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THE VARIATIONAL APPROACH TO THE DIRICHLET PROBLEM IN C^*-ALGEBRAS

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Abstract. The aim of this work is to develop the variational approach to the Dirichlet problem for generators of sub-Markovian semigroups on C^*-algebras. KMS symmetry and the KMS condition allow the introduction of the notion of weak solution of the Dirichlet problem. We will then show that a unique weak solution always exists and that a generalized maximum principle holds true.

1. The Dirichlet problem in C^*-algebras. The extension of the Dirichlet problem to noncommutative C^*-algebras has been considered some years ago by J.-L. Sauvageot [Sau1]. To formulate the problem the structure required consists of:

i) a C^*-algebra A
ii) a two-sided closed ideal $I \subset A$
iii) a strongly continuous, sub-Markovian semigroup, $\{\Phi_t : t \geq 0\}$ on A.

By double duality, $\{\Phi_t : t \geq 0\}$ can be canonically extended to a semigroup $\{\overline{\Phi}_t : t \geq 0\}$ of sub-Markovian normal maps on the enveloping von Neumann algebra A^{**}. The generator

$$\Delta x := \lim_{t\to 0} \frac{1}{t}(x - \Phi_t(x)),$$

defined on the domain $\mathrm{Dom}(\Delta)$ where the above limit exists in the norm of A, is then extended by

$$\overline{\Delta} x := \lim_{t\to 0} \frac{1}{t}(x - \overline{\Phi}_t(x)),$$

on the domain $\mathrm{Dom}(\overline{\Delta})$ of A^{**}, where the limit exists in the norm topology of A^{**}. Notice that the extended semigroup is not necessarily strongly nor weakly* continuous on A^{**}.

The formulation of the Dirichlet problem in the noncommutative setting is based on the notions of localized convergence and harmonicity, which we recall in a slight more

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restrictive form than the original one [Sau1, 2.1, 3.1].

Definition 1.1. i) A net $\{y_\alpha\} \subset A^{**}$ is said to converge to 0 in A^{**}, *uniformly over the compact sets of* I, if, for all $a \in I$ and for all $\varepsilon > 0$, there exists $a' \in A$ such that

$$\begin{cases} \|a - a'\| < \varepsilon, \\ \lim_\alpha \Big(\|y_\alpha a'\| + \|a' y_\alpha\|\Big) = 0. \end{cases}$$

ii) An element $x \in A^{**}$ is said to be *harmonic on* I, *w.r.t.* $\{\Phi_t : t \geq 0\}$, if

$$\frac{1}{t}[\overline{\Phi}_t(x) - x] \to 0$$

uniformly over the compact sets of I, as $t \to 0$.

Let $M(A)$ denote the multipliers algebra of A: $M(A) := \{x \in A^{**} : ax,\, xa \in A \quad \forall a \in A\}$.

Remark 1.2. It is easily recognized that on norm bounded subsets of $M(A)$ the uniform convergence over the compact sets of I introduced above coincides with the convergence in the *strict topology* of $M(A)$ (see [Ped, 3.12]). Consequently, an element $x \in \mathrm{Dom}(\overline{\Delta})$ is harmonic on I iff

$$a(\overline{\Delta}x) = 0, \qquad (\overline{\Delta}x)a = 0 \qquad \forall\, a \in I;$$

for $x \in \mathrm{Dom}(\Delta)$ this means that x is harmonic iff Δx belongs to the (closed, two sided) *annihilator ideal* $I' := \{b \in A : ba = 0 \quad \forall\, a \in I\}$ of I in A (see [Ric, II-8]).

The next definition makes precise what we mean by a Dirichlet problem in the C^*-algebra setting. With respect to the original definition proposed in [Sau1], there are two main differences. The first is that, as mentioned above, we use localized convergence in a stronger form. The second is that we take into account possibly non vanishing inner and boundary data.

Let A/I be the quotient C^*-algebra of A by the two-sided closed ideal I. By [Ped, Proposition 3.12.10], the canonical surjection of A onto A/I extends to a surjective morphism $p : M(A) \to M(A/I)$.

Definition 1.3 (Dirichlet problem). Let $\alpha \geq 0$, $y \in M(I)$, $z \in M(A/I)$ be fixed data. An element $x := L(y, z) \in M(A)$ is said to be a *solution of the Dirichlet problem*, with *inner condition* y and *boundary condition* z, if the class of x in $M(A/I)$ is z and

$$\frac{1}{t}[\overline{\Phi}_t(x) - x] + \alpha x \to y, \qquad t \to 0 \tag{1.1}$$

uniformly over the compact sets of I.

Globally, a solution of the Dirichlet problem is a completely sub-Markovian map

$$L : M(I) \oplus M(A/I) \to M(A) \qquad (y, z) \to x := L(y, z) \tag{1.2}$$

such that composed with the projection $p : M(A) \to M(A/I)$, $p \circ L$ gives the second coordinate map and such that $x = L(y, z)$ solves (1.1).

Notice that if the solution x of the Dirichlet problem belongs to $\mathrm{Dom}(\overline{\Delta})$, equation (1.1) reads as follows

$$a(-\overline{\Delta}x + \alpha x - y) = 0, \qquad (-\overline{\Delta}x + \alpha x - y)a = 0 \qquad \forall\, a \in I.$$

EXAMPLE 1.4. To recover a classical (commutative) Dirichlet problem consider as A the algebra $C_0(X)$ of continuous function, vanishing at infinity, on a Riemannian manifold (X, g). The ideal I will correspond to the algebra $C_0(\Omega)$, for some fixed open set $\Omega \subset X$. The quotient A/I will represent the algebra $C_0(\Omega^c)$ of continuous functions on Ω^c vanishing at infinity and $M(A)$ (resp. $M(I), M(A/I)$) the algebra $C_b(X)$ (resp. $C_b(\Omega), C_b(\Omega^c)$) of all continuous bounded functions on X (resp Ω, Ω^c). Clearly, the notion of convergence introduced above reduces to the uniform convergence over the compact subsets of Ω. If we choose for $\Phi_t = \exp(t\Delta_g)$ the heat semigroup whose generator is the Laplacian operator Δ_g associated to the metric g, a C^2 function $x \in C_b(X)$ is a solution of the Dirichlet problem (see [Bre, IX.5]) with data $y \in C_b(\Omega)$ and $z \in C_b(\Omega^c)$ if

$$\begin{cases} -\Delta_g x + \alpha x = y & \text{on} \quad \Omega \\ \qquad\qquad x = z & \text{on} \quad \Omega^c. \end{cases}$$

Finally, the complete sub-Markovianity of the lift L (which in the commutative case simply reduces to sub-Markovianity) translates into the algebraic setting what is known as the *maximum principle* for solutions of the Dirichlet problem (see [Bre, IX.7]).

