KNOT THEORY
BANACH CENTER PUBLICATIONS, VOLUME 42
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

THE p_1-CENTRAL EXTENSION OF THE MAPPING CLASS GROUP OF ORIENTABLE SURFACES

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Abstract. Topological Quantum Field Theories are closely related to representations of Mapping Class Groups of surfaces. Considering the case of the TQFTs derived from the Kauffman bracket, we describe the central extension coming from this representation, which is just a projective extension.

1. Introduction. A Topological Quantum Field Theory (TQFT) is a way of extending an invariant $< >$ defined on oriented closed 3-manifolds to manifolds with boundary. It consists of a functor on a cobordism category: to a surface Σ we associate a module $V(\Sigma)$ and to a cobordism† M from Σ_1 to Σ_2, we associate a linear map Z_M from $V(\Sigma_1)$ to $V(\Sigma_2)$. The reader can refer to Atiyah [A1] for details.

The TQFT-functors are related to representations of mapping class group of surfaces in the following way. Let Σ be an oriented closed surface, Γ_Σ its mapping class group, that is to say the group of isotopy classes of orientation preserving diffeomorphisms of Σ. If f is an element of Γ_Σ, then its mapping cylinder C_f can be seen as a cobordism from Σ to Σ. So we obtain an endomorphism Z_{C_f} of $V(\Sigma)$ and we get a representation of Γ_Σ. Generaly, this representation is just projective (because of what is called the framing anomaly) and so, linearizing this representation, one obtains a central extension of Γ_Σ. Masbaum and Roberts describe some of these extensions in [M-R].

The aim of this note is to study the central extension arising from the TQFT-functors constructed in [BHMV3] from the Kauffman bracket. First, we will recall some facts about this TQFT.

1991 *Mathematics Subject Classification*: Primary 57M25; Secondary 57N10, 20F05.

The paper is in final form and no version of it will be published elsewhere.

† Since this functor must satisfy certain properties, one has to define carefully the cobordism category. So, '*manifold*' means manifold possibly equipped with structure or provided with a banded link.

2. TQFT derived from the Kauffman bracket. The purpose of this section is not to go into the details of the construction, but just to try to explain why we consider p_1-structures.

Considering an appropriate renormalization of the invariant θ_p defined in [BHMV1,2] (invariant of a closed 3-manifold with banded link) and using the universal construction, Blanchet, Habegger, Masbaum and Vogel constructed in [BHMV3] a family (V_p) of TQFT-functors. These can be defined in the following way: let $\mathcal{V}_p(\Sigma)$ be the free module generated by $\{M \,/\, M \textit{ is a } 3\textit{-manifold such that } \partial M = \Sigma\}$, and let $<\,,\,>_\Sigma$ be the bilinear form on $\mathcal{V}_p(\Sigma)$ defined by $< M_1, M_2 >_\Sigma = < M_1 \cup_\Sigma (-M_2) >_p$ (where $<\,>_p$ denotes the renormalized invariant). Each cobordism M from Σ_1 to Σ_2 induces a linear map Z_M from $\mathcal{V}_p(\Sigma_1)$ to $\mathcal{V}_p(\Sigma_2)$ defined by $Z_M(M_1) = M_1 \cup_{\Sigma_1} M$. If $V_p(\Sigma)$ is defined to be $\mathcal{V}_p(\Sigma)$ divided by the left kernel of $<\,,\,>_\Sigma$, then Z_M induces a linear map Z_M from $V_p(\Sigma_1)$ to $V_p(\Sigma_2)$. With these definitions, the authors show that V_p satisfies the TQFT axioms. Furthermore, since the invariants θ_p come from the Kauffman bracket, the functors V_p satisfy the Kauffman relation, that is to say, for all 3-manifolds M, there is a linear map $\mathcal{K}(M) \to V_p(\partial M)$ which associates to each link L in M (modulo the Kauffman relations) the class of (M, L) $\big(\mathcal{K}(M)$ is the Kauffman module of $M\big)$.

Since we want to compute the modules $V_p(\Sigma)$, we ask the invariants to satisfy surgery axioms (see [BHMV3]). The main one is the index two surgery axiom, which can be stated as follows: there is a linear combination $\omega = \sum \lambda_i L_i$ of banded links in the solid torus $-(S^1 \times D^2)$ such that, for any closed 3-manifold M and any banded link L in M, one has $< M(L) > = < \big(M, L(\omega)\big) >$, where $M(L)$ is the 3-manifold obtained from M by surgery on L and $L(\omega)$ is the linear combination of banded links in M obtained by inserting a copy of ω in a neighborhood of each component of L.

