbf{p})$ in $\mathbf{p}$ alone where*

$$\tilde{f} = \sum_{n=0}^{\infty} (n!)^{-1} \left(\frac{t}{2}\right)^n f^{(2n)}. \tag{2.6}$$

P r o o f. The Weyl quantization maps each polynomial $g(p)$ to $g(\mathbf{p})$. Since $\exp[\frac{t}{2}(\partial_1^2 + \partial_1^2)]$ evidently maps $f(p)$ to the polynomial

$$g(p) = \sum_{n=0}^{\infty} (n!)^{-1} \left(\frac{t}{2}\right)^n f^{(2n)}(p),$$

the result follows from (2.1). ■

We define the *Bargmann transformation* $\mathcal{B}_t$ initially from the space of complex polynomials in one variable to itself by $\mathcal{B}_t(f) = \tilde{f}$ where $\tilde{f}(\mathbf{p})$ is the anti-normal ordered quantization of $f(p)$. Equivalently, in view of Theorem 2.1,

$$\tilde{f} = \exp\left(\frac{t}{2}D^2\right) f \tag{2.7}$$

where D denotes differentiation. Note that this can also be expressed as the convolution $\tilde{f} = f * d\gamma_t$ where $d\gamma_t = (2\pi t)^{-\frac{1}{2}} e^{-(2t)^{-1}x^2} dx$. Evidently $\mathcal{B}_t$ inherits translation invariance from the anti-normal ordered quantization; if, for $x \in \mathbb{R}$, $f_x(p) = f(x+p)$ and $\tilde{f}_x(\mathbf{p}) = \tilde{f}(x+\mathbf{p})$, then, with $\tilde{f} = \mathcal{B}_t(f)$, we have $\tilde{f}_x = f_x$. That it is an L^2 isometry for the appropriate Gaussian measures on $\mathbb{R}$ and $\mathbb{C}$ can be deduced from the L^2 isometry property of the Weyl quantization; it will also follow from more general considerations in Section 4. That it intertwines the annihilation operators in the respective L^2 spaces (and hence also their adjoints, the creation operators), both of which act by differentiation on polynomials, is clear from (2.7); indeed this is the infinitesimal form of translation invariance. It is evident that it maps one to the other the respective identity polynomials.

3. Two stochastic flows. Let G be a Lie group whose Lie algebra $\mathcal{L}$ is equipped with an *ad*-invariant inner product. Such a group is necessarily unimodular, being the product of a compact Lie group with some $\mathbb{R}^N$ [GrMa].

We take the complexification $\tilde{\mathcal{L}}$ of $\mathcal{L}$, equipped with the sesqui-linear inner product $\langle\cdot,\cdot\rangle$ extending the original one as the circumambient space of a quantum stochastic calculus. Thus, in the Fock space $\mathcal{F}(L^2(\mathbb{R}_+ \bigotimes \tilde{\mathcal{L}}))$ we have creation and annihilation processes $A^\dagger(K)$ and $A(K)$ labelled by elements of $\tilde{\mathcal{L}}$ and gauge processes $\Lambda(S)$ labelled by linear transformations on $\tilde{\mathcal{L}}$. These satisfy the Ito product rules

$$dA(K)dA^\dagger(L) = \langle K, L\rangle dT, \quad dA(K)d\Lambda(S) = dA(S^*K),$$

$$d\Lambda(S)d\Lambda(T) = d\Lambda(ST), \quad d\Lambda(S)dA^\dagger(K) = dA^\dagger(SK),$$

[HuPa] all other products being zero.