EXAMPLE 1.5. Let G be a locally compact group with identity e. To any continuous, negative definite function $\psi : G \to \mathbb{R}$ such that $\psi(e) = 0$ is associated a strongly continuous (completely) sub-Markovian semigroup $\{\Phi_t : t \geq 0\}$ on the reduced C^*-algebra $C^*_{\mathrm{red}}(G)$, which extends to the semigroup $u \to e^{-t\psi}u$ on the algebra $\mathcal{K}(G)$ of continuous functions with compact support on G. Ideals of this C^*-algebra correspond to kernels of unitary representations of G.

EXAMPLE 1.6. Combining, in the natural way, the heat semigroup $e^{+t\Delta_g}$ of Example 1.4 and the semigroup $e^{-t\psi}$ of Example 1.5 one can construct a strongly continuous sub-Markovian semigroup on the crossed product C^*-algebra $\mathrm{C}^*(\alpha, G, C_0(X))$ [Ped, 7.6], associated to a continuous action $\alpha : G \times C_0(X) \to C_0(X)$ of isometries. The only condition one has to require is that the action commutes with the semigroup $e^{+t\Delta_g}$. This is an unpublished result due to J.-L. Sauvageot, who also proved the Feller's property (see below) for these kind of semigroups. Typical ideals in the above crossed product are in correspondence with *saturated subsets* of the manifold X, i.e. closed set which are union of orbits of the action α.

EXAMPLE 1.7. Let $(V, \mathcal{F})$ a Riemannian foliation of the compact manifold V and consider the associated C^*-algebra $C^*(V, \mathcal{F})$, constructed by A. Connes (see [Con2]). On this algebra J.-L. Sauvageot [Sau4] has recently constructed the so called *transverse heat semigroup* of the Riemannian foliation $(V, \mathcal{F})$. In this case closed ideals correspond to *saturated subsets* of V, i.e. closed subsets which are unions of leaves of the foliation.

EXAMPLE 1.8. In the notations of Example 1.4, consider the C^*-algebra

$$A := \{u \in C_0(X, M_n(\mathbb{C})) : u(x) \text{ is diagonal } \forall\, x \in \Omega'^c\},$$

where $M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices over $\mathbb{C}$ and $\Omega' \subset X$ is a fixed open set. Typical closed ideals in this algebra are those whose functions vanish outside an open set $\Omega' \subset \Omega$. A strongly continuous sub-Markovian semigroup can be realized by reducing to A the tensor product semigroup $e^{+t\Delta_g} \otimes \Psi_t$, where ψ is any sub-Markovian semigroup on the full matrix algebra $M_n(\mathbb{C})$.

To stress the differences between our approach and the one followed by J.-L. Sauvageot, we end this section discussing the main assumptions used in [Sau1] to solve, in the sense of Definition 1.3, the noncommutative Dirichlet problem. These were:

i) the complete sub-Markovianity of the semigroup;
ii) its Feller property: $\overline{\Phi}_t(A^{**}) \subset M(A) \qquad \forall\, t > 0$;
iii) the locality of its generator in the ideal I;
iv) the regularity of the ideal I.

The requirement i) is directly connected to Sauvageot's approach based on two typical quantum probabilistic tools: the construction of the quantum stochastic process associated with the semigroup (see [Sau2]) and the theory of quantum stopping times developed in [Sau3]. Properties ii) and iii) are instead connected with the solubility of the problem within the multiplier algebra $M(I)$ (as stated in Definition 1.3); property iv), expressed in terms of the regularity of the stopping time associated with I, is a sufficient condition to find the solution in $M(A)$ than in some other larger subalgebra of A^{**}.

2. KMS-symmetric Markovian semigroups and Dirichlet forms. To formulate the notion of weak solution of the Dirichlet problem, the main assumption we use is the following notion of symmetry.

Definition 2.1 (KMS-Symmetric Markovian Semigroups). Let $\omega \in A^*_+$ be a state satisfying the KMS condition at $\beta \in \mathbb{R}$ w.r.t. a strongly continuous automorphisms group $\{\alpha_t\}_{t\in\mathbb{R}}$ of A (see [Ped, 8.12]). The semigroup $\{\Phi_t\}_{t\geq 0}$ is said to be *β-KMS-symmetric* w.r.t. $\{\alpha_t\}_{t\in\mathbb{R}}$ and ω if

$$\omega(\Phi_t(x)\alpha_{-\beta\frac{i}{2}}(y)) = \omega(\alpha_{+\beta\frac{i}{2}}(x)\Phi_t(y)) \qquad \forall\, t \geq 0 \tag{2.1}$$

and for all $x, y \in A^a$ (the *-subalgebra of analytic elements for $\{\alpha_t\}_{t\in\mathbb{R}}$).

Clearly the above definition still makes sense not only for semigroups but also for single maps, even if these are not everywhere defined (in this case (2.1) will be verified for the analytic elements in the domain of the map only). In this form KMS symmetry has been introduced in [Cip, Definition 2.1] for the particular case of von Neumann algebras and modular automorphisms groups. See also [GL] for an equivalent formulation in the context of the Haagerup's standard form of the von Neumann algebra. The importance of KMS-symmetry for semigroups is that, combined with the KMS condition, it allows us to study semigroups in the GNS representation $(\pi_\omega, H_\omega, \xi_\omega)$ of the state ω.

Lemma 2.2. *A KMS-symmetric map Φ on A leaves globally invariant the kernel* $\ker(\pi_\omega)$ *of the GNS representation of* ω.