Now, if $\mathcal{U}_\varepsilon$ is the unknot with framing ε in the 3-sphere S^3, one can see, using the Kauffman relations, that $< S^3, \mathcal{U}_\varepsilon(\omega) > = < S^3 > \{\mathcal{U}_\varepsilon(\omega)\}$ where $\{\,\}$ denote the Kauffman bracket. Thus, the index two surgery axiom implies that $< S^3(\mathcal{U}_\varepsilon) > = < S^3 > \{\mathcal{U}_\varepsilon(\omega)\}$. But $S^3(\mathcal{U}_\varepsilon)$ is diffeomorphic to S^3 and computations of [BHMV1] show that $\{\mathcal{U}_1(\omega)\}$ and $\{\mathcal{U}_{-1}(\omega)\}$ cannot be both equal to 1 (this problem is the so-called *framing anomaly*). Thus, since $\{\mathcal{U}_1(\omega)\} = \{\mathcal{U}_{-1}(\omega)\}^{-1} (= \mu)$, doing surgery on $\mathcal{U}_\varepsilon(\omega)$ multiplies the invariant by μ^ε. But under this surgery, $S^3(\mathcal{U}_\varepsilon)$ is the boundary of $\mathbf{CP}^2 \setminus D^4$ and ε is precisely the signature of this 4-manifold. Therefore, we see that $<\,>$ depends on the signature of the trace of the surgery. So, we shall consider an additional structure on manifolds such that doing a surgery modifies the structure and makes the invariant independent of the signature. Hirzebruch's signature theorem leads us to consider p_1-structure on manifolds (see [BHMV3]).

3. p_1-structure. Let ξ be a real oriented vector bundle over a CW-complex B and denote by $\xi_{\mathbf{C}}$ its complexification. The first obstruction to trivialise a complex vector bundle is its first Chern class. Since $\xi_{\mathbf{C}}$ is the complexification of a real oriented vector bundle, one has $c_1(\xi_{\mathbf{C}}) = 0$. Thus, the first obstruction we meet to trivialise $\xi_{\mathbf{C}}$ is its second Chern class, which is nothing but $p_1(\xi)$, the first Pontryagin class of ξ. This leads us to give the following definition.

Definition 1. A p_1-structure on ξ is a trivialisation of the stabilisation of $\xi_{\mathbf{C}}$ over the 3-skeleton of B which extends to the 4-skeleton of B.

If B' is a subcomplex of B and ξ' is the restriction of ξ to B', a p_1-structure on ξ induces one on ξ' by restriction. Conversely, if α is a p_1-structure on ξ', we ask if it can be extended to ξ. The machinery of obstruction theory (see [St], §32]) proves the following.

Proposition 1. *There exists a cohomology class $p_1(B,\alpha) \in H^4(B,B';\mathbf{Z})$ such that α extends to ξ if and only if $p_1(B,\alpha)=0$.*

Remark 1. When B' is empty, $p_1(B,\alpha)$ is equal to the first Pontryagin class $p_1(\xi)$.

Now, let α_0 and α_1 be two p_1-structure on ξ which coïncide with a given p_1-structure φ on ξ'.

Definition 2. A homotopy rel ξ' between α_0 and α_1 is a p_1-structure on the product bundle $\xi \times I$ which coincides with α_0 on $\xi \times \{0\}$, with α_1 on $\xi \times \{1\}$ and with φ on $\xi' \times \{t\}$ for all $t \in I$.

Considering the difference cochain given by obstruction theory (see [St], §33), one gets:

Proposition 2. *The set of homotopy classes rel ξ' of p_1-structure on ξ is affinely isomorphic to $H^3(B,B';\mathbf{Z})$.*

Now, let M be a compact oriented manifold and define a p_1-structure on M to be a p_1-structure on its tangent bundle. Suppose that N is a submanifold of ∂M. Choosing the normal vector of ∂M to be outward, one can see $\tau_M = \tau_N \oplus \varepsilon$. Thus, a p_1-structure on M induces one on N by restriction. In this situation, the preceeding result gives the following.

