

TRACE FUNCTIONS ON IWAHORI–HECKE ALGEBRAS

MEINOLF GECK

*UFR de Mathématiques, Université Paris 7
2 Place Jussieu, F-75251 Paris Cedex 05, France
E-mail: geck@math.jussieu.fr*

Abstract. This paper is an expanded version of a talk given at the Banach Center Symposium on Knot Theory in July/August 1995. Its aim is to provide a general survey about trace functions on Iwahori–Hecke algebras associated with finite Coxeter groups. The so-called Markov traces are relevant to knot theory as they can be used to construct invariants of oriented knots and links. We present a classification of Markov traces for the classical types A , B and D .

1. Introduction. The algebras in the title first came up in Iwahori’s work (cf. [CR], §67) on endomorphism algebras of certain representations of finite Chevalley groups. From this point of view, they are relevant to the representation theory of finite groups. Alternatively, such an algebra can be described as a finite dimensional quotient of the group algebra of an Artin-Tits braid group. In the case where the underlying Coxeter group is a symmetric group, Jones [Jo] used this approach to construct invariants for isotopy classes of oriented knots and links in the 3-sphere. (For generalizations to Coxeter groups of type B , see [La].) In this article, we shall regard these algebras from a purely algebraic point of view, as deformations of the group algebras of finite Coxeter groups.

Our aim is to describe a general plan for studying trace functions on Iwahori–Hecke algebras associated with any given type of finite Coxeter group. This plan has been developed in joint work with G. Pfeiffer [GP]. As an application, we present a classification of so-called Markov traces on Iwahori–Hecke algebras of classical type. For type A , this is originally due to Ocneanu; for type B , this is joint work with S. Lambropoulou [GL]. We shall give a self-contained account of the uniqueness part in the latter result but refer the reader to [GL] for the technically more complicated details of the existence part. We then show how the classification of Markov traces for type B can be used to classify Markov traces for type D . These results are new; they are motivated by discussions with S. Lambropoulou which are gratefully acknowledged.

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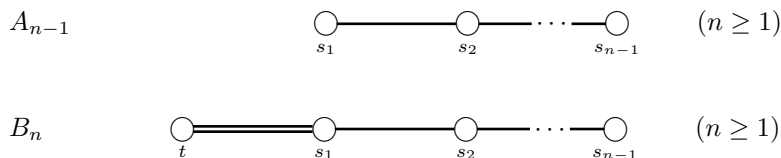
The paper is in final form and no version of it will be published elsewhere.

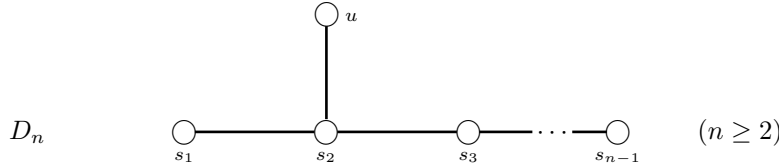
In Section 2 we define and state the main properties of finite Coxeter groups and describe normal forms of the elements for the classical types A , B and D . In Section 3 we study conjugation in these groups by examining the signed cycle type of elements, following Carter's description in [Ca]. The associated Iwahori–Hecke algebras and their trace functions are the subject of Section 4. We show how the operation of conjugation in the Coxeter group translates to the Iwahori–Hecke algebra. This will be the key step in describing our general plan for trace functions, along the lines of [GP]. In Section 5, we introduce Markov traces for Iwahori–Hecke algebras of classical type, and study some of their basic properties. Finally, in Section 6, we give a classification of Markov traces in the case of type D . This is followed by some concluding remarks and open questions.

We have tried to keep the level of our exposition as elementary as possible. We assume known the notions of finitely presented groups and associative algebras over commutative rings, as well as some basic facts from the classical representation theory of associative algebras over fields. The classical reference for Coxeter groups is N. Bourbaki [Bo]. Especially suited for the things that we need is [CR], § 64, 68.

