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MARKOVIAN PROCESSES ON MUTUALLY COMMUTING VON NEUMANN ALGEBRAS

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1. The aim of this paper is to study markovianity for states on von Neumann algebras generated by the union of (not necessarily commutative) von Neumann subagebras which commute with each other. This study has been already begun in [2] using several a priori different notions of noncommutative markovianity. In this paper we assume to deal with the particular case of states which define odd stochastic couplings (as developed in [3]) for all couples of von Neumann algebras involved. In this situation these definitions are equivalent, and in this case it is possible to get the full noncommutative generalization of the basic classical Markov theory results. In particular we get a correspondence theorem, and an explicit structure theorem for Markov states.

2. Let M be a von Neumann algebra acting on an Hilbert space H. For ξ in H we denote by ω_ξ the vector state on $B(H)$ implemented by ξ. In order to simplify our notations we shall often write $(\omega_\xi)_M$ for $\omega_\xi|_M$ or simply $(\omega_\xi)_\alpha$ if the von Neumann algebra involved is endowed with an index α.

We shall say C is a self–dual positive cone for M in H if there is a separating vector Ω for M in H, such that C is the selfdual positive cone for EME in EH (in the sense of the modular theory for von Neumann algebras) which contains Ω, with E the orthogonal projection from M to the closure of $\{a\Omega, a \in M\}$.

Let γ be an index, M_γ be a von Neumann algebra acting on a Hilbert space H, and let Ω be a vector in H which is separating for M_γ. We shall denote by H_γ the closure of $\{a\Omega, a \text{ in } M_\gamma\}$, by E_γ the orthogonal projection from H to H_γ, and with the usual notations endow with an index γ the objects of the modular theory for the action of

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$E_\gamma M_\gamma E_\gamma$ on H_γ; in particular we denote by J_γ the isometrical involution for $E_\gamma M_\gamma E_\gamma$ on H_γ which leaves C_γ invariant. If M_μ, M_ν are von Neumann algebras with M_ν contained in M_μ we shall denote by $\varepsilon_{\mu,\nu}$ the ω– (or generalized) conditional expectation (cf. [1]) from M_μ to M_ν which preserves ω_Ω. We shall denote by $F(\varepsilon_{\mu,\nu})$ the set of the fixed points under $\varepsilon_{\mu,\nu}$. If the ω–conditional expectation is a norm one projection (i.e. $F(\varepsilon_{\mu,\nu}) = M_\nu$) we shall say M_ν to be expected in M_μ with respect to $(\omega_\Omega)_\mu$.

We shall start by considering two mutually commuting (not necessarily commutative) von Neumann algebras M_1 and M_2 (i.e. for all a_1 in M_1, a_2 in M_2 we assume $a_1a_2 = a_2a_1$) and assume Ω in H to be separating for both M_1 and M_2.

Let a_1 be in M_1. Then $E_2a_1E_2$ commutes with M_2; so there is a unique operator $\lambda_{1,2}(a_1)$ such that $E_2a_1\xi = J_2\lambda_{1,2}(a_1^+)J_2\xi$ for all ξ in H_2. It is immediate to check that the mapping $\lambda_{1,2} : M_1 \mapsto M_2$ is a linear, ultraweakly continuous, completely positive contraction. If ξ, η are in C_1 we have, for all a_1 in M_1:

$$\begin{aligned}
&\omega_{\eta+i\xi}(\lambda_{1,2}(a_1)) = \langle \eta + i\xi, \lambda_{1,2}(a_1)(\eta + i\xi)\rangle \\
&= \langle \eta + i\xi, J_2E_2a_1^+J_2(\eta + i\xi)\rangle = \langle E_2a_1^+(\eta - i\xi), J_2(\eta + i\xi)\rangle \\
&= \langle E_2a_1^+(\eta - i\xi), \eta - i\xi\rangle = \langle \eta - i\xi, a_1(\eta - i\xi)\rangle \\
&= \omega_{\eta-i\xi}(a_1).
\end{aligned}$$

This proves that the mapping $\lambda_{1,2}$ is the dual mapping of an odd stochastic transition $\varrho_{2,1}$ from $(M_2)^*$ to $(M_1)^*$ (cf. [3]). If we define $\varrho_{2,1}$ symmetrically the same proof yields that $(\varrho_{2,1}, \varrho_{1,2})$ is an Ω implemented odd stochastic coupling for M_1 and M_2 as defined in [3] provided we assume the following

Condition. Let σ_i^t be the modular authomorphism group on M_i for $(\omega_\Omega)_i$ $(i = 1, 2)$. For any t real and a_1 in M_1 we have $\lambda_{1,2}(\sigma_1^t(a_1)) = \sigma_2^t(\lambda_{1,2}(a_1))$.