Corollary 3. (i) *If M is a compact oriented manifold of dimension 1 or 2, there is a unique p_1-structure on M up to homotopy.*

(ii) *If M is a compact oriented manifold of dimension 3, the set of homotopy classes rel ∂M of p_1-structure on M is affinely isomorphic to* $\mathbf{Z}$.

Remark 1. The definition of p_1-structure given in [BHMV3] and [G2] is not the same as here. In dimension less than or equal to 4, it is equivalent to ours. But in higher dimensions, the notion of p_1-structure introduced in [BHMV3] and [G2] is not canonical. To explain this, let us recall briefly the definition of p_1-structure given in [BHMV3] and [G2].

Denote by X_{p_1} the homotopy fiber of the map $\tilde{p}_1 : BSO \to K(\mathbf{Z},4)$ corresponding to the first Pontryagin class of the universal stable bundle γ_{SO} over BSO and let γ_X be the pullback of γ_{SO} to X_{p_1}. A p_1-structure on an oriented manifold M is a bundle morphism from the stable tangent bundle of M to γ_X which is an orientation preserving linear isomorphism on each fiber. One can see that this definition depends in the general case on the choice of the map $\tilde{p}_1$. More precisely, the dependence comes from an action of $\beta\big(w_2(M)\big) \in H^3(M;\mathbf{Z})$ on the set of homotopy classes of p_1-structure on M, where β is the Bockstein homomorphism and $w_2(M)$ the second Stiefel-Whitney class of M.

When the dimension of M is less than or equal to 4, one has $\beta\big(w_2(M)\big)=0$. This is why the two definitions are equivalent in this case.

4. The Mapping Class Group with p_1-structure: definition. First, let us look at the induced projective representation of Γ_Σ in the case of the TQFT above. Consider the genus g Heegaard splitting $S^3 = H \cup_\Sigma H'$. Then, since the functors V_p are cobordism generated and satisfy the Kauffman and surgery axioms, $V_p(\Sigma)$ is isomorphic to the left kernel of the bilinear form $\{\ ,\ \}$ induced on $\mathcal{K}(H) \times \mathcal{K}(H')$ by the Kauffman bracket. With this point of view, the projective action of Γ_Σ on $V_p(\Sigma)$ can be seen in the following way. If f is a diffeomorphism of Σ which extends to H, then f induces an endomorphism of $\mathcal{K}(H)$ which descends to $V_p(\Sigma)$. If f extends to H', we get the action by considering the adjoint of the endomorphism induced on $\mathcal{K}(H')$.

Now, let us suppose that Σ is the torus $S^1 \times S^1$ and a (resp. b) the Dehn twist along the curves $S^1 \times \{1\}$ (resp. $\{1\} \times S^1$). It is well known that Γ_Σ is generated by a and b, with the two relations $aba = bab$ and $(aba)^4 = Id$. Denote by $\tilde{a}$ and $\tilde{b}$ the linear transformations of $V_p(S^1 \times S^1)$ induced by a and b as described above. Then, using methods of [BHMV1], one can check that these two endomorphisms satisfy the following relations (see [G1]):

$$\tilde{a}\tilde{b}\tilde{a} = \tilde{b}\tilde{a}\tilde{b} \qquad \text{and} \qquad (\tilde{a}\tilde{b}\tilde{a})^4 = \lambda Id$$

where λ is a scalar different from 1. Thus, the action of $\Gamma_{S^1 \times S^1}$ is not linear, but just projective (this is another way to see the framing anomaly). In order to linearize this action, and following what we have seen in the second section to solve the framing anomaly, we will provide the mapping cylinder C_f of an element f of Γ_Σ with a p_1-structure. The precise definition is the following.

Let Σ be an oriented, connected, closed surface and let φ be a given p_1-structure on Σ. For $f \in \Gamma_\Sigma$, we provide ∂C_f with the p_1-structure φ. This one can be extended to C_f, and P_f, the set of homotopy classes rel ∂C_f of such extensions, is affinely isomorphic to $\mathbf{Z}$ (corollary 3).

Definition 3. The mapping class group with p_1-structure, denoted by $\widetilde{\Gamma}_\Sigma$, is the set of all pairs (f, α) where $f \in \Gamma_\Sigma$ and $\alpha \in P_f$, together with the obvious composition.

Remark 1. Atiyah ([A2]) has previously defined this group in a different way.