2. Finite Coxeter groups of classical type. A Coxeter group W is a group given by a presentation of a very special form: We have a finite set of generators $S \subseteq W$ which we write in the form $S = \{s_i \mid i \in I\}$ for some finite index set I , and the defining relations are just relations of the form $(s_i s_j)^{m(i,j)} = 1$ where $i, j \in I$ and $m(i, j) > 1$ for $i \neq j$, $m(i, i) = 1$. Since each generator is equal to its inverse, we can rewrite the set of relations $(s_i s_j)^{m(i,j)} = 1$, for $i \neq j$, equivalently in the form of the equalities $s_i s_j \cdots = s_j s_i \cdots$, with $m(i, j)$ factors on both sides. These relations will be called braid relations, or homogeneous relations. Thus, a Coxeter group is a group generated by a set of involutions which satisfy a defining set of homogeneous relations (where the number of factors on each side of the equation has been prescribed).

We shall assume throughout that the integers $m(i, j)$ are chosen so that W is a finite group. This is the case if and only if the matrix $(-\cos(\pi/m(i, j)))_{i,j \in I}$ is positive-definite. We only remark that all matrices $(m(i, j))$ with this property can be classified, and that this classification leads to the well-known list of classical types A_{n-1} , B_n , D_n (for $n \geq 1$); of exceptional types G_2 , F_4 , E_6 , E_7 , E_8 ; and of non-crystallographic types $I_2(p)$ (for $p \geq 5$, $p \neq 6$), H_3 and H_4 . Moreover, every such group has a geometrical description as a finite subgroup of the orthogonal group of an Euclidean space of dimension $|I|$, where the generators s_i are represented by reflections. We shall not need this classification result in this paper, and we shall only be dealing with the classical types A_{n-1} , B_n , D_n whose diagrams are given as follows.





These diagrams encode the relations among the generators by the following scheme. The nodes of the diagram are in bijection with the generators of W . If two generators label nodes which are not directly linked to each other then their product has order 2 (that is, they commute with each other), if they are joined by a single bond then the product has order 3, in the case of a double bond (which here only occurs for type B_n) the order is 4.

On W , we have a length function defined as follows. Let $w \in W$. Then it is possible to write w as a product of generators, $w = s_{i_1} \cdots s_{i_k}$ say, where k is a non-negative integer and $i_1, \dots, i_k \in I$. (Note that we don't have to take into account the inverses of the generators.) If we choose k minimal with this property, then the corresponding product of generators will be called a reduced expression for w and k will be the length of w , denoted $l(w)$. Clearly, we have $l(1) = 0$ and $l(s_i) = 1$ for all $i \in I$. On the other hand, for any $w \in W$ and $i \in I$ we have either $l(s_i w) = l(w) + 1$ or $l(s_i w) = l(w) - 1$ (and similarly for right multiplication by a generator). A key property of a Coxeter group is the fact (Matsumoto's Theorem, see [CR], (64.20)) that any two reduced expressions for a given element can be transformed into each other by applying a finite sequence of braid relations.

We will see below how we can find reduced expressions for the elements of the Coxeter groups of classical type A_{n-1} , B_n and D_n .

2.1. Normal form for type A_{n-1} . We consider the Coxeter group of type A_{n-1} ($n \geq 1$) with generators and relations given by the above diagram. Let us take an arbitrary product of generators $s_{i_1} \cdots s_{i_r}$, for some $r \geq 0$. We now use the braid relations and the relations $s_i^2 = 1$ as rewriting rules for subexpressions, that is, if our product contains a subword which looks like the left hand side of any one of these relations then we can replace it by the right hand side, and repeat this process. By a simple induction on r it then follows that we can rewrite our expression in the form $s_{i_1} \cdots s_{i_r} = s_{j_1} \cdots s_{j_t}$ where $j_1, \dots, j_t \in I$ for some $0 \leq t \leq r$, and at most one factor in the product on the right hand side equals s_{n-1} . Now let us define the set

$$\mathcal{R}_n^+ := \{1, s_{n-1}, s_{n-1}s_{n-2}, \dots, s_{n-1}s_{n-2} \cdots s_1\}.$$

(This definition works for all $n \geq 1$, with $\mathcal{R}_1^+ = \{1\}$.) Note that \mathcal{R}_n^+ has cardinality n . A second induction then implies that an arbitrary product of s_i 's can be written in the form $r_1 \cdots r_n$ with $r_i \in \mathcal{R}_i^+$ for all i . Since there are only $n!$ expressions of this form we conclude that our group has order at most $n!$.