In the following we shall always assume, without recalling it explicitly, this intertwining condition to be satisfied for all pairs of mutually commuting algebras we consider with reference to the given vector.

It is proved in [3] that in this situation there is an antiunitary operator $\mathbf{J}$ on the Hilbert space $H_{\{1,2\}}$ spanned by $H_1 \cup H_2$ which commutes with both E_1 and E_2 and such that the action of $\mathbf{J}E_i$ on H_i coincides with the action of J_i $(i = 1, 2)$. This implies immediately the following lemma, basic for our development.

Lemma 2.1. *In the above situation* $J_1E_1E_2 = E_1J_2E_1 = E_2J_1E_1$.

Proof. $J_1E_1E_2 = \mathbf{J}E_1E_2 = E_1\mathbf{J}E_2 = E_1J_2E_2$ and symmetrically. ■

We recall also [3] for the following

Lemma 2.2. *In the above situation let* R_1 (R_2) *be the von Neumann subalgebra of* M_1 *(resp.* M_2*) generated by the range of* $\lambda_{2,1}$ *(resp.* $\lambda_{1,2}$*). Then there are norm one projections* ε_i *from* M_i *to* R_i *which preserve* $(\omega_\Omega)_i$ $(i = 1, 2)$.

Proof. Cf. Lemma 5.1, [3]. ■

3. Markovianity on triples of mutually commuting von Neumann algebras. In this section we consider a triple M_1, M_2, M_3 of mutually commuting von Neumann algebras which act on a Hilbert space H containing a vector Ω separating for the von Neumann algebra M generated by $M_1 \cup M_2 \cup M_3$. For $i,j = 1,2,3, i \neq j$, we denote by $M_{\{i,j\}}$ the von Neumann algebra generated by $M_i \cup M_j$, and endow with the index $\{i,j\}$ all the already introduced objects when referred to $M_{\{i,j\}}$. If $k = 1,2,3, k \neq i,j$, then $M_{\{i,j\}}$ commutes with M_k. We shall generalize this notation in the natural way when dealing with more than three mutually commuting von Neumann algebras.

THEOREM 3.1. *The following statements are equivalent*:
a. $\lambda_{3,\{1,2\}}(M_3)$ *is contained in* M_2.
b. $\lambda_{3,\{1,2\}} = \lambda_{3,2}$.
c. $E_{\{1,2\}}E_3 = E_2E_3$.
d. $\lambda_{3,\{1,2\}}(M_3)$ *is contained in* $F(\varepsilon_{\{1,2\},2})$.

Proof. In the following we take a_3 in M_3, and use the fact that by Lemma 2.1, $J_2E_2a_3\Omega = E_2J_3a_3\Omega$ and $E_{\{1,2\}}J_3a_3\Omega = J_{\{1,2\}}E_{\{1,2\}}a_3\Omega$.
a. ⇒ b.

$$\begin{aligned}\lambda_{3,\{1,2\}}(a_3)\Omega &= E_2\lambda_{3,\{1,2\}}(a_3)\Omega\\ &= E_2J_{\{1,2\}}E_{\{1,2\}}a_3^+\Omega = E_2E_{\{1,2\}}J_3a_3^+\Omega\\ &= E_2J_3a_3^+\Omega = J_2E_2a_3^+\Omega = \lambda_{3,2}(a_3)\Omega,\end{aligned}$$

which is b. since Ω is separating for $M_{1,2}$.
b. ⇒ c.

$$\begin{aligned}E_{\{1,2\}}J_3a_3\Omega &= J_{\{1,2\}}E_{\{1,2\}}a_3\Omega\\ &= \lambda_{3,\{1,2\}}(a_3^+)\Omega = \lambda_{3,2}(a_3^+)\Omega\\ &= J_2E_2a_3\Omega = E_2J_3a_3\Omega,\end{aligned}$$

which implies c.
c. ⇒ d.