Proof. For $\beta \neq 0$ it is enough, by rescaling, to consider the case $\beta = 1$. We have to prove that for $x \in A$, $\pi_\omega(x) = 0$ implies $\pi_\omega(\Phi(x)) = 0$. Equivalently, we have to show that, for $x \in A$,

$$\omega(zxy) = 0 \qquad \forall\, y, z \in A \tag{2.2}$$

implies

$$\omega(z\Phi(x)y) = 0 \qquad \forall\, y, z \in A. \tag{2.3}$$

By the density of A^a in A (see [Ped, 8.12]) and the continuity of the map Φ, it is enough to prove (2.3) for $y, z \in A^a$ and $x \in A^a \cap \ker(\pi_\omega)$.

Let $\alpha_w(x)$ be, for $w \in \mathbb{C}$, the (entire) analytic extension of the function $\alpha_t(x)$ of the real variable t. Notice that $\{\alpha_w\}_{w\in\mathbb{C}}$ is an automorphism group on A^a indexed by $\mathbb{C}$ and, by [Ped, 8.12.4], $\omega(\alpha_w(x)) = \omega(x)$ for all $w \in \mathbb{C}$. Moreover, (2.2) implies:

$$\omega(z\alpha_w(x)y) = \omega(\alpha_{-w}(\alpha_w(z)x\alpha_w(y))) = \omega(\alpha_w(z)x\alpha_w(y)) = 0 \qquad \forall\, w \in \mathbb{C}. \tag{2.4}$$

By the KMS-condition satisfied by ω w.r.t. $\{\alpha_t\}_{t\in\mathbb{R}}$ we have:

$$\omega(z\Phi(x)y) = \omega(\Phi(x)y\alpha_i(z)) = \omega(\Phi(x)\alpha_{-i/2}(\alpha_{+i/2}(y\alpha_i(z)))).$$

Using the KMS-symmetry (2.1) and (2.4) we then have

$$\omega(z\Phi(x)y) = \omega(\alpha_{+i/2}(x)\Phi(\alpha_{+i/2}(y\alpha_i(z)))) = 0$$

as required. For $\beta = 0$ the proof is similar and easier. ∎

Since $\alpha_t(\ker(\pi_\omega)) \subset \ker(\pi_\omega)$ for all $t \in \mathbb{R}$, we can extend the automorphism group on the C^*-algebra $\pi_\omega(A)$ (see [Dop, V.4]):

$$\alpha_t^\omega : \pi_\omega(A) \to \pi_\omega(A) \qquad \alpha_t^\omega(\pi_\omega(x)) := \pi_\omega(\alpha_t(x)) \qquad \forall\, x \in A, \quad \forall\, t \in \mathbb{R}.$$

Let us denote by $\mathcal{M}$ the weak closure $\pi_\omega(A)''$ in $B(H_\omega)$. Since no confusion can arise, ω will also denote the normal extension to $\mathcal{M}$ of the state ω on A: $\omega(\cdot) = (\cdot\xi_\omega; \xi_\omega)$. The vector ξ_ω is cyclic and separating for $\mathcal{M}$ and the associated modular automorphisms group $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ on $\mathcal{M}$, when reduced on $\pi_\omega(A)$, coincides with $\{\alpha_{\beta t}^\omega\}_{t\in\mathbb{R}}$ (see [Dop, V.4]).

THEOREM 2.3. *Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a strongly continuous automorphisms group on the C^*-algebra A and let $\omega \in A_+^*$ be an associated KMS state at $\beta \in \mathbb{R}$. Let $(\pi_\omega, H_\omega, \xi_\omega)$ be the GNS representation of ω. Let $\{\Phi_t\}_{t\ge 0}$ be a strongly continuous, sub-Markovian semigroup on A and suppose it is β-KMS symmetric w.r.t. $\{\alpha_t\}_{t\in\mathbb{R}}$ and $\omega \in A_+^*$. Then there exists on $\mathcal{M} = \pi_\omega(A)''$ a unique $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous, sub-Markovian semigroup $\{\Phi_t^\omega\}_{t\ge 0}$ such that*

$$\Phi_t^\omega(\pi_\omega(x)) = \pi_\omega(\Phi_t(x)) \qquad \forall\, x \in A, \qquad \forall\, t \ge 0. \tag{2.5}$$

Moreover $\{\Phi_t^\omega\}_{t\ge 0}$ is KMS symmetric w.r.t. the modular automorphisms group $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ of the normal extension of ω on $\mathcal{M}$.

Proof. By Lemma 2.2, formula (2.5) defines a semigroup of maps on $\pi_\omega(A)$. Since $\{\Phi_t\}_{t\ge 0}$ is β-KMS symmetric w.r.t. $\{\alpha_t\}_{t\in\mathbb{R}}$ and ω, $\{\Phi_t^\omega\}_{t\ge 0}$ is KMS symmetric w.r.t. $\{\sigma_t^\omega\}_{t\in\mathbb{R}}$ (see [Cip, Definition 2.1)]. By [Cip, Proposition 2.3] these maps are $\sigma(\mathcal{M}, \mathcal{M}_*)$-densely defined and $\sigma(\mathcal{M}, \mathcal{M}_*)$-closable. By [Cip, Proposition 2.3 ii)], to prove that they can be uniquely extended to everywhere defined $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous (hence norm bounded) maps, it is enough to show that the domain of their closures is $\mathcal{M}$. Fix $t \ge 0$ and $a \in \mathcal{M}$ and consider a net $\{\pi_\omega(x_\beta)\}_\beta \subset \pi_\omega(A)$ converging to a in the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology. The net is then norm bounded and, by the sub-Markovianity of Φ_t^ω, $\{\Phi_t^\omega(x_\beta)\}_\beta$ is norm bounded too, hence a $\sigma(\mathcal{M},\mathcal{M}_*)$-relatively compact set. Hence, possibly considering a suitable subnet, $\{\pi_\omega(x_\beta)\}_\beta$ converges to a in the $\sigma(\mathcal{M},\mathcal{M}_*)$-topology and $\{\Phi_t^\omega(x_\beta)\}_\beta$ converges in $\mathcal{M}$, which proves that a is in the domain of the closure of Φ_t^ω.