On the other hand, we know that the symmetric group \mathcal{S}_n on n letters is generated by the transpositions σ_i ($1 \leq i \leq n-1$) which interchange the letters $i, i+1$ and leave all others unchanged, and that these elements σ_i satisfy the defining relations for type A_{n-1} . Thus we conclude that \mathcal{S}_n is in fact a realization of the Coxeter group of type A_{n-1} . It

also follows that the above expressions of the elements of \mathcal{S}_n are reduced expressions; we shall call them the normal form for the elements of \mathcal{S}_n .

2.2. Normal form for type B_n . We consider the Coxeter group of type B_n ($n \geq 1$) and denote it by W_n . In order to obtain a normal form for its elements we define

$$t_j := s_j s_{j-1} \cdots s_1 t s_1 \cdots s_{j-1} s_j \in W_n \quad \text{for } j = 0, 1, 2, \dots$$

By a similar induction as before we see that every element of W_n can be written as a product of factors s_i ($1 \leq i \leq n-1$) and t_j ($0 \leq j \leq n-1$) in such a way that s_{n-1} and t_{n-1} occur at most once, and if one of these two elements occurs the other doesn't. Now let us define the set

$$\mathcal{R}_n := \{s_{n-1} \cdots s_{i+1}, s_{n-1} s_{n-2} \cdots s_{i+1} t_i \mid 0 \leq i \leq n-1\}.$$

This set has cardinality $2n$ (where $\mathcal{R}_1 = \{1, t\}$). As above, we can then conclude that every element in W_n can be written in the form $r_1 \cdots r_n$ with $r_i \in \mathcal{R}_i$ for all i . Since there are only $2 \cdot 4 \cdots 2n = 2^n n!$ expressions of this form, we deduce that our group has order at most $2^n n!$.

On the other hand, we know that the hyperoctahedral group, that is, the symmetry group of an n -dimensional cube, is a quotient of B_n . A good way to think about this group is as follows. We consider the symmetric group of degree $2n$ where we denote the objects being permuted by $\{n, \dots, 1, 1', \dots, n'\}$. Then the hyperoctahedral group is the subgroup of all elements which commute with the map $i \mapsto i'$, $i' \mapsto i$ ($1 \leq i \leq n$). Thus, it is generated by the permutations $t = (1, 1')$, $s_1 = (1, 2)(1', 2')$, \dots , $s_{n-1} = (n-1, n)((n-1)', n')$. It is readily checked that these elements indeed satisfy the defining relations for a Coxeter group of type B_n , and that it has the correct order.

It can be further shown that t_i commutes with t_j for all i, j , hence the set of elements $\{t_{i_1} \cdots t_{i_r} \mid 0 \leq i_1 < \dots < i_r \leq n-1\}$ is a subgroup of W_n isomorphic to the direct product of n copies of the cyclic group C_2 of order 2. This subgroup clearly is invariant under conjugation by all generators. On the other hand, the subgroup of W_n generated by $\{s_1, \dots, s_{n-1}\}$ is isomorphic to the symmetric group \mathcal{S}_n , and we obtain a decomposition of W_n as the wreath product $C_2 \wr \mathcal{S}_n$ where \mathcal{S}_n acts on C_2^n by permutation of the components.

2.3. Normal form for type D_n . The Coxeter group of type D_n (for $n \geq 2$) is best described as a subgroup of index 2 in the Coxeter group W_n of type B_n considered before. First note that the defining relations for W_n show that there exists a unique group homomorphism $\varepsilon : W_n \rightarrow \{1, -1\}$ such that $\varepsilon(t) = -1$ and $\varepsilon(s_i) = 1$ for all i . The kernel of this map, which we will denote by W'_n , is generated by the elements $u := t s_1 t, s_1, \dots, s_{n-1}$, which satisfy the relations given by the above Dynkin diagram of type D_n . Thus, the group W'_n is seen to be a quotient of the Coxeter group of type D_n . Formally, this also works for $n = 1$ where it leads to the convention that $W'_n = \{1\}$ and u is considered as being equal to the identity element. On the other hand, one can check (using the Reidemeister-Schreier procedure for finding a presentation of a subgroup of a finitely presented group) that the subgroup of W_n generated by $\{u, s_1, \dots, s_{n-1}\}$ indeed is a realization of the Coxeter group of type D_n . An element of W_n belongs to W'_n if and

only if it can be written as a product of various generators t, s_1, \dots, s_{n-1} with an even number of t 's.