$$\begin{aligned}\varepsilon_{\{1,2\}}(\lambda_{3,\{1,2\}}(a_3))\Omega &= J_2E_2a_3^+\Omega\\ &= E_2J_3a_3^+\Omega = E_{\{1,2\}}J_3a_3^+\Omega\\ &= J_{\{1,2\}}E_{\{1,2\}}a_3^+\Omega = \lambda_{2,\{1,2\}}(a_3)\Omega;\end{aligned}$$

so $\varepsilon_{\{1,2\},2}(\lambda_{3,\{1,2\}}(a_3)) = \lambda_{3,\{1,2\}}(a_3)$.
d. ⇒ a. Trivial. ∎

DEFINITION 3.2. We shall say Ω to be a *Markov vector* (and ω_Ω to be a *Markov state*) for M with respect to the localization (M_1, M_2, M_3) if the equivalent conditions of Theorem 3.1 are met.

Since in the abelian case $\lambda_{3,\{1,2\}}$ is nothing else than the restriction to M_3 of the $(\omega_\Omega)_M$ preserving conditional expectation from M to $M_{\{1,2\}}$ by a. in Theorem 3.1 our definition is a generalization of the classical notion of markovianity.

THEOREM 3.3. *The state* ω_Ω *is a Markov state with respect to the localization* (M_1, M_2, M_3) *iff for all* $a_1 \in M_1$, $a_3 \in M_3$ *we have*

$$\lambda_{\{1,3\},2}(a_1a_3) = \lambda_{1,2}(a_1)\lambda_{3,2}(a_3). \qquad (*)$$

If so then $\lambda_{1,2}(M_1)$ *commutes with* $\lambda_{3,2}(M_3)$.

P r o o f. We have:

$$\begin{aligned}\lambda_{\{1,3\},2}(a_1a_3)\Omega &= J_2E_2a_3^+a_1^+\Omega = j_2E_2a_1^+a_3^+\Omega\\ &= J_2E_2E_{\{1,2\}}a_1^+a_3^+\Omega = J_2E_2a_1^+E_{\{1,2\}}a_3^+\Omega\\ &= J_2E_2a_1^+J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega.\end{aligned}$$

On the other hand:

$$\lambda_{1,2}(a_1)\lambda_{3,2}(a_3)\Omega = J_2E_2a_1^+J_2\lambda_{3,2}(a_3)\Omega.$$

If Ω is markovian then by c. in Theorem 3.1:

$$\begin{aligned}J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega &= E_{\{1,2\}}a_3^+\Omega\\ &= E_2a_3^+\Omega = J_2\lambda_{3,2}(a_3)\Omega\end{aligned}$$

and $(*)$ follows.

Conversely, $(*)$ implies for all $a_i \in M_i$ $(i = 1, 2, 3)$:

$$\begin{aligned}&\langle a_1a_2\Omega, J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega\rangle = \langle a_2\Omega, E_2a_1^+J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega\rangle\\ &= \langle a_2\Omega, E_2a_1^+J_2\lambda_{3,2}(a_3)\Omega\rangle = \langle a_1a_2\Omega, J_2\lambda_{3,2}(a_3)\Omega\rangle.\end{aligned}$$

As both $J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega$ and $J_2\lambda_{3,2}(a_3)\Omega$ are in $H_{\{1,2\}}$ we get:

$$E_{\{1,2\}}a_3^+\Omega = J_{\{1,2\}}\lambda_{3,\{1,2\}}(a_3)\Omega = J_2\lambda_{3,2}(a_3)\Omega = E_2a_3^+\Omega,$$

which is c. in Theorem 3.1.

Let $(*)$ be satisfied. Then

$$\begin{aligned}&\lambda_{1,2}(a_1)\lambda_{3,2}(a_3) = \lambda_{\{1,3\},2}(a_1a_3) = \lambda_{\{1,3\},2}(a_3a_1)\\ &= \lambda_{\{1,3\},2}((a_1^+a_3^+)^+) = (\lambda_{1,2}(a_1^+)\lambda_{3,2}(a_3^+))^+ = \lambda_{3,2}(a_3)\lambda_{1,2}(a_1). \ \blacksquare\end{aligned}$$

COROLLARY 3.4 (reversibility). *The state ω_Ω is a Markov state for M with respect to the localization (M_1, M_2, M_3) iff it is a Markov state for M with respect to the localization (M_3, M_2, M_1).*

