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L_∞-KHINTCHINE–BONAMI INEQUALITY IN FREE PROBABILITY

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Abstract. We prove the norm estimates for operator-valued functions on free groups supported on the words with fixed length ($f = \sum_{|w|=l} a_w \otimes \lambda(w)$). Next, we replace the translations by the free generators with a free family of operators and prove inequalities of the same type.

1. Introduction. The constants in the classical Khintchine inequality and in its extensions (for the products of Rademacher functions R_i) are unbounded at infinity (see [Bn], see also [H1] where the best constant for $k = 1$ was found):

$$\Big\| \sum_{i_1 \ldots i_k} a_{i_1,\ldots,i_k} R_{i_1} \ldots R_{i_k} \Big\|_{L_{2p}} \le (2p-1)^{\frac{k}{2}} \Big\| \sum_{i_1 \ldots i_k} a_{i_1,\ldots,i_k} R_{i_1} \ldots R_{i_k} \Big\|_{L_2}.$$

In the free case (for free generators, or more generally for Leinert sets) the constants are bounded at infinity. It was shown by M. Leinert in [M] that the square summable functions supported on Leinert's subsets of discrete groups are convolvers (the symbols of the convolution operators):

$$\|f\|_{VN(G)} \le \sqrt{5}\, \|f\|_{l^2(G)}.$$

In the paper [Bo1] by Bożejko the same result (with the best constant) was obtained as the limit version of Khintchine inequality:

$$\|f\|_{L_{2p}} \le (C_p)^{\frac{1}{2p}} \|f\|_{L_2}$$

where $\|f\|_{L_{2p}}^{2p} = (f * \tilde{f})^{*p}(e)$, $\tilde{f}\,(g) = \overline{f\,(g^{-1})}$ and $C_p = \binom{2p}{p}\frac{1}{p+1}$ are the Catalan numbers.

The same kind of norm equivalence (with bounded constant) holds for the functions supported on words with fixed length (on free groups). The following estimate was proved

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by U. Haagerup in [H2]:

$$\|f\|_{VN(\mathbb{F})} \le (l+1)\,\|f\|_{l^2(\mathbb{F})}$$

where supp $f \subset \{w \in \mathbb{F} : |w| = l\}$ and $\mathbb{F}$ is a free group.

In the paper [HP] some estimate for operator-valued functions was shown. For a function $f : \mathbb{F} \to B(H)$ supported on the words of length one with the values in the algebra of bounded operators on a Hilbert space H Haagerup and Pisier obtained:

$$\|f\|_{B(H)\otimes VN(\mathbb{F})} \le 2\max\Big\{\Big\| \sum_{g\in supp\ f} f(g)^* f(g)\Big\|^{1/2}, \Big\| \sum_{g\in supp\ f} f(g)\, f(g)^*\Big\|^{1/2}\Big\}. \tag{1}$$

In the same paper the above estimate was extended to the case of a family of operators G_i which is free in the Voiculescu's sense (for the notion of freeness see [V] or [VDN]):

$$\Big\|\sum_i a_i \otimes G_i\Big\| \le 2\max\Big\{\Big\|\sum_i a_i^* a_i\Big\|^{1/2}, \Big\|\sum_i a_i a_i^*\Big\|^{1/2}\Big\} \tag{2}$$

where a_i are elements of $B(H)$ and G_i form the free family. See also [BSp] for another generalization of (2).

The two above estimates are extended in this note. In Section 2 we show the following inequality for operator-valued functions supported on the words of fixed length:

$$\Big\|\sum_{|w|=l} a_w \otimes \lambda(w)\Big\| \le (l+1)\max\{\|\,(a_{pq})\,\|_{X_k} \ : \ k \in \{0,\dots,l\}\}\,,$$

where p, q are the words of the length $l-k$ and k respectively, $a_{\underline{i}} \in B(H)$, $(a_{pq}) : \oplus_{\{q\in\mathbb{F}:|q|=l-k\}} H \to \oplus_{\{p\in\mathbb{F}:|p|=k\}} H$, and $a_{pq} = 0$ if $|pq| \ne l$.