To prove the $\sigma(\mathcal{M},\mathcal{M}_*)$-continuity of the semigroup $\{\Phi_t^\omega\}_{t\geq 0}$, we start to observe that for normal functionals of the form $\omega_\xi = (\cdot\xi;\xi)$, $\xi = \pi_\omega(y)\xi_\omega$ for some $y \in A$, the strong continuity of $\{\Phi_t\}_{t\geq 0}$ gives:

$$\omega_\xi(\Phi_t^\omega(\pi_\omega(x))) = (\pi_\omega(\Phi_t(x))\xi;\xi) = \omega(y^*\Phi_t(x)y) \to \omega(y^*xy), \qquad t \to 0.$$

Every $\psi \in \mathcal{M}_{*+}$ can be represented as $\psi(\cdot) = (\cdot\eta;\eta)$ for some $\eta \in H_\omega$ (see [Ara]). Since $\|\psi-\omega_\xi\| \leq \|\eta-\xi\|\cdot\|\eta+\xi\|$, the density of $\pi_\omega(A)\xi_\omega$ in H_ω implies that $\{\omega_\xi : \xi \in \pi_\omega(A)\xi_\omega\}$ span a norm dense subset of $\mathcal{M}_*$. This and the fact that sub-Markovian semigroups are necessarily contractive ([Cip, Proposition 2.6]) implies the statement. ∎

By Tomita-Takesaki theory (see [Ped, 8.13]) we consider the standard form of $\mathcal{M} = \pi_\omega(A)''$ in the GNS representation of the state ω on the C^*-algebra A (see [Ara], [Con1], [Haa]). This consists of the triple $(\mathcal{M}, L^2(A,\omega), L^2_+(A,\omega))$, where

$$\mathcal{M} = \pi_\omega(A)'', \qquad L^2(A,\omega) := H_\omega, \qquad L^2_+(A,\omega) := \overline{\Delta_{\xi_\omega}^{1/4}\mathcal{M}_+\xi_\omega}.$$

Here Δ_{ξ_ω} denotes the modular operator associated to the cyclic and separating vector ξ_ω. Among the main properties of $L^2_+(A,\omega)$, whose elements are called *positive*, the following ones will be crucial for us: $L^2_+(A,\omega)$ is a *closed*, *convex*, *selfdual cone* in the sense that

$$L^2_+(A,\omega) = \{\xi \in L^2(A,\omega) : (\xi,\eta) \geq 0 \ \forall \eta \in L^2_+(A,\omega)\}.$$

Furthermore, $L^2(A,\omega)$ is the complexification of the subspace

$$L^2_{\mathbb{R}}(A,\omega) := \{\xi \in L^2(A,\omega) : (\xi,\eta) \in \mathbb{R} \ \forall \eta \in L^2_+(A,\omega)\}$$

whose elements are called *real* and on which the cone induces a structure of ordered real Hilbert space (denoted by $\leq$). It also gives rise to an isometric conjugation J on $L^2(A,\omega)$ which leaves $L^2_+(A,\omega)$ and $L^2_{\mathbb{R}}(A,\omega)$ invariant: $J(\xi+i\eta) := \xi - i\eta$ for all $\xi,\eta \in L^2_{\mathbb{R}}(A,\omega)$. An element ξ is real iff $J\xi = \xi$. Any real element ξ can be uniquely decomposed as a difference $\xi = \xi_+ - \xi_-$ of two orthogonal positive elements, the *positive* and *negative* parts: $\xi_\pm \in L^2_+(A,\omega)$, $(\xi_+,\xi_-) = 0$. The positive part ξ_+ is identified with the hilbertian projection of ξ onto the closed convex cone $L^2_+(A,\omega)$. This is the *Jordan decomposition* which characterizes selfdual cones among the convex and closed ones (see [Ioc]). By the general theory of [Cip], we can then extend $\{\Phi_t^\omega\}_{t\geq 0}$ to a well behaved semigroup on the space $L^2(A,\omega)$.

THEOREM 2.4. *There exists a unique, strongly continuous semigroup $\{T_t^\omega\}_{t\geq 0}$ on the Hilbert space $L^2(A,\omega)$ such that*

$$T_t^\omega i_\omega(x) = i_\omega(\Phi_t^\omega(x)) \qquad \forall x \in \mathcal{M} \qquad (2.6),$$

where $i_\omega : \mathcal{M} \to L^2(A,\omega)$ denotes the symmetric embedding: $i_\omega(x) = \Delta_{\xi_\omega}^{1/4} x\xi_\omega$.

Moreover $\{T_t^\omega\}_{t\geq 0}$ is symmetric, contractive and sub-Markovian in the sense that

$$0 \leq T_t^\omega(\xi) \leq \xi_\omega \qquad \text{whenever} \qquad 0 \leq \xi \leq \xi_\omega, \quad \xi \in L^2(A,\omega), \qquad (2.7)$$

the order relation in $L^2(A,\omega)$ being defined by the (closed and convex) cone $L^2_+(A,\omega)$.

P r o o f. It is a straightforward application of [Cip, Definition 2.8, Theorem 2.11]. ∎

Associated to the sub-Markovian semigroup $\{T_t^\omega\}_{t\geq 0}$, let us consider the following symmetric, quadratic form on $L^2(A,\omega)$:

$$\begin{cases} \mathcal{F}^\omega := \{\xi \in L^2(A,\omega) : \lim_{t\to 0} \dfrac{1}{t}(\xi - T_t^\omega \xi;\xi) \quad \text{exists}\} \\ \mathcal{E}^\omega[\xi] := \lim_{t\to 0} \dfrac{1}{t}(\xi - T_t^\omega \xi;\xi) \qquad \xi \in \mathcal{F}^\omega, \end{cases} \tag{2.8}$$

which will be central in next section. It is the form of the selfadjoint generator of $\{T_t^\omega\}_{t\geq 0}$ and, by a general result [Cip, Theorem 4.11], it is a Dirichlet form in the following sense [Cip, §4].