P r o o f. If Ω is Markovian for M with respect to the localization (M_1, M_2, M_3) then by Theorem 3.3

$$\begin{aligned}\lambda_{\{3,1\},2}(a_3a_1) &= \lambda_{\{1,3\},2}(a_1a_3)\\ &= \lambda_{1,2}(a_1)\lambda_{3,2}(a_3) = \lambda_{3,2}(a_3)\lambda_{1,2}(a_1).\end{aligned}$$

Now by the converse implication of Theorem 3.3 our claim follows. $\blacksquare$

4. Noncommutative Markov chains

LEMMA 4.1. *Let M_i $(i = 1, \ldots, n;\ n \geq 4)$ be mutually commuting von Neumann algebras acting on a Hilbert space H and Ω be markovian for $M_{\{1,\ldots,k\}}$ with respect to the localization $(M_{\{1,\ldots,k-2\}}, M_{k-1}, M_k)$ for all $k = 3, \ldots, n$. Then for all $a_k \in M_k$ $(k = 3, \ldots, n)$ we have*:

$$\begin{aligned}&\lambda_{\{3,4,\ldots,n\},\{1,2\}}(a_3a_4\ldots a_n)\Omega\\ &= E_2J_3a_3^+J_3E_3\ldots E_{n-2}J_{n-1}a_{n-1}^+J_{n-1}\lambda_{n,n-1}(a_n)\Omega.\end{aligned}$$

Proof. By induction. For $n = 3$ our equality is the Markov property of Ω for $M_{\{1,2,3\}}$ with respect to the localization (M_1, M_2, M_3). On the other hand by the induction hypothesis applied to $\lambda_{\{4,\ldots,n\},\{1,2,3\}}$ $(a_3a_4 \ldots a_n)$ Ω we get:

$$\begin{aligned}
&\lambda_{\{3,4,\ldots,n\},\{1,2\}}(a_3a_4 \ldots a_n)\Omega = J_{\{1,2\}}E_{\{1,2\}}a_3^+a_4^+ \ldots a_n^+\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}E_{\{1,2,3\}}a_3^+a_4^+ \ldots a_n^+\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}a_3^+E_{\{1,2,3\}}a_4^+ \ldots a_n^+\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}a_3^+J_{\{1,2,3\}}\lambda_{\{4,\ldots,n\},\{1,2,3\}}(a_4 \ldots a_n)\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}a_3^+J_{\{1,2,3\}}E_3J_4a_4^+J_4E_4 \ldots E_{n-2}J_{n-1}a_{n-1}^+J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}a_3^+E_3a_4^+J_4E_4 \ldots E_{n-2}J_{n-1}a_{n-1}^+J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= E_{\{1,2\}}J_3a_3^+E_3a_4^+J_4E_4 \ldots E_{n-2}J_{n-1}a_{n-1}^+J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= E_2J_3a_3^+J_3E_3J_4a_4^+J_4E_4 \ldots E_{n-2}J_{n-1}a_{n-1}^+J_{n-1}\lambda_{n,n-1}(a_n)\Omega. \ \blacksquare
\end{aligned}$$

PROPOSITION 4.2. *Let the hypothesis of Lemma 4.1 be satisfied. Then Ω is markovian for $M_{\{1,\ldots,n\}}$ with respect to the localization $M_{\{1,\ldots,k-1\}}, M_k, M_{\{k+1,\ldots,n\}})$ for $k = 2, \ldots, n-1$. Moreover for $j = 1, \ldots n-3$, $a_n \in M_n$ we have*:

$$\lambda_{n-j,n-j-1}(\ldots(\lambda_{n-1,n-2}(\lambda_{n,n-1}(a_n)))\ldots) = \lambda_{n,n-j-1}(a_n),$$

when j is even (chain rule) and

$$\lambda_{n-j,n-j-1}(\ldots(\lambda_{n-1,n-2}(\lambda_{n,n-1}(a_n)))\ldots)\Omega = J_{n-j-1}\lambda_{n,n-j-1}(a_n^+)\Omega$$

when j is odd.