Now let $(L_1, \ldots, L_d)$ be an orthonormal basis of $\mathcal{L}$ and denote by $\mathbb{L}$ the basis independent Laplacian

$$\mathbb{L} = \sum_{j=1}^{d} L_j^2,$$

regarded as a left-invariant second order differential operator on G. We consider the diffusion X on G, starting at time 0 at the neutral element e, whose generator is $\mathbb{L}$ (sometimes called the Brownian motion on G). Thus X is a G-valued random variable distributed according to the heat-kernel measure $d\gamma_t$ whose density γ_t is the solution of the partial differential equation

$$\frac{\partial}{\partial t}\gamma = \frac{1}{2}\mathbb{L}\gamma, \quad \gamma_0 = \delta_e. \tag{3.1}$$

For smooth f, $f(X_t)$ can be identified with the evaluation at e of $J_t(f)$ where the stochastic flow J is defined by

$$dJ(f) = \sum_{j=0}^{d} J(L_j f) dP_j - \frac{1}{2} J(\mathbb{L} f) dT, \quad J_0(f) = f, \tag{3.2}$$

where P_j is the momentum process (Brownian motion) $i(A^\dagger(L_j) - A(L_j))$. The solution of (3.2) may be expressed through the stochastic Dyson perturbation expansion, in which the stochastic terms in the differential equation are regarded as perturbations of the time terms, as

$$J_t(f) = \sum_{n=o}^{\infty} \sum_{j_1,\dots,j_n=1}^{d} \int_{\Delta_n(t)} e^{-\frac{1}{2}t_1\mathbb{L}} L_{j_1} e^{-\frac{1}{2}(t_2-t_1)\mathbb{L}} L_{j_2} \dots L_{j_n} e^{-\frac{1}{2}(t-t_n)\mathbb{L}} (f) dP_{j_1}(t_1) \dots dP_{j_n}(t_n)$$

where $\Delta_n(t)$ is the increasing subset of $\mathbb{R}^n$,

$$\Delta_n(t) = (0 < t_1 < \dots < t_n < t)\,.$$

Because $\mathbb{L}$ commutes with each L_j the integrand in (3.3) may be collapsed to give

$$J_t(f) = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=1}^{d} \prod_{k=1}^{n} L_{j_k} e^{-\frac{1}{2}t\mathbb{L}}(f) \int_{\Delta_n(t)} dP_{j_1}(t_1) \dots dP_{j_n}(t_n). \tag{3.3}$$

We may evaluate at the neutral element e to obtain

$$f(X_t) = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=1}^{d} \prod_{k=1}^{n} L_{j_k} e^{-\frac{1}{2}t\mathbb{L}} f(e) \int_{\Delta_n(t)} dP_{j_1}(t_1) \dots dP_{j_n}(t_n). \tag{3.4}$$

The space $\tilde{\mathcal{L}}$ is the Lie algebra of a group $\tilde{G}$ into which G is embedded via the natural embedding $\mathcal{L} \subset \tilde{\mathcal{L}}$. $\tilde{G}$ inherits a complex structure from that of $\tilde{\mathcal{L}}$. We define a stochastic flow $\tilde{J}$ on holomorphic functions $\tilde{f}$ on $\tilde{G}$ by

$$d\tilde{J}(\tilde{f}) = \sum_{j=0}^{d} \tilde{J}(\tilde{L}_j(\tilde{f})) dA_j^\dagger, \quad \tilde{J}_0(\tilde{f}) = \tilde{f}. \tag{3.5}$$

where $A_j^\dagger$ is the creation process $A^\dagger(L_j)$ and $\tilde{L}_j$ is the holomorphic action of L_j as left-invariant holomorphic vector fields on the Lie group $\tilde{G}$.

The iterative solution of (3.6) is

$$\tilde{J}_t(\tilde{f}) = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n=1}^{d} \prod_{k=1}^{n} \tilde{L}_{j_k}(\tilde{f}) \int_{\Delta_n(t)} dA_{j_1}^\dagger \dots dA_{j_n}^\dagger. \tag{3.6}$$

Again we can evaluate at e to obtain

$$\tilde{J}_t(\tilde{f})(e) = \sum_{n=0}^{\infty} \sum_{j_1,\dots,j_n}^{d} \prod_{k=1}^{n} \tilde{L}_{j_k} \tilde{f}(e) \int_{\Delta_n(t)} dA_{j_1}^\dagger \dots dA_{j_n}^\dagger. \tag{3.7}$$