DEFINITION 2.5 (Dirichlet Form). Let $(\mathcal{E},\mathcal{F})$ be a symmetric, nonnegative, quadratic form on $L^2(A,\omega)$. It is said *J-real* if

$$J\xi \in \mathcal{F} \qquad \text{and} \qquad \mathcal{E}[J\xi] = \mathcal{E}[\xi], \qquad \text{whenever} \quad \xi \in \mathcal{F}, \tag{2.9}$$

it is said *Markovian* if:

$$\xi \wedge \xi_\omega \in \mathcal{F} \qquad \text{and} \qquad \mathcal{E}[\xi\wedge\xi_\omega] \leq \mathcal{E}[\xi], \qquad \text{whenever} \quad \xi \in \mathcal{F} \quad \text{and} \quad J\xi = \xi, \tag{2.10}$$

where $\xi \wedge \xi_\omega$ denotes the hilbertian projection of the vector ξ onto the closed and convex set $\xi_\omega - L^2_+(A,\omega)$.

A nonnegative, closed, Markovian quadratic form is called a *Dirichlet form*.

3. Weak solution of the Dirichlet problem. To motivate the introduction of the notion of weak solution of the Dirichlet problem posed in Definition 2.1, notice that the C^*-algebra $M(A)$ is not always C^*-isomorphic to $M(I)\oplus M(A/I)$ for generic two sided closed ideals I. Analogously, the existence of the completely sub-Markovian lift $L : M(I)\oplus M(A/I) \to M(A)$, solving the Dirichlet problem, necessarily requires strong regularity properties on the ideal and on the sub-Markovian semigroup, as for example those adopted by J.-L. Sauvageot in [Sau1] and discussed after Definition 1.3. On the other hand, $M(A)$ and $M(I)\oplus M(A/I)$ are always *Borel isomorphic*, in the sense that their enveloping Borel *-algebras are isomorphic ([Ped, 4.6]). This suggests that the Dirichlet problem (1.1)-(1.2) could be solved under less pressing assumptions, when considered in the Borel or W*-category ([Ped, 4.5]). Under the KMS symmetry assumption, we are going to show that this program can be in fact carried out, once we will have suitably adapted Definition 1.3 at the level of standard form of von Neumann algebras.

Let us consider the supporting central projection $z_I \in A^{**}$ of the closed, two sided ideal I of A. It is easy to see that I^{**} and $(A/I)^{**}$ can be identified with $z_I A^{**}$ and $(1-z_I)A/I^{**}$, respectively. Let us denote by e_I the image of z_I under the canonical surjection $\widetilde{\pi_\omega} : A^{**} \to \mathcal{M}$ which extends $\pi_\omega : A \to \mathcal{M}$ ([Ped, Theorem 3.7.7]). Then e_I is a central projection in $\mathcal{M}$ and the image by $\widetilde{\pi_\omega}$ of the $\sigma(A,A_*)$-closed two sided ideals $z_I I^{**}$ and $(1-z_I)(A/I)^{**}$ coincide with the $\sigma(\mathcal{M},\mathcal{M}_*)$-closed two sided ideals $\mathcal{M}e_I$ and $\mathcal{M}(1-e_I)$, respectively.

Notice also that while $\mathcal{M}e_I$ coincide with $\pi_\omega(I)''$, it is not difficult to see that $\mathcal{M}(1-e_I)$ can be naturally identified with $\pi_\omega(A/I)''$, where now π_ω denotes the GNS representation of the reduction of ω to A/I (which, by abuse of notation, we still denote by ω):

$$\omega : A/I \to \mathbb{C} \qquad \omega(a+I) := \lim_{\lambda} \omega((1-u_\lambda)a(1-u_\lambda))$$

for all $(a+I) \in A/I$ and some fixed approximate unit $\{u_\lambda\}_\lambda$ in the ideal I. It is not difficult to see that the above definition is independent of the particular approximate unit involved. Recall that a *face* F in the selfdual, convex cone $\mathcal{P} := L^2_+(A,\omega)$ is a convex subcone which is hereditary in the sense that $\xi \in \mathrm{F}$, $\eta \in \mathcal{P}$ and $\xi - \eta \in \mathcal{P}$ imply $\eta \in \mathrm{F}$; the face F is said to be *splitting* if $\mathcal{P} = \mathrm{F} \oplus \mathrm{F}^\perp$, where the orthogonal face is defined as $\mathrm{F}^\perp := \{\eta \in \mathcal{P} : (\eta, \xi) = 0 \quad \forall \xi \in \mathrm{F}\}$ (see [Con1], [Ioc]). In the following we will adopt the following notations:

$$L^2(I,\omega) := e_I L^2(A,\omega), \qquad L^2(A/I,\omega) := (1-e_I)L^2(A,\omega),$$
$$L^2_+(I,\omega) := e_I L^2_+(A,\omega), \qquad L^2_+(A/I,\omega) := (1-e_I)L^2_+(A,\omega).$$

LEMMA 3.1. *$L^2_+(I,\omega)$ and $L^2_+(A/I,\omega)$ are closed, splitting faces of the seldual convex cone $L^2_+(A,\omega)$. Moreover, e_I and $e_{A/I}$ are sub-Markovian projections.*

Proof. By the properties of standard forms of von Neumann algebras, each closed face of $L^2_+(A,\omega)$ is of the form $eJeJ(L^2_+(A,\omega))$ for a unique projection $e \in \mathcal{M}$, and $xJxJ$ is positivity preserving for all $x \in \mathcal{M}$ (see [Con1]). Moreover if e is central in $\mathcal{M}$, then $eJeJ = ee^* = e$, the face $L^2_+(I,\omega)$ is splitting and the orthogonal face is $L^2_+(A/I,\omega)$. Let $\xi \in L^2_{\mathbb{R}}(A,\omega)$ be such that $0 \le \xi \le \xi_\omega$. Then, since $e = eJeJ$ and $(1-e) = (1-e)J(1-e)J$, e and $e^\perp$ are positivity preserving, so that

$$0 \le e\xi \le e\xi_\omega \le e\xi_\omega + (1-e)\xi_\omega = \xi_\omega,$$

which shows the two projections to be sub-Markovian. ∎

We can now introduce the following notion of solution of the noncommutative Dirichlet problem.