Proof. By Lemma 4.1 we have for $a_k \in M_k$ $(k = 3, \ldots, n)$:

$$\begin{aligned}
&E_{\{1,2\}}a_3a_4 \ldots a_n\Omega = J_{\{1,2\}}\lambda_{\{3,4,\ldots,n\},\{1,2\}}(a_3^+a_4^+ \ldots a_n^+)\Omega \\
&= J_{\{1,2\}}E_2J_3a_3J_3E_3 \ldots E_{n-2}J_{n-1}a_{n-1}J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= J_{\{1,2\}}E_{\{1,2\}}J_3a_3J_3E_3 \ldots E_{n-2}J_{n-1}a_{n-1}J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= E_{\{1,2\}}a_3J_3E_3 \ldots E_{n-2}J_{n-1}a_{n-1}J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= J_2E_2J_3a_3J_3E_3 \ldots E_{n-2}J_{n-1}a_{n-1}J_{n-1}\lambda_{n,n-1}(a_n)\Omega \\
&= J_2\lambda_{\{3,4,\ldots,n\},\{1,2\}}(a_3^+a_4^+ \ldots a_n^+)\Omega = E_2a_3a_4 \ldots a_n\Omega;
\end{aligned}$$

so c. in Theorem 3.1 is satisfied for Ω and $M_{\{1,\ldots,n\}}$ with respect to the localization $(M_1, M_2,\ M_{\{3,\ldots,n\}})$.

Our hypothesis now allows us to let $M_{\{1,\ldots,k-1\}}$ play the role of M_1 above, M_k the role of M_2 and $M_{\{k+1,\ldots,n\}}$ of $M_{\{3,\ldots,n\}}$ and we get our first claim.

Let us now prove our second claim for j even. Then, applying Lemma 4.1 to the localization $(M_{\{1,\ldots,n-j-2\}}, M_{n-j-1}, M_{\{n-j,\ldots,n\}})$ and setting $a_{n-1} = a_{n-2} = \ldots = a_{n-j} = l$, we get:

$$\begin{aligned}
&\lambda_{n,n-j-1}(a_n)\Omega = \lambda_{\{n-j,\ldots,n\},\{1,\ldots,n-j-1\}}(a_n)\Omega \\
&= E_{n-j-1}E_{n-j} \ldots E_{n-2}\lambda_{n,n-1}(a_n)\Omega \\
&= J_{n-j-1}E_{n-j-1}J_{n-j}E_{n-j}J_{n-j+1} \ldots j_{n-3}E_{n-3}J_{n-2}E_{n-2}\lambda_{n,n-1}(a_n)\Omega \\
&= \lambda_{n-j,n-j-1}(\ldots(\lambda_{n-1,n-2}(\lambda_{n,n-1}(a_n)^+)^+)\ldots^+)\Omega,
\end{aligned}$$

and our claim follows since Ω is separating for M_{n-j-1}.

If j is odd we have, using the preceding case:

$$\begin{aligned}
&\lambda_{n,n-j-1}(a_n)\Omega \\
&= E_{n-j-1}J_{n-j}E_{n-j}J_{n-j+1}\ldots J_{n-3}E_{n-3}J_{n-2}E_{n-2}\lambda_{n,n-1}(a_n)\Omega \\
&= E_{n-j-1}\lambda + n - j + 1, n - j(\ldots(\lambda_{n-1,n-2}(\lambda_{n,n-1}(a_n)^+)^+)\ldots+)\Omega \\
&= J_{n-j-1}\lambda_{n-j,n-j-1}(\ldots(\lambda_{n-1,n-2}(\lambda_{n,n-1}(a_n)))\ldots)\Omega. \ \blacksquare
\end{aligned}$$

PROPOSITION 4.3. *Let M_i (i integer) be the von Neumann algebras acting on a Hilbert space H. We call $M_{[i}$ (M) the von Neumann algebra generated by the union of M_k with $k \geq i$ (by the union of all M_k). We assume the vector Ω in H to be markovian for $M_{\{1,\ldots,k\}}$ with respect to the localization $(M_{\{1,\ldots,k-2\}}, M_{k-1}, M_k)$ for all integers k. Then for all integers k Ω is markovian for M with respect to the localization $(M_{\{1,\ldots,k-2\}}, M_{k-1}, M_{[k})$.*