In the last section the same inequality was shown for the products of operators G_i which form a free family with respect to the state $\phi(\cdot) = \langle \cdot\xi|\xi\rangle$:

$$\Big\|\sum_{|\underline{i}|=l} a_{\underline{i}} \otimes G_{\underline{i}}\Big\| \le \Delta^{\max\{0,l-1\}}(l+1)\max\{\|(a_{\underline{pq}})\|_{X_k} \ : \ k\in\{0,\dots,l\}\}, \tag{3}$$

where $\Delta = \max_i\{\|G_i\xi\|, \|G_i^*\xi\|\}$, $G_{\underline{i}} = G_{i_1}\dots G_{i_l}$, $a_{\underline{i}} \in B(H)$ and $a_{\underline{pq}} = 0$ if $p_k = q_1$. In the scalar case the inequality (3) was obtained by M. Bożejko in [Bo2].

2. The free groups' case. In this section $E_l(G)$ denotes the subset of words of length l of a free group G (considered with a fixed family of free generators $g_1,\dots,g_N$). For $w \in E_l(G)$ we denote by w_i the i-th letter in the reduced word w, i.e.

$$w = w_1 \dots w_l \text{ , where } w_i \ \in \ \{g_1,\dots,g_N, g_1^{-1},\dots,g_N^{-1}\},$$

and by ${}_{k|}w$ ($w_{k|}$) we denote the product of the first k-letters (last $l-k$-letters):

$${}_{k|}w = \begin{cases} w & \text{for } k > l,\\ \prod_{i=1}^k w_i & \text{for } l \ge k > 0,\\ e & \text{for } k \le 0,\end{cases} \qquad w_{k|} = \begin{cases} w & \text{for } k < 0,\\ \prod_{i=k+1}^l w_i & \text{for } 0 \le k < l,\\ e & \text{for } k \ge l.\end{cases}$$

First we recall a decomposition of the left translation operator by a free generator g (see the proof of Proposition 1.1 in [HP]):

$$\lambda(g) = P_g\lambda(g) \ + \ \lambda(g)P_{g^{-1}}. \tag{4}$$

Here and in what follows P_w denotes the orthogonal projection of $l^2(G)$ onto the closed span of $\{\delta_{wg} \in l^2(G) \ : \ |wg| = |w| + |g|\}$ ($P_e = Id_{l^2(G)}$). In an obvious way (4) gives a decomposition for any word:

$$\lambda(w) = \sum_{k=0}^{|w|} P_{{}_{k|}w}\lambda(w)P_{w_{k|}^{-1}}. \tag{5}$$

From now on we write $T_k(w)$ instead of $P_{{}_{k|}w}\lambda(w)P_{w_{k|}^{-1}}$. For $w, v \in E_1(G)$ it is easy to verify that:

$$T_0(w)T_0(v)^* = \delta_{w,v}\,(Id \ - \ P_w)\,, \qquad T_1(w)^*T_1(v) = \delta_{w,v}\,(Id \ - \ P_{w^{-1}})\,.$$

The above observation in an elementary way implies the following lemma:

LEMMA 1. *Let $|w| = |v|$ then*:

i) $T_k(w)T_k(v)^* = \delta_{w_{k|},v_{k|}}T_k({}_{k|}w)(Id \ - \ P_{w_{k+1}})T_k({}_{k|}v)^*$,

ii) $T_k(w)^*T_k(v) = \delta_{{}_{k|}w,{}_{k|}v}T_0(w_{k|})^*(Id \ - \ P_{w_k^{-1}})T_0(v_{k|})$.

In the case when $|w| = |v|$ and $w_{k|} \ \neq \ v_{k|}$ (resp. ${}_{k|}w \ \neq \ {}_{k|}v$), the operators $T_k(w)$ and $T_k(v)$ have orthogonal domains (resp. images), thus we get:

$$\sum_{w\in E_l(G)} T_k(w) = (\mathcal{X}_l(pq)T_k(pq))_{(p,q)\in E_k(G)\times E_{l-k}(G)}\,, \tag{6}$$

where $\mathcal{X}_l$ is the characteristic function of the set $E_l(G)$.