DEFINITION 3.2 (Weak Solution of the Dirichlet Problem). Let $\{\Phi_t\}_{t\ge0}$ be a strongly continuous sub-Markovian semigroup on the C^*-algebra A. Let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a strongly continuous automorphisms group of A and $\omega \in A^*_+$ an associated KMS state at $\beta \in \mathbb{R}$ with respect to which the semigroup is β-KMS symmetric. Let us consider the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on the Hilbert space $L^2(A,\omega)$ associated with $\{\Phi_t\}_{t\ge0}$ and fix $\alpha > 0$, $\eta \in L^2(I,\omega)$ and $\zeta \in \mathcal{F}^\omega$. An element $\xi \in \mathcal{F}^\omega$ is said to be a *weak solution of the Dirichlet problem* posed in Definition 1.3, with *inner* and *exterior data* η and ζ respectively, if

$$\begin{cases} \mathcal{E}^\omega(\xi,\xi') + \alpha(\xi,\xi')_{L^2(A,\omega)} = (\eta,\xi')_{L^2(A,\omega)} \qquad \forall\, \xi' \in \mathcal{F}^\omega_I \\ \quad \xi \in \zeta + \mathcal{F}^\omega_I \end{cases} \tag{3.1}$$

where $\mathcal{F}^\omega_I := \mathcal{F}^\omega \cap L^2(I,\omega)$. Notice that the solution ξ depends upon the exterior condition ζ only through $(1-e_I)\zeta \in L^2(A/I,\omega)$.

The next theorem is our noncommutative version of the *Dirichlet Principle* for weak solutions.

THEOREM 3.3 (Existence and Uniqueness of the Weak Solution). *With the assumptions of Definition 3.2, there exists a unique weak solution of the noncommutative Dirichlet* *problem. It can be characterized as the unique minimizer of the following functionals*:

$$\begin{aligned} E_1 : \mathcal{F}^\omega \to \mathbb{R} \qquad & E_1(\xi) := \frac{1}{2}\Big(\mathcal{E}^\omega[\xi] + \alpha\|\xi\|^2_{L^2(A,\omega)}\Big) - \alpha(\eta,\xi)_{L^2(A,\omega)}, \\ E_2 : \mathcal{F}^\omega \to \mathbb{R} \qquad & E_2(\xi) := \alpha^{-1}\mathcal{E}^\omega[\xi] + \|\eta - \xi\|^2_{L^2(A,\omega)} \end{aligned} \tag{3.2}$$

over the set $\zeta + L^2(I,\omega)$.

P r o o f. Let $Q =: \zeta + L^2(I,\omega)$. As in the classical case (e.g. [Bre, Proposition IX.22]), one easily verifies that $\xi \in Q$ is a weak solution if and only if

$$\mathcal{E}^\omega(\xi, \xi' - \xi) + \alpha(\xi, \xi' - \xi) \ge (\eta, \xi' - \xi) \qquad \forall \xi' \in Q.$$

Since Q is closed and convex and $\mathcal{E}^\omega_\alpha[\cdot] := \mathcal{E}^\omega[\cdot] + \alpha\|\cdot\|^2_{L^2(A,\omega)}$ is coercive (for $\alpha > 0$), we can then apply Stampacchia's theorem on $(\mathcal{E}^\omega_\alpha, \mathcal{F}^\omega)$ [Bre, Theorem V.6] to get the existence and uniqueness of the weak solution as unique minimizer of E_1. The proof is completed as soon as one notices that $E_2[\cdot] = 2\alpha^{-1}E_1[\cdot] + \|\eta\|^2$. ∎

THEOREM 3.4 (Maximum Principle for Weak Solutions). *Let $\xi \in \mathcal{F}^\omega$ be a weak solution of the noncommutative Dirichlet problem with inner and exterior conditions $\eta \in L^2(I,\omega)$ and $e_I^\perp\zeta \in L^2(A/I,\omega)$ (for some $\zeta \in \mathcal{F}^\omega$), respectively. Then, for $\lambda \ge 0$ we have*:

$$\xi \in L^2_{\mathbb{R}}(A,\omega) \qquad \textit{whenever} \qquad \eta \in L^2_{\mathbb{R}}(I,\omega), \quad e_I^\perp\zeta \in L^2_{\mathbb{R}}(A/I,\omega), \tag{3.3}$$

$$\xi \in L^2_+(A,\omega) \qquad \textit{whenever} \qquad \eta \in L^2_+(I,\omega), \quad e_I^\perp\zeta \in L^2_+(A/I,\omega), \tag{3.4}$$

$$0 \le \xi \le \lambda\xi_\omega \qquad \textit{whenever} \qquad 0 \le \eta \le \lambda e_I\xi_\omega, \quad 0 \le e_I^\perp\zeta \le \lambda e_I^\perp\xi_\omega. \tag{3.5}$$

More globally, the map $L : L^2(I,\omega) \oplus (I - e_I)\mathcal{F}^\omega \to L^2(A,\omega)$, where $L(\eta,\zeta)$ is the weak solution of the Dirichlet problem with data η and ζ, is sub-Markovian.

The proof of the theorem relies on the following lemmas.

LEMMA 3.5. *Let e be a projection onto a closed, convex, splitting face of $L^2_+(A,\omega)$. We then have*:

i) $(e\xi)_\pm = e\xi_\pm$ *and* $(e^\perp\xi)_\pm = e^\perp\xi_\pm$ *for all* $\xi \in L^2_{\mathbb{R}}(A,\omega)$;
ii) $e(\xi_1 \wedge \xi_2) = e\xi_1 \wedge e\xi_2$ *for all* $\xi_i \in L^2_{\mathbb{R}}(A,\omega)$ $i = 1, 2$;
iii) $e^\perp(\xi_1 \wedge \xi_2) = e^\perp\xi_1 \wedge e^\perp\xi_2$ *for all* $\xi_i \in L^2_{\mathbb{R}}(A,\omega)$ $i = 1, 2$.