Proof. The projection $E_{[k}$ on H is the supremum of the projections $E_{\{k,\ldots,k+n\}}$ for n natural. So by Proposition 4.2 we have (the limit is taken in the strong operator topology):

$$\begin{aligned}
E_{\{1,\ldots,k-1\}}E_{[k} &= \lim E_{\{1,\ldots,k-1\}}E_{\{k,\ldots,k+n\}} \\
&= \lim E_{k-1}E_{\{k,\ldots,k+n\}} = E_{k-1}E_{[k},
\end{aligned}$$

and c. in Theorem 3.1 is proved for our localization. $\blacksquare$

THEOREM 4.4. *Let for all natural numbers i M_i be a von Neumann algebra acting on a Hilbert space H and Ω in H be Markovian for $M_{\{1,\ldots,k\}}$ with respect to the localization $(M_{\{1,\ldots,k-2\}}, M_{k-1}, M_k)$ for k natural. Let A, B, C be subsets of the natural numbers such that for a in A, b in B, c in C we have $a < b < c$. Then Ω is Markovian for $M_{A\cup B\cup C}$ with respect to the localization (M_A, M_B, M_C).*

Proof. Let $b = \max B$, By prop. 4.3 for all a_C in M_C we have $E_{A\cup B}a_C\Omega = E_b a_C\Omega$, which implies $E_{A\cup B}a_c\Omega = E_B a_c\Omega$ and c. in th. 3.1 is satisfied for our localization. $\blacksquare$

5. A structure theorem for markovian states

THEOREM 5.1. *Let Ω be a Markov state for M with respect to the localization (M_1, M_2, M_3). We set $M_{2,3}$ ($M_{2,1}, M_{1,2}, M_{3,2}$) to be the von Neumann subalgebra of M_2 generated by the range of $\lambda_{3,2}$ (resp. of $\lambda_{1,2}, \lambda_{2,1}, \lambda_{2,3}$), N_1 (N_2, N_3, N) the von Neumann algebra generated by $M_{2,1} \cup M_{1,2}$ (resp. $M_{2,1} \cup M_{2,3}$, $M_{2,3} \cup M_{3,2}$, $M_{2,1} \cup M_{1,2} \cup M_{2,3} \cup M_{3,2}$).*

Then N_1 and N_3 mutually commute and there are ω_Ω preserving norm one projections $\varepsilon : M \mapsto N$, $\varepsilon_1 : M_1 \mapsto M_{1,2}$, $\varepsilon_2 : M_2 \mapsto N_2$ and $\varepsilon_3 : M_3 \mapsto M_{3,2}$ such that for all a_i in M_i ($i = 1, 2, 3$)

$$\varepsilon(a_1a_2a_3) = \varepsilon_1(a_1)\varepsilon_2(a_2)\varepsilon_3(a_3).$$

Further let us denote by $\lambda^{N_{3,1}}$ ($\lambda^{N_{1,3}}$) the dual of the Ω implemented odd stochastic coupling for N_3 and N_1 (resp. for N_1 and N_3), and by $Z_{2,1}$ ($Z_{2,3}$) the center of $M_{2,1}$ ($M_{2,3}$). Then $\lambda^{N_{3,1}}(N_3) \subseteq Z_{2,1}$ and $\lambda^{N_{1,3}}(N_1) \subseteq Z_{2,3}$.

Proof. By Theorem 3.3 $M_{2,3}$ and $M_{2,1}$ mutually commute; this implies that N_1 and N_3 also mutually commute. We note also that Lemma 2.2 gives the existence of ε_1 and ε_3 as above, as well as the existence of ω_Ω preserving norm one projections $\varepsilon_{2,1}$ and $\varepsilon_{2,3}$ from M_2 to $M_{2,1}$ and to $M_{2,3}$. This implies the existence of ε_2.

We have now, if $b_1 \in M_{1,2}, b_2 \in N_2$ and $b_3 \in M_{3,2}$:

$$\begin{aligned}
&\langle b_1 b_2 b_3 \Omega, a_1 a_2 a_3 \Omega\rangle = \langle a_1^+ a_2^+ b_1 b_2 \Omega, b_3^+ a_3 \Omega\rangle \\
&= \langle J_3 \lambda_{\{1,2\},3}(a_1^+ a_2^+ b_1 b_2)\Omega, b_3^+ a_3 \Omega\rangle \\
&= \langle J_3 \lambda_{\{1,2\},3}(a_1^+ a_2^+ b_1 b_2)\Omega, \varepsilon_3(b_3^+ a_3)\Omega\rangle \\
&= \langle J_3 \lambda_{\{1,2\},3}(a_1^+ a_2^+ b_1 b_2)\Omega, b_3^+ \varepsilon_3(a_3)\Omega\rangle \\
&= \langle a_2^+ \varepsilon_3(a_3)^+ b_2 b_3 \Omega, b_1^+ a_1 \Omega\rangle = \langle a_2^+ \varepsilon_3(a_3)^+ b_2 b_3 \Omega, b_1^+ \varepsilon_1(a_1)\Omega\rangle \\
&= \langle \varepsilon_1(a_1)^+ b_1 \varepsilon_3(a_3)^+ b_3 \Omega, b_2^+ a_2 \Omega\rangle \\
&= \langle J_2 \lambda_{\{1,3\},2}(b_1^+ \varepsilon_1(a_1) b_3^+ \varepsilon_3(a_3)^+)\Omega, b_2^+ a_2 \Omega\rangle \\
&= \langle J_2 \lambda_{1,2}(b_1^+ \varepsilon_1(a_1))\lambda_{3,2}(b_3^+ \varepsilon_3(a_3)^+)\Omega, b_2^+ a_2 \Omega\rangle \\
&= \langle J_2 \lambda_{1,2}(b_1^+ \varepsilon_1(a_1))\lambda_{3,2}(b_3^+ \varepsilon_3(a_3)^+)\Omega, \varepsilon_2(b_2^+ a_2)\Omega\rangle \\
&= \langle J_2 \lambda_{1,2}(b_1^+ \varepsilon_1(a_1))\lambda_{3,2}(b_3^+ \varepsilon_3(a_3)^+)\Omega, b_2^+ \varepsilon_2(a_2)\Omega\rangle \\
&= \langle \varepsilon_1(a_1)^+ b_1 \varepsilon_3(a_3)^+ b_3 \Omega, b_2^+ \varepsilon_2(a_2)\Omega\rangle \\
&= \langle b_1 b_2 b_3 \Omega, \varepsilon_1(a_1)\varepsilon_2(a_2)\varepsilon_3(a_3)\Omega\rangle,
\end{aligned}$$

so our first claim follows.

The vector Ω is obviously markovian with respect to the localization $(N_1, M_{2,3}, M_{3,2})$ for the von Neumann algebra N; it is also markovian with respect to the localization $(M_{1,2}, M_{2,1}, M_{2,3})$ for the von Neumann algebra generated by the union of these latter algebras (it is obvious that the von Neumann algebras involved in the above triples mutually commute). It follows then by Proposition 4.2 that it is markovian with respect to the localization $(M_{1,2}, M_{2,1}, N_3)$ for the von Neumann algebra generated by their union. This implies by Theorem 3.1 a. the range of $\lambda^{N_{3,1}}$ to be contained in $M_{2,1}$. We also note that the dual of the Ω implemented odd stochastic transition for $M_{1,2}$ and $M_{2,1}$ coincides with the restriction of $\lambda_{1,2}$ to $M_{1,2}$ and that its range generates $M_{2,1}$ By the first part of this theorem the ranges of $\lambda^{N_{3,1}}$ and of this latter mapping commute; $\lambda^{N_{3,1}}(N_3) \subseteq Z_{2,1}$. Symmetrically we prove that $\lambda^{N_{1,3}}(N_1) \subseteq Z_{2,3}$. ∎

Example 5.2. Let us assume in Theorem 5.1 $\lambda_{1,2}$ and $\lambda_{3,2}$ to be surjective, $M_{2,1}$ and $M_{2,3}$ to be factors and M_2 to be generated by their union. Then Theorem 5.1 implies that $(\omega_\Omega)_M$ is a state product of its restrictions to the von Neumann subalgebras generated by the union of M_1 and $M_{2,1}$ and of $M_{2,3}$ and M_3.

Theorem 5.3. *Let Ω be a Markov state for M with respect to the localization (M_1, M_2, M_3) and σ_t be the modular authomorphism group for $(\omega_\Omega)_M$ on M. Then $\sigma_t(M_1) \subseteq M_{\{1,2\}}$ for all real t.*

Proof. We shall use the notations established in Theorem 5.1 and prove that the von Neumann algebra L_1 generated by the union of M_1 and $M_{2,1}$ is $(\omega_\Omega)_M$ expected in M. This will imply $\sigma_t(M_1) \subseteq L_1$ and therefore our claim.