The *Hall transformation* $\mathcal{H}_t : f \mapsto \tilde{f}$ can be defined by equating the coefficients of the iterated integrals in (3.5) and (3.8). Thus f is related to $\tilde{f}$ by

$$f(X_t) = \sum_{n=0}^{\infty} \sum_{j_1,\ldots,j_n=1}^{d} \prod_{k=1}^{n} \tilde{L}_{j_k} \tilde{f}(e) \int_{\Delta_n(t)} dP_{j_1}(t_1)\ldots dP_{j_n}(t_n). \tag{3.8}$$

It is evident from its genesis that $\mathcal{H}_t$ has the translation-invariance property

$$\mathcal{H}_t(f_x) = (\mathcal{H}_t(f))_x \tag{3.9}$$

where, for $x \in G \subset \tilde{G}$, f_x denotes the left translation $f_x(y) = f(xy)$.

In the case $G = (\mathbb{R}, +)$, $\mathcal{H}_t$ reduces to the Bargmann transformation $\mathcal{B}_t$.

4. Isometry properties and convolution formula. Since X_t has distibution $d\gamma_t$ and the map $f \mapsto f(X_t)$ is multiplicative and respects complex conjugation, the proof of the following theorem is immediate.

THEOREM 4.1. *The map $\mathcal{F}_t : f \mapsto f(X_t)$ is isometric from $L^2(d\gamma_t)$ to the space $\mathcal{W}_t$ of square-integrable complex-valued functions of the random variable X_t equipped with the norm*

$$\| f(X_t) \|^2 = \mathbb{E}\left[| f(X_t) |^2 \right].$$

For the next isometry property we use (3.5) to write

$$f(X_t) = \sum_{n=0}^{\infty} \sum_{j_1,\ldots,j_n=1}^{d} c_{j_1,\ldots j_n}(t) \int_{\Delta_n(t)} dP_{j_1}\ldots dP_{j_n}. \tag{4.1}$$

where the $c_{j_1,\ldots,j_n}(t)$ are complex numbers.

THEOREM 4.2.

$$\| f(X_t) \|^2 = \sum_{n=0}^{\infty} (n!)^{-1} t^n \sum_{j_1,\ldots,j_n=1}^{d} | c_{j_1,\ldots,j_n}(t) |^2 . \tag{4.2}$$

P r o o f. This follows from (4.1) together with the relation

$$\mathbb{E}\Big[\int_{\Delta_m(t)} dP_{k_1}\ldots dP_{k_m} \int_{\Delta_n(t)} dP_{j_1}(t_1)\ldots dP_{j_n}(t_n)\Big] = (n!)^{-1}\delta_{m,n} \prod_{l=1}^{n} \delta_{k_l,j_l}$$

which is easily deduced from the Ito formula $dP_k dP_j = \delta_{k,j} dT$. ∎

Similarly, if we write (3.8) in the form

$$\tilde{\mathcal{F}}_t(\tilde{f}) = \sum_{n=0}^{\infty} \sum_{j_1,\ldots,j_n=1}^{d} \tilde{c}_{j_1,\ldots,j_n}(t) \int_{\Delta_n(t)} dA_{j_1}^{\dagger}\ldots dA_{j_n}^{\dagger} \tag{4.3}$$

where $\tilde{\mathcal{F}}_t(\tilde{f}) = \tilde{J}_t(\tilde{f})(e)$, we may use the Ito formula $dA_k dA_j^{\dagger} = \delta_{k,j} dT$ to prove

THEOREM 4.3.

$$\| \tilde{\mathcal{F}}_t(\tilde{f})\Omega \|^2 = \sum_{n=0}^{\infty} (n!)^{-1} t^n \sum_{j_1,\ldots,j_n=1}^{d} | \tilde{c}_{j_1,\ldots,j_n}(t) |^2 .$$

where Ω is the Fock vacuum vector.