P r o o f. i) Let $\xi \in L^2_{\mathbb{R}}(A,\omega)$ and $\xi = \xi_+ - \xi_-$ its Jordan decomposition into the positive and negative parts. Since e is positivity preserving $e\xi_\pm \in L^2_+(A,\omega)$. Moreover, since the face is splitting, $e^\perp$ is the orthogonal projection onto the orthogonal face, hence a positive preserving projection too. By selfduality of the cone $L^2_+(A,\omega)$ we have

$$0 \le (e\xi_+, e\xi_-) = (\xi_+, e\xi_-) \le (\xi_+, \xi_-) = 0,$$

so that $e\xi = e\xi_+ - e\xi_-$ is the Jordan decomposition of $e\xi$. The statement involving $e^\perp$ can be proved analogously.

ii) By [Cip, Lemma 4.4], $\xi_1 \wedge \xi_2 = \xi_2 - (\xi_1 - \xi_2)_-$ and by part i) we have:

$$e(\xi_1 \wedge \xi_2) = e(\xi_2 - (\xi_1 - \xi_2)_-) = e\xi_2 - e(\xi_1 - \xi_2)_- = e\xi_2 - (e\xi_1 - e\xi_2)_- = e\xi_1 \wedge e\xi_2. \ ∎$$

LEMMA 3.6. *Let e be the projection onto a closed, convex, splitting face of $L^2_+(A,\omega)$ and consider $\xi_i \in L^2_{\mathbb{R}}(A,\omega)$ $i = 1, 2$, $\xi_3 \in L^2(A,\omega)$.*

i) If $e^\perp\xi_3 \ge 0$ we have $\xi_{1+} \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$ whenever $\xi_1 \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$;

ii) if $e^{\perp}\xi_1 \le e^{\perp}\xi_3$ *we have* $\xi_1 \wedge \xi_2 \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$ *whenever* $\xi_1 \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$.

Proof. i) Since $\xi_1 \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$ we have $e^{\perp}\xi_1 = e^{\perp}\xi_3$, so that, by Lemma 3.5 i), we have also: $e^{\perp}(\xi_{1+} - \xi_3) = e^{\perp}(\xi_{1+}) - e^{\perp}\xi_3 = (e^{\perp}\xi_1)_+ - e^{\perp}\xi_3 = (e^{\perp}\xi_3)_+ - e^{\perp}\xi_3 = 0$.

ii) Since $\xi_1 \in \xi_3 + L^2_{\mathbb{R}}(I,\omega)$ we have: $e^{\perp}\xi_1 = e^{\perp}\xi_3$. By Lemma 3.5 iii) we have:

$$e^{\perp}(\xi_1 \wedge \xi_2 - \xi_3) = e^{\perp}(\xi_1 \wedge \xi_2) - e^{\perp}\xi_3 = e^{\perp}\xi_1 \wedge e^{\perp}\xi_2 - e^{\perp}\xi_3 = \\ = e^{\perp}\xi_3 \wedge e^{\perp}\xi_2 - e^{\perp}\xi_3 = e^{\perp}\xi_3 - e^{\perp}\xi_3 = 0. \blacksquare$$

LEMMA 3.7. *For all real* $\xi \in \mathcal{F}^\omega$ *we have*: $\mathcal{E}^\omega[\xi \wedge \lambda\xi_\omega] \le \mathcal{E}^\omega[\xi] \qquad \forall\, \lambda \ge 0$.

Proof. For $\lambda = 0$ apply [Cip, Theorem 4.11, Theorem 4.7]. By [Cip, Theorem 6.1], it suffices to prove that $T^\omega_t(\lambda\xi_\omega - L^2_+(A,\omega)) \subseteq \lambda\xi_\omega - L^2_+(A,\omega)$ for all $t \ge 0$. In fact, by sub-Markovianity of the semigroup we have:

$$T^\omega_t(\lambda\xi_\omega - L^2_+(A,\omega)) = \lambda T^\omega_t(\xi_\omega - \lambda^{-1}L^2_+(A,\omega)) = \lambda T^\omega_t(\xi_\omega - L^2_+(A,\omega)) \subseteq \\ \subseteq \lambda(\xi_\omega - L^2_+(A,\omega)) = \lambda\xi_\omega - L^2_+(A,\omega). \blacksquare$$

Proof of Theorem 3.4. Let us denote by e the projection e_I and let ξ be the weak solution of the noncommutative Dirichlet problem with data $\eta \in L^2(I,\omega)$ and $e^{\perp}\zeta \in L^2(A/I,\omega)$, $\zeta \in \mathcal{F}^\omega$. In order to prove (3.3), let us assume $\eta \in L^2_{\mathbb{R}}(I,\omega)$, $e^{\perp}\zeta \in L^2_{\mathbb{R}}(A/I,\omega)$ and define $Q := \zeta + L^2(I,\omega)$. By Theorem 3.3 it is enough to show that $P\xi \subset Q$ and $E_2(P\xi) \le E_2(\xi)$ where $P := (1+J)/2$ is the projection onto the real part $L^2(A,\omega)$. We have $e^{\perp}(\xi - \zeta) = 0$ by hypothesis. Since e is a projection in the center of $\mathcal{M}$, it commutes with J and P ([Con1]). Since moreover, by hypothesis, $Je^{\perp}\zeta = \zeta$ we have $P\xi = \xi$ and

$$e^{\perp}(P\xi - \zeta) = Pe^{\perp}\xi - e^{\perp}\zeta = P(e^{\perp}\xi - e^{\perp}\zeta) = Pe^{\perp}(\xi - \zeta),$$

so that $P\xi \in Q$. Since $\mathcal{E}^\omega$ is J-real (Definition 2.5), $\mathcal{E}^\omega[P\xi] \le \mathcal{E}^\omega[\xi]$ by [Cip, Lemma 4.2, Theorem 6.1]. Since, by hypothesis $P\eta = \eta$, we finally have:

$$E_2(P\xi) = \alpha^{-1}\mathcal{E}^\omega[P\xi] + \|\eta - P\xi\|^2 \le \alpha^{-1}\mathcal{E}^\omega[\xi] + \|P\eta - P\xi\|^2 \le \alpha^{-1}\mathcal{E}^\omega[\xi] + \|\eta - \xi\|^2 = E_2(\xi).$$