Let L_3 be the von Neumann algebra generated by $M_3 \cup M_{2,3}$, and L the von Neumann algebra generated by $L_1 \cup L_3$. We prove first that L is $(\omega_\Omega)_M$ expected in M. Let $a_1, b_1 \in M_1$, $a_3, b_3 \in M_3$, $a_2 \in M_2$ and $b_2 \in N_2$. Then:

$$\begin{aligned}&\langle b_1b_2b_3\Omega, a_1a_2a_3\Omega\rangle = \langle a_1^+b_1a_3^+b_3\Omega, b_2^+a_2\Omega\rangle\\&= \langle J_2\lambda_{\{1,3\},2}(b_1^+a_1b_3^+a_3^+)\Omega, b_2^+a_2\Omega\rangle\\&= \langle J_2\lambda_{\{1,3\},2}(b_1^+a_1b_3^+a_3^+)\Omega, \varepsilon_2(b_2^+a_2)\Omega\rangle\\&= \langle J_2\lambda_{\{1,3\},2}(b_1^+a_1b_3^+a_3^+)\Omega, b_2^+\varepsilon_2(a_2)\Omega\rangle\\&= \langle a_1^+b_1a_3^+b_3\Omega, b_2^+\varepsilon_2(a_2)\Omega\rangle\\&= \langle b_1b_2b_3\Omega, a_1\varepsilon_2(a_2)a_3\Omega\rangle,\end{aligned}$$

so the required projection ε_L is obtained by setting

$$\varepsilon_L(a_1a_2a_3) = a_1\varepsilon_2(a_2)a_3$$

and extending it then by linearity and continuity to M.

Let $\lambda^{L_{3,1}}$ be the dual of the stochastic coupling for (L_3, L_1) implemented by Ω. Then $\lambda^{L_{3,1}}(L_3) \subseteq Z_{2,1}$. As $M_{2,3} \supseteq \lambda_{3,\{1,2\}}(M_{\{1,2\}})$, Ω is markovian on L with respect to the localization $(L_1, M_{2,3}, M_3)$, Ω is also markovian on the von Neumann algebra generated by the union of L_1 and $M_{2,3}$ with respect to the localization $(L_1, Z_{2,1}, M_{2,3})$. Indeed if we take a_1 in M_1, $a_{2,1}$ in $M_{2,1}$ and $a_{2,3}$ in $M_{2,3}$, and remember that $E_{2,3}E_{2,1} = E^{Z_{2,3}}E_{2,1}$; this follows from:

$$\begin{aligned}&\langle a_1a_{2,1}\Omega, a_{2,3}\Omega\rangle = \langle a_{2,1}E_{2,1}a_1\Omega, a_{2,3}\Omega\rangle\\&= \langle E_{2,1}a_{2,1}a_1\Omega, a_{2,3}\Omega\rangle = \langle E^{Z_{2,3}}E_{2,1}a_{2,1}a_1\Omega, a_{2,3}\Omega\rangle\\&= \langle E_{2,1}a_{2,1}a_1\Omega, E^{Z_{2,3}}a_{2,3}\Omega\rangle = \langle E^{Z_{2,1}}a_{2,1}a_1\Omega, a_{2,3}\Omega\rangle\\&= \langle a_{2,1}a_1\Omega, E^{Z_{2,1}}a_{2,3}\Omega\rangle.\end{aligned}$$

Now Lemma 4.1 yields that Ω is markovian on L for the localization $(L_1, Z_{1,2}, M_{2,3})$, which implies $\lambda^{L_{3,1}}(L_3)$ to be contained in $Z_{2,1}$.

Let now a_1 be in L_1, a_3 in L_3 and E^{L_1} the projection on the closure of $\{a_1\Omega : a_1 \in L_1\}$. We have:

$$\begin{aligned}&E^{L_1}a_1a_3\Omega = a_1E^{L_1}a_3\Omega\\&= a_1J_1\lambda^{L_{3,1}}(a_3^+)\Omega = a_1\lambda^{N_{3,1}}(a_3)\Omega,\end{aligned}$$

and the mapping ε^{L_1} obtained by setting $\varepsilon^{L_1}(a_1a_3) = a_1\lambda^{N_{3,1}}(a_3)$ and extending it once more to L_1 by linearity and continuity is the required $(\omega_\Omega)_L$ preserving generalized conditional expectation from L to L_1. This completes the proof. ∎

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