Let us also derive a normal form for the elements in W'_n , directly in terms of the generators $\{u, s_1, \dots, s_{n-1}\}$. We have $W_n = W'_n \cup W'_n t$ and $W_n = \bigcup_{r \in \mathcal{R}_n} W_{n-1} r$. These two equalities imply that an element of W'_n either is of the form wr (with $w \in W'_{n-1}$ and where $r \in \mathcal{R}_n$ does not contain t) or of the form wtr (with $w \in W'_{n-1}$ and where $r \in \mathcal{R}_n$ does contain t). We can rewrite such expressions in terms of the generators of W'_n and are lead to define the set

$$\mathcal{R}'_n := \{s_{n-1}s_{n-2} \cdots s_{i+1}, s_{n-1}s_{n-2} \cdots s_2 u s_1 \cdots s_i \mid 0 \leq i \leq n-1\}.$$

This set has cardinality $2n$ (where $\mathcal{R}'_2 = \{1, u, s_1, u s_1\}$ and, as a convention, $\mathcal{R}'_1 = \{1\}$). Then every element in W'_n can be written uniquely in the form $r_1 \cdots r_n$ with $r_i \in \mathcal{R}'_i$ for all i , and these expressions are reduced. (There are exactly $1 \cdot 4 \cdot 6 \cdots 2n = 2^{n-1} n!$ such expressions.)

To summarize the above descriptions we introduce the following notation. The symbol X_n (for $n \geq 1$) denotes one of the above classical types A_{n-1} , B_n , or D_n . Then $W(X_n)$ is the corresponding Coxeter group and $\mathcal{R}(X_n)$ the subset of $W(X_n)$ called \mathcal{R}_n^+ for type A_{n-1} , \mathcal{R}_n for type B_n , and \mathcal{R}'_n for type D_n . For $n \geq 2$, the group $W(X_{n-1})$ is naturally embedded into $W(X_n)$, and the set $\mathcal{R}(X_n)$ is nothing but the set of so-called distinguished right coset representatives of $W(X_n)$ with respect to $W(X_{n-1})$. Thus, it satisfies the following properties (cf. [CR], (64.38)).

* $\mathcal{R}(X_1) = W(X_1)$.

* If $n \geq 2$ then every element $w \in W(X_n)$ can be written uniquely in the form $w = w' r$ with $w' \in W(X_{n-1})$ and $r \in \mathcal{R}(X_n)$, and we have $l(w) = l(w') + l(r)$.

In particular, every element $w \in W(X_n)$ can be written uniquely in the form $w = r_1 \cdots r_n$ with $r_i \in \mathcal{R}(X_i)$ for all i (and these expressions are reduced). Thus we have a uniform inductive description of the elements in Coxeter groups of classical type.

3. Conjugation in the Coxeter groups of classical type. We shall now use the above inductive description as an essential tool to derive the crucial result about the form of the conjugacy classes in Coxeter groups of classical types. Let again $n \geq 1$ and X_n one of the types A_{n-1} , B_n or D_n . Recalling the definition of the set $\mathcal{R}(X_n)$ we see that each of its elements (for $n \geq 2$) has a very special form: It is either a product of s_{n-1} and an element lying in $W(X_{n-1})$ or it is equal to 1, respectively $t_{n-1} = s_{n-1} \cdots s_1 t s_1 \cdots s_{n-1}$ (in type B_n) or $u_{n-1} := s_{n-1} \cdots s_2 u s_1 s_2 \cdots s_{n-1}$ (in type D_n). In order to obtain a uniform description we introduce subsets $\mathcal{D}(X_n) \subseteq \mathcal{R}(X_n)$ for all $n \geq 1$ as follows.

DEFINITION. If $n = 1$ we let $\mathcal{D}(X_1) := \mathcal{R}(X_1) = W(X_1)$. For $n \geq 2$ we give the definition case by case.

Type A_{n-1} : $\mathcal{D}(X_n) := \{1, s_{n-1}\}$.

Type B_n : $\mathcal{D}(X_n) := \{1, s_{n-1}, t_{n-1}\}$.

Type D_n : $\mathcal{D}(X_n) := \{1, s_{n-1}, u_{n-1}\}$ for $n \geq 3$ and $\mathcal{D}(X_2) := W(X_2)$.

(Recall that $t_{n-1} = s_{n-1} \cdots s_1 t s_1 \cdots s_{n-1}$ respectively $u_{n-1} = s_{n-1} \cdots s_2 u s_1 s_2 \cdots s_{n-1}$.)