To prove (3.4) assume now $\eta \in L^2_+(I,\omega)$ and $e^{\perp}\zeta \in L^2_+(A/I,\omega)$ and define $Q := \zeta + L^2_+(I,\omega)$. By the first part of the proof we know ξ to be real. By Lemma 3.6, $\xi_+ \in Q$ and, since $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ is a Dirichlet form, we have also $\mathcal{E}^\omega[\xi_+] \le \mathcal{E}^\omega[\xi]$ by [Cip, Theorem 4.7 iii)]. Since the projection $\xi \to \xi_+$ is nonexpansive and η is positive, we get

$$E_2(\xi_+) = \alpha^{-1}\mathcal{E}^\omega[\xi_+] + \|\eta - \xi_+\|^2 \le \alpha^{-1}\mathcal{E}^\omega[\xi] + \|\eta_+ - \xi_+\|^2 \le \alpha^{-1}\mathcal{E}^\omega[\xi] + \|\eta - \xi\|^2 = E_2(\xi)$$

so that $\xi = \xi_+$ by Theorem 3.3.

To complete the proof of the theorem we have to show that $\xi \le \lambda\xi_\omega$ assuming now $\eta \le \lambda e\xi_\omega$ and $e^{\perp}\zeta\lambda e^{\perp}\xi_\omega$. Notice that the hypothesis on η is equivalent to $\eta \wedge \lambda e\xi_\omega = \eta$. Define $Q := \zeta + L^2_{\mathbb{R}}(I,\omega)$. By Lemma 3.6 ii), $\xi \wedge \lambda\xi_\omega \in Q$. Applying Lemma 3.7 and the fact that the hilbertian projection $\xi \to \xi \wedge \lambda\xi_\omega$ is non expansive, we obtain

$$E_2(\xi \wedge \lambda\xi_\omega) = \alpha^{-1}\mathcal{E}^\omega[\xi \wedge \lambda\xi_\omega] + \|\eta - \xi \wedge \lambda\xi_\omega\|^2 \le \\ \le \alpha^{-1}\mathcal{E}^\omega[\xi] + \|\eta \wedge \lambda\xi_\omega - \xi \wedge \lambda\xi_\omega\|^2 \le E_2(\xi),$$

which concludes the proof of the theorem. $\blacksquare$

4. Conclusions and prospects. In this work the variational approach to the noncommutative Dirichlet problem has been developed and applied to a given sub-Markovian semigroup, or its generator, on a C^*-algebra A. As it results quite clearly from the proofs of Theorem 3.3 and Theorem 3.4, the method provided has a wider range of applications. For example, using the methods of [Cip], one can deal with semigroups and generators associated to a general noncommutative Dirichlet form on any standard form of a von Neumann algebra. Moreover, while closed two sided ideals, $I \subset A$, are associated, in the Jacobson topology, to closed subsets of the primitive spectrum $\mathrm{Prim}(A)$ (see [Ped, Theorem 4.1.3]), our method allows to solve the Dirichlet problem, in the weak sense, on Borel sets of several Borel structures on $\mathrm{Prim}(A)$ [Ped, 4.7]. In particular Theorem 3.3 and Theorem 3.4 remain valid for general closed splitting faces of the selfdual cone.

The important problem one has to face when dealing with weak solutions is their regularity. In other words, if the inner and boundary data η and ζ belong to $M(I)$ and $M(A/I)$, respectively, does the corresponding weak solution belong to $M(A)$? In [Sau1] this problem has been posed and solved using a probabilistic approach. In particular, beside analytic regularity properties of the semigroup, such as locality and Feller's properties, a probabilistic regularity property of the ideal I is required in terms of the quantum stopping time associated with I. For the time being we prefer to conclude the paper at this stage, leaving to subsequent works the study of the regularity of weak solutions of noncommutative Dirichlet problems, such as those arising in contexts of Noncommutative Geometry of Examples 1.5, 1.6, 1.7.

References

[Ara] H. Araki, *Some properties of the modular conjugation operator of von Neumann algebras and noncommutative Radon-Nikodym theorem with chain rule*, Pacific J. Math. 50 (1974), 309-354.

[Bre] H. Brezis, *Analyse Fonctionnelle*, Masson S.A., Paris, Milan, Barcelone 1993.

[Cip] F. Cipriani, *Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras*, J. Funct. Anal. 147 (1997), 259-300.

[Con1] A. Connes, *Caractérisation des espaces vectoriels ordonnés sous jacents aux algèbres de von Neumann*, Ann. Inst. Fourier (Grenoble) 24 No. 4 (1974), 121-155.

[Con2] A. Connes, *Noncommutative Geometry*, Academic Press, London New York San Francisco, 1994.

[GL] S. Goldstein and J. M. Lindsay, *Beurling-Deny conditions for KMS-symmetric dynamical semigroups*, C. R. Acad. Sci. Paris, Ser. *I* 317 (1993), 1053-1057.

[Haa] U. Haagerup, *The standard form of a von Neumann algebra*, Math. Scand. 37 (1975), 271-283.

[Ioc] B. Iochum, *Cones autopolaires et algèbres de Jordan*, Lecture Notes in Mathematics 1049, Springer-Verlag, Berlin-Heidelberg-New York, 1984.

[Ped] G. K. Pedersen, *C^*-Algebras and their Automorphism Groups*, Academic Press, London New York San Francisco, 1979.

[Ric] C. E. Rickart, *General Theory of Banach Algebras*, D. Van Nostrand Company, Inc., Princeton, New Jersey, Toronto New York London, 1960.

[Sau1] J.-L. Sauvageot, *Le problème de Dirichlet dans les C^*-algèbres*, J. Funct. Anal. 101 (1991), 50-73.

[Sau2] J.-L. Sauvageot, *Markov quantum semigroups admits covariant Markov C^*-dilations*, Comm. Math. Phys. 106 (1986), 91-103.

[Sau3] J.-L. Sauvageot, *First exit time: A theory of stopping times in quantum processes*, in: Quantum Probability and Applications III, Lect. Notes in Math. 1303, Springer-Verlag, New York, 1988.

[Sau4] J.-L. Sauvageot, *Semi-groupe de la chaleur transverse sur la C^*-algèbre d'un feuilletage riemannien*, J. Funct. Anal. 142 (1996), 511-538.