Our aim is to estimate the norm of the following operator:

$$\sum_{w\in E_l(G)} a_w \otimes \lambda(w),$$

where $a_w \in B(H)$ (H - a Hilbert space). For the upper estimate it is enough to compute the norm of the operator:

$$(a_{pq} \otimes T_k(pq))_{(p,q)\in E_k(G)\times E_{l-k}(G)}\,.$$

Here, of course, $a_{pq} \ = 0$ if $|pq| < l$. By Lemma 1 we have:

$$\begin{aligned}
&\|\,(a_{pq}\otimes T_k(pq)) \times (a_{pq}\otimes T_k(pq))^*\,\| = \\
&= \ \|\,(a_{pq}\otimes T_k(p)(Id - P_{q_1})) \times (a_{pq}\otimes T_k(p)(Id - P_{q_1}))^*\,\| \\
&= \ \|\,(a_{pq}\otimes T_k(p)(Id - P_{q_1}))^* \times (a_{pq}\otimes T_k(p)(Id - P_{q_1}))\,\| \\
&= \ \|\,(a_{pq}\otimes (Id - P_{p_k})(Id - P_{q_1}))\,\|^2 \leq \|\,(a_{pq})\,\|^2.
\end{aligned}$$

In order to prove the opposite inequality we observe that for any vector of the form $\oplus_{q\in E_{l-k}(G)}v_q$ we have:

$$\begin{aligned}
\|\oplus v_q\|_{\oplus_{q\in E_{l-k}(G)}H} &= \|\oplus (v_q\otimes\delta_{q^{-1}})\|_{\oplus_{q\in E_{l-k}(G)}H\otimes_2 l^2(G)}, \\
\|\,(a_{pq})\oplus v_q\| &= \|\,(a_{pq}\otimes T_k(pq))\oplus(v_q\otimes\delta_{q^{-1}})\|.
\end{aligned} \tag{7}$$

The above considerations prove the second statement of the following theorem:

THEOREM 2. *Let G be a free group generated by the set $\{g_1,\ldots,g_N\}$ of free generators, and a_w ($w \in E_l(G)$) be a family of bounded operators on a Hilbert space. Then the following statements hold*:

i) $\| \sum_{w\in E_l(G)} a_w \otimes \lambda(w)\| \geq \max\{\|\,(a_{pq})\,\|_{X_k} \ : \ k \in \{0,\dots,l\}\}$,
ii) $\| \sum_{w\in E_l(G)} a_w \otimes \lambda(w)\| \leq (l+1)\max\{\|\,(a_{pq})\,\|_{X_k} \ : \ k \in \{0,\dots,l\}\}$,

where $(a_{pq}) \ : \ \oplus_{q\in E_{l-k}(G)}H \ \to \ \oplus_{p\in E_k(G)}H$ *and* $a_{pq}=0$ *if* $pq \notin E_l(G)$.

The statement (i) can be proved similarly as in (7) but instead of $\oplus(v_q\otimes\delta_{q^{-1}})$ we take $v=\sum v_q\otimes\delta_{q^{-1}}$ and observe that $a_w\otimes T_m(w)v \perp a_w\otimes T_n(w)v$ if $m\neq n$.

3. The case of the reduced free product of $B(H)$'s. We use the following notations throughout this section. (H_i,ξ_i) denotes a Hilbert space H_i with a distinguished normalized vector ξ_i. For any space H_i we consider a bounded operator $g_i \in B(H_i)$ with the property: $\langle g_i\xi_i|\xi_i\rangle = 0$ and we denote by G_i the extension of g_i onto the free product $*(H_i,\xi_i)$. By $\underline{i}$ we denote the sequence $(i_1,\dots,i_n)$ with the property $i_1\neq i_2\neq\dots\neq i_n$ and by ${}_{k|}\underline{i}$, $\underline{i}_{k|}$ the restrictions of $\underline{i}$ to the $(i_1,\dots,i_k)$ and $(i_{k+1},\dots,i_n)$ respectively.

As in the previous section we first recall a decomposition of the operators G_i (see proof of Proposition 4.9 in [HP]):

$$G_i = e_iG_i(1-e_i) \ + \ G_ie_i, \tag{8}$$

where e_i denotes the orthogonal projection of $*(H_i,\xi_i)$ onto the space

$$\mathbb{C}\xi \oplus \oplus_{n=1}^{\infty} \oplus_{i=i_1\neq i_2\neq\dots\neq i_n} H_{i_1}^{\circ}\otimes_2\dots\otimes_2 H_{i_n}^{\circ}.$$

This gives a decomposition for the product of G_i (under the assumption that $i_1\neq i_2\neq\dots\neq i_n$):

$$G_{i_1}G_{i_2}\dots G_{i_n} = \sum_{k=0}^{n}\prod_{p=1}^{k} G_{i_p}(1-e_{i_p})\prod_{q=k+1}^{n} G_{i_q}e_{i_q}. \tag{9)}$$

From now on the operators which arise under the sum on the right hand side we denote by $S_k(\underline{i})$.