Remark 2. Up to canonical isomorphism, this group does not depend on the choice of φ: if ψ is another p_1-structure on Σ, the isomorphism is given by the conjugation by $\Sigma \times I$ equipped with a p_1-structure which realizes a homotopy between φ and ψ.

The forgetful map μ is an epimorphism from $\widetilde{\Gamma}_\Sigma$ to Γ_Σ which defines a central extension of Γ_Σ by $\mathbf{Z}$. Since an element of $V_p(\Sigma)$ can be represented by a 3-manifold M provided with a p_1-structure and with boundary Σ, we have a linear action of $\widetilde{\Gamma}_\Sigma$ by gluing $\Sigma \times I$ along $\Sigma \times \{0\}$ to M. Thus, the problem of linearizing the action of Γ_Σ is solved.

5. Presentation of $\widetilde{\Gamma}_\Sigma$. Now, let us give a presentation of this extended group. It is well known that Γ_Σ is generated by Dehn twists. So, we shall construct a canonical lifting $\widetilde{\tau_\alpha} = (\tau_\alpha, A)$ of τ_α, the twist along a simple closed curve (s.c.c.) α on Σ. To do this, we have to define the p_1-structure A on C_{τ_α}. Consider a neighborhood V of α in Σ and define A outside $V \times I$ to be equal to φ. Then it remains to extend it on $V \times I$. But $V \times I$ is diffeomorphic to $S^1 \times I \times I$ and so, we want to extend a given p_1-structure on $\partial(S^1 \times I \times I)$. The corollary 3 tells us that the set of such extensions is parametrized by $\mathbf{Z}$. We will take the one which extends to $D^2 \times I \times I$. More precisely, note that φ, which is by restriction a p_1-structure on $V \approx S^1 \times I$, can be extended to $D^2 \times I$ in a unique way up to homotopy by proposition 2. The twist τ_α, which can be seen as a diffeomorphism of V, extends to $D^2 \times I$. By proposition 2, there is a unique p_1-structure $\mathcal{A}$ on $D^2 \times I \times I$ up to homotopy such that $\mathcal{A}_{|D^2\times\partial I\times I} = \varphi$, $\mathcal{A}_{|D^2\times I\times 0} = \varphi$ and $\mathcal{A}_{|D^2\times I\times 1} = \varphi \circ \tau_\alpha$. We define A on $V \times I$ to be the restriction of $\mathcal{A}$.

Now, if $\mathcal{C}$ is a set of s.c.c. in Σ such that $\{\tau_\alpha \,/\, \alpha \in \mathcal{C}\}$ generates Γ_Σ, then the set $\{\widetilde{\tau_\alpha} \,/\, \alpha \in \mathcal{C}\}$ together with a generator u of $\ker \mu$ generate $\widetilde{\Gamma}_\Sigma$. Let us look at the relations.

Since the extension is central, u commutes with all the $\widetilde{\tau_\alpha}$. To obtain the other relations, one just needs to lift the relations of Γ_Σ. These are the following.

The braid relations. It is well known that if α is a s.c.c. in Σ and h is a diffeomorphism of Σ, then $\tau_{h(\alpha)} = h\tau_\alpha h^{-1}$. More specifically, if h is a twist τ_β, one has the relation (called a braid relation)

$$\tau_\gamma = \tau_\beta \tau_\alpha \tau_\beta^{-1} \quad (T)$$

where $\gamma = \tau_\beta(\alpha)$.

The lantern relations. Let us consider a subsurface of Σ which is homeomorphic to a disc with three holes. Let α_1, α_2, α_3 and α_4 be the boundary components and β_1, β_2 and β_3 curves as shown in figure 1. The relation (called a lantern relation) is:

$$\tau_{\alpha_1}\tau_{\alpha_2}\tau_{\alpha_3}\tau_{\alpha_4} = \tau_{\beta_3}\tau_{\beta_2}\tau_{\beta_1} \quad (L).$$

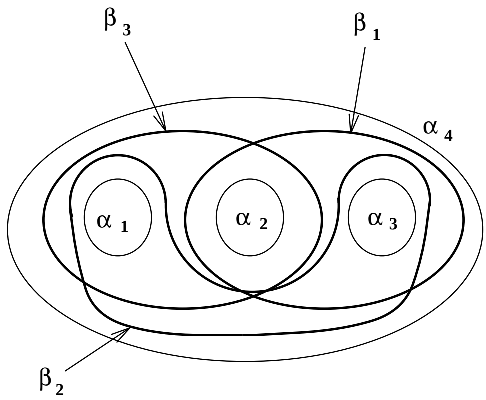

Fig. 1

The chain relations. We consider a subsurface of Σ which is homeomorphic to a surface of genus one with two boundary components. The relation is

$$(\tau_{\alpha_1}\tau_\beta\tau_{\alpha_2})^4 = \tau_{\delta_1}\tau_{\delta_2} \quad (C)$$

where the curves are described in figure 2.