Our last isometry is more subtle. We introduce the Laplacian on $\tilde{G}$

$$\tilde{\mathbb{L}} = \sum_{j=1}^{d} \left((L_j)^2 + (iL_j)^2\right).$$

Here $(L_1, \ldots, L_d, iL_1, \ldots, iL_d)$ is a basis of $\tilde{\mathcal{L}} = \mathcal{L} \otimes_{\mathbb{R}} \mathbb{C}$ regarded as a real Lie algebra. The heat kernel measures $d\tilde{\gamma}_t$ are defined by a renormalized analogue of (3.1):

$$\frac{\partial}{\partial t}\tilde{\gamma} = \frac{1}{4}\tilde{\mathbb{L}}\tilde{\gamma}, \quad \tilde{\gamma}_0 = \delta_e.$$

THEOREM 4.4. *For holomorphic* $\tilde{f}$,

$$\| \tilde{\mathcal{F}}_t(\tilde{f})\Omega \|^2 = \int_{\tilde{G}} | \tilde{f} |^2 \, d\tilde{\gamma}_t. \tag{4.4}$$

P r o o f. Using (3.7) we have

$$(\tilde{J}(\tilde{f}))^\dagger = \bar{J}(\bar{\tilde{f}})$$

where $\bar{J}$ is the flow defined on conjugate-holomorphic functions on $\tilde{G}$ by

$$d\bar{J}(\bar{\tilde{f}}) = \sum_{j=1}^{d} (\bar{J}(\bar{L}_j(\bar{\tilde{f}})))^\dagger dA_j, \, \bar{J}_0(\bar{\tilde{f}}) = \bar{\tilde{f}} \tag{4.5}$$

where $A_j = A(L_j)$ and, for $L \in \mathcal{L}$, $\bar{L}$ denotes the action of L as a conjugate-holomorphic vector field. Evaluating at e the iterated solution corresponding to (3.7) of (4.5) and using (3.8), we find that

$$d\tilde{\mathcal{F}}(\tilde{f}) = \sum_{j=1}^{d} \tilde{\mathcal{F}}(\tilde{L}_j(\tilde{f}))dA_j^\dagger,$$

$$d(\tilde{\mathcal{F}}(\tilde{f}))^\dagger = \sum_{j=1}^{d} \bar{\mathcal{F}}(\bar{L}_j \bar{\tilde{f}})dA_j$$

where $\bar{\mathcal{F}}_t(\bar{\tilde{f}}) = \bar{J}_t \bar{\tilde{f}}(e)$. By the quantum Ito formula we have

$$\begin{aligned} d((\tilde{\mathcal{F}}(\tilde{f}))^\dagger \tilde{\mathcal{F}}(\tilde{f})) &= \sum_{j=1}^{d} (\bar{\mathcal{F}}(\bar{L}_j \bar{\tilde{f}})\tilde{\mathcal{F}}(\tilde{f})dA_j + (\tilde{\mathcal{F}}(\tilde{f}))^\dagger \tilde{\mathcal{F}}(\tilde{L}_j \tilde{f})dA_j^\dagger) \\ &+ \sum_{j=1}^{d} \bar{\mathcal{F}}(\bar{L}_j \bar{\tilde{f}})\tilde{\mathcal{F}}(\tilde{L}_j \tilde{f})dT. \end{aligned}$$

Taking vacuum expectations we obtain

$$\frac{d}{dt} \| \tilde{\mathcal{F}}_t(\tilde{f})\Omega \|^2 = \sum_{j=1}^{d} \| \tilde{\mathcal{F}}_t(\tilde{L}_j \tilde{f})\Omega \|^2 . \tag{4.6}$$

We may solve (4.6), together with the initial value

$$\| \tilde{\mathcal{F}}_0(\tilde{f})\Omega \| = | \tilde{f}(e) |^2,$$

iteratively, to obtain

$$\| \tilde{\mathcal{F}}_t(\tilde{f})\Omega \|^2 = \sum_{n=0}^{\infty}(n!)^{-1}t^n \sum_{j_1,\dots,j_n=1}^{d} \Big| \prod_{k=1}^{n} \tilde{L}_{j_k}\tilde{f}(e)\Big|^2. \tag{4.7}$$