The S_k's have properties analogous to (i) and (ii) from Lemma 1:

LEMMA 3. *Let* $\underline{i}$ *and* $\underline{j}$ *have the same length. Then the following statements hold*:

i) $S_k(\underline{i})S_k(\underline{j})^* = \delta_{\underline{i}_{k|},\underline{j}_{k|}} S_k({}_{k+1|}\underline{i}) \left(\prod_{p=k+2}^{n}\langle G_{i_p}G_{i_p}^*\xi|\xi\rangle\right) S_k({}_{k+1|}\underline{j})^*$,
ii) $S_k(\underline{i})^*S_k(\underline{j}) = \delta_{{}_{k|}\underline{i},{}_{k|}\underline{j}} S_k(\underline{i}_{k|})^* \left(\prod_{p=1}^{k}\langle G_{i_p}^*G_{i_p}\xi|\xi\rangle\right) S_k(\underline{j}_{k|})$.

Hence S_k's have the same orthogonality properties as T_k's. This gives the equality:

$$\sum_{|\underline{i}|=n} S_k(\underline{i}) = \big((1-\delta_{p_k,q_1})S_k(\underline{pq})\big)_{(\underline{p},\underline{q})}, \tag{10}$$

where $\big((1-\delta_{p_k,q_1})S_k(\underline{pq})\big) : \oplus_{|\underline{q}|=n-k} *(H_i,\xi_i) \to \oplus_{|\underline{p}|=k} *(H_i,\xi_i)$. In the same way as in the previous section we obtain:

$$\|(a_{\underline{pq}}\otimes S_k(\underline{pq}))\| = \|(Id\otimes E_{\underline{p},\underline{p}})\times(a_{\underline{pq}}\otimes Id)\times(Id\otimes F_{\underline{q},\underline{q}})\|,$$

where the first and last matrices on the right hand side are diagonal of order $|\underline{p}|=k$ and $|\underline{q}|=n-k$ respectively and such that:

$$E_{\underline{p},\underline{p}} = (1-e_{p_k})\prod_{m=1}^{k}\|G_{p_m}\xi\|, \qquad F_{\underline{q},\underline{q}} = S_0(q_1)\prod_{m=2}^{l-k}\|G_{q_m}^*\xi\|.$$

Finally we obtain the inequality:

$$\|(a_{\underline{pq}} \otimes S_k(\underline{pq}))\| \le \max_{\underline{p}} \Big\{ \prod_{m=1}^{k} \|G_{p_m}\xi\| \Big\} \max_{\underline{q}} \Big\{ \|G_{q_1}\| \prod_{m=2}^{l-k} \|G^*_{q_m}\xi\| \Big\} \|(a_{\underline{pq}})\|.$$

Now the second statement of the following theorem is clear:

THEOREM 4. *Let G_i $(i \in \{1,\ldots,m\})$ be as above. Then under the assumption that G_i are contractions we have*:

i) $\|\sum_{|\underline{i}|=n} a_{\underline{i}} \otimes G_{\underline{i}}\| \ge \delta^n \max\{\|(a_{\underline{pq}})\|_{X_k} : k \in \{0,\ldots,n\}\}$,

ii) $\|\sum_{|\underline{i}|=n} a_{\underline{i}} \otimes G_{\underline{i}}\| \le \Delta^{\max\{0,n-1\}}(n+1) \max\{\|(a_{\underline{pq}})\|_{X_k} : k \in \{0,\ldots,n\}\}$,

where $\delta = \min_i \{\|G_i\xi\|, \|G_i^*\xi\|\}$, $\Delta = \max_i \{\|G_i\xi\|, \|G_i^*\xi\|\}$, $G_{\underline{i}} = G_{i_1}\ldots G_{i_n}$ *and* $a_{\underline{pq}} = 0$ *if* $p_k = q_1$.

The first statement can be verified by substitutions of vectors (as in the proof of Theorem 2 (i)).

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