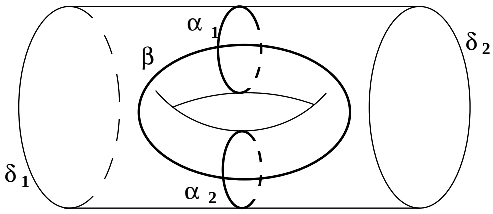

Fig. 2

Lifting these relations, we obtain a presentation of $\widetilde{\Gamma}_\Sigma$.

Notations. If $m \in \Gamma_\Sigma$ is a word $\tau_{\alpha_1}\cdots\tau_{\alpha_n}$, we will denote by $\tilde{m}$ the word $\widetilde{\tau_{\alpha_1}}\cdots\widetilde{\tau_{\alpha_n}} \in \widetilde{\Gamma}_\Sigma$.

THEOREM ([G2]). *For any oriented surface* Σ, $\widetilde{\Gamma}_\Sigma$ *is generated by the set of all* $\widetilde{\tau_\alpha}$ (*where* α *is a simple closed curve on* Σ) *together with* u, *and is defined by the relations* (I), (II), (III) *and* (IV) *below*:

(I) *for any s.c.c.* α, $\widetilde{\tau_\alpha}\, u = u\, \widetilde{\tau_\alpha}$,

(II) $\tilde{T} = 1$ *for all the braids* T *between the curves* α *and* β *such that* α *and* β *intersect in zero*, *one or two points with opposite signs*,

(III) $\tilde{C} = u^{12}$ *for all chains* C,

(IV) $\tilde{L} = 1$ *for all lanterns* L.

Remark 1. One can obtain a finite presentation of $\widetilde{\Gamma}_\Sigma$ using the presentation of Γ_Σ given by Wajnryb in [W].

Remark 2. This presentation allows us to extend $\tau_\alpha \mapsto \widetilde{\tau_\alpha}$ to a section $s : \Gamma_\Sigma \to \widetilde{\Gamma}_\Sigma$ and to compute the associated cocycle. This is equal to 12 times the generator of $H^2(\Gamma_\Sigma; \mathbf{Z})$ (see also [A2]).

Acknowledgements. I want to thank C. Blanchet and P. Vogel for many useful discussions. I would like to thank also the Stefan Banach Center and the organizers of the mini-semester for their hospitality.

References

[A1] M. Atiyah, *Topological quantum field theories*, Publ. Math. IHES 68 (1989), 175-186 .

[A2] M. Atiyah, *On framings of 3-manifolds*, Topology 29 (1990), 1-7.

[BHMV1] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Three-manifold invariants derived from the Kauffman bracket*, Topology 31 (1992), 685-699.

[BHMV2] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Remarks on the Three-manifold Invariants* θ_p, in 'Operator Algebras, Mathematical Physics, and Low

Dimensional Topology' (NATO Workshop July 1991) Edited by R. Herman and B. Tanbay, Research Notes in Mathematics Vol 5, 39-59.

[BHMV3] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological Quantum Field Theories derived from the Kauffman bracket*, Topology 34 (1995), 883-927.

[G1] S. Gervais, *Etude de certaines extensions centrales du "mapping class group" des surfaces orientables*, thèse, Université de Nantes, 1994.

[G2] S. Gervais, *Presentation and central extensions of Mapping Class Groups*, Trans. of Amer. Math. Soc. 348 (1996), 3097-3132.

[M-R] G. Masbaum and J. Roberts, *On Central Extensions of Mapping Class Groups*, Math. Ann. 302 (1995), 131-150.

[St] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951.

[W] B. Wajnryb, *A simple presentation for the Mapping Class Group of an orientable surface*, Israel Journal of Math. 45 (1983), 157-174 .