Now we follow Driver [Driv], noting that, for holomorphic $\tilde{f}$, each $\bar{L}_j\tilde{f} = \tilde{L}_j\bar{\tilde{f}} = 0$, and that each $\tilde{L}_j$ commutes with each $\bar{L}_j$ so that (4.7) can be rewritten as

$$\| \tilde{\mathcal{F}}_t(\tilde{f})\Omega \|^2 = \sum_{n=0}^{\infty}(n!)^{-1}t^n \Big(\sum_{j=1}^{d} \bar{L}_j\tilde{L}_j\Big)^n |\tilde{f}|^2(e). \tag{4.8}$$

But $\sum_{j=1}^{d} \bar{L}_j\tilde{L}_j$ is just twice the Laplacian $\tilde{\mathbb{L}}$. Hence (4.4) follows from (4.8). ∎

Since the Hall transformation $\mathcal{H}_t : f \mapsto \tilde{f}$ is defined by identifying the expansion coefficients $c_{j_1,\dots,j_n}(t)$ and $\tilde{c}_{j_1,\dots,j_n}(t)$ in (4.1) and ((4.2) it follows from Theorems 4.1,...,4.4 that

THEOREM 4.5. *$\mathcal{H}_t$ is an isometry from $L^2(d\gamma_t)$ to the holomorphic subspace $HL^2(d\tilde{\gamma}_t)$ of $L^2(d\tilde{\gamma}_t)$.*

Finally we must prove the convolution formula (1.1).

THEOREM 4.6. *If $\tilde{f} = \mathcal{H}_t f$ then, for arbitrary $x \in G \subset \tilde{G}$ we have*

$$\tilde{f}(x) = \int_{y\in G} f(xy^{-1})d\gamma_t(y) = \mathbb{E}[f(xX_t)]. \tag{4.9}$$

Proof. By (3.9) we have $\mathbb{E}[f(X_t)] = \tilde{f}(e)$, that is, since G is necessarily unimodular, (4.9) holds when $x = e$. Its validity for general x follows from this together with the translation-invariance property (3.10). ∎

5. Conclusion. The approach to the Hall transformation described here appears to be a natural one in that it gives direct probabilistic descriptions of the norms $\| \ \|_t$ on the dual of the universal enveloping algebra which render isometric the factors $\mathcal{D}_t$ and $\tilde{\mathcal{D}}_t$ of $\mathcal{H}_t$ discovered by Gross, Driver and their collaborators. It is essential that a *quantum* stochastic calculus be used to understand in this sense the Driver-Gross isometry $\tilde{\mathcal{D}}_t$. However our derivation of this isometry does not use the *ad*-invariance of the inner product on $\mathcal{L}$, so that this isometry extends to this more general case as noted in [DrGr]. In contrast, *ad*-invariance is essential for the centrality of the Laplacian (which is the quadratic Casimir element), required for our derivation of the isometry of the Gross map $\mathcal{D}_t$.

Our approach offers some interesting possibilities for further work and extension. The first of these is to construct a non-linear (in the Hall sense) generalisation of Segal's Fermionic analogue of the Bargmann transformation [Sega], making use of the $\mathbb{Z}_2$-graded quantum stochastic calculus [EyHu] which generalises the quantum stochastic Boson-Fermion unification [HuPa$_2$] and which lives naturally [Eyre] on the universal enveloping algebra of a Lie super-algebra in the same way that a quantum stochastic calculus lives naturally on the universal enveloping algebra of a Lie algebra [HuPu].

Another extension of the present work is to the study of deformations of Poisson Lie algebras [ChPr] and associated solutions of the quantum Yang-Baxter equation using quantum stochastic calculus. This work makes use of both holomorphic and conjugate-holomorphic derivative operators to construct deformed products which generalise the Weyl-Moyal product (2.4) of the case $G = \mathbb{R}$, $\tilde{G} = \mathbb{C}$. The corresponding solution of the quantum Yang-Baxter equation is constructed in the tensor product of two copies of the completed universal enveloping algebra using the tensor product operator of $\tilde{\mathcal{D}}_t$ with its conjugate-holomorphic counterpart.

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