The form of the elements in the sets $\mathcal{D}(X_n)$ and $\mathcal{R}(X_n)$ then immediately implies the following result.

LEMMA 3.1 (Double coset decomposition). *Let $n \geq 2$ and $\mathcal{D}(X_n)$ as defined above. Then every element $w \in W(X_n)$ can be written in the form $w = w_1 d w_2$ with $w_1, w_2 \in W(X_{n-1})$, $d \in \mathcal{D}(X_n)$ and $l(w) = l(w_1) + l(d) + l(w_2)$.*

Now recall the following general definition from [GP]. Let W be an arbitrary Coxeter group, with generating set $S = \{s_i \mid i \in I\}$. Given $w, w' \in W$ we write $w \rightarrow w'$ if there exists a sequence of elements $w = w_1, \dots, w_m = w'$ (for some $m \geq 1$) and indices i_1, \dots, i_{m-1} such that $w_{k+1} = s_{i_k} w_k s_{i_k}$ and $l(w_{k+1}) \leq l(w_k)$ for all k . Thus, the relation $w \rightarrow w'$ means that we can conjugate w to w' step by step by a sequence of generators such that the length of the elements remains the same at each step or decreases.

Now we can state our crucial result.

PROPOSITION 3.2 (Cf. [GP], Proposition 2.3). *Let $n \geq 1$ and $w \in W(X_n)$. Then there exist some $d_i \in \mathcal{D}(X_i)$, for $1 \leq i \leq n$, such that $w \rightarrow d_1 \cdots d_n$ and such that the generators performing the sequence of conjugations lie in $W(X_{n-1})$.*

Proof. For $n = 1$ there is nothing to prove due to the definition of $\mathcal{D}(X_1)$. Now let $n \geq 2$ and $w \in W(X_n)$. By Lemma 3.1 we can write $w = w' d_n w''$ with $w', w'' \in W(X_{n-1})$ and $d_n \in \mathcal{D}(X_n)$. Taking any reduced expression for w'' and conjugating step by step by the generators involved in this reduced expression yields that

$$w \rightarrow w_{n-1} d_n \quad \text{for some } w_{n-1} \in W(X_{n-1}).$$

If $n = 2$ we are done, otherwise we can again write $w_{n-1} = w' d_{n-1} w''$ with $w', w'' \in W(X_{n-2})$ and $d_{n-1} \in \mathcal{D}(X_{n-1})$.

Now it readily follows using the braid relations that, for $n \geq 3$, each element in $\mathcal{D}(X_n)$ commutes with the generators for $W(X_{n-2})$, with the only exception that in type D_n we only have $u_{n-1} u = s_1 u_{n-1}$ and $u_{n-1} s_1 = u u_{n-1}$. In any case the following relation holds. For each generator x of $W(X_{n-2})$ and each $d \in \mathcal{D}(X_n)$ there exists some generator x' of $W(X_{n-2})$ such that $xd = dx'$.

Returning to our above equation $w_{n-1} = w' d_{n-1} w''$ we take a reduced expression for w'' , conjugate step by step by the generators involved in it, and conclude that

$$w_{n-1} d_n = w' d_{n-1} w'' d_n \rightarrow w_{n-2} d_{n-1} d_n \quad \text{for some } w_{n-2} \in W(X_{n-2}).$$

Continuing in this way, we obtain the desired result after finitely many steps. Note that the conjugating elements all lie in $W(X_{n-1})$. This completes the proof.

An element of the form $d_1 \cdots d_n$, with $d_i \in \mathcal{D}(X_i)$ for all i , will be called an element in signed block form. Such elements play an essential role later on in our classification of Markov traces. If we group together consecutive factors s_i we obtain a decomposition of our given element as a product of ‘‘blocks’’. To explain this and the notion of ‘‘positive blocks’’ and ‘‘negative blocks’’ in more detail we consider in turn the classification of conjugacy classes for type A_{n-1} , B_n and D_n .

3.3. Conjugacy classes for type A_{n-1} . By Proposition 3.2, any element of $W(A_{n-1})$ (for $n \geq 1$) is conjugate to an element of the form $d_1 \cdots d_n$ with $d_i \in \mathcal{D}(A_{i-1})$ for all i .

Let us fix such an element. Since each factor d_i either equals 1 or equals s_{i-1} we see that our element is uniquely determined by the sequence of non-negative integers defined by the condition that $d_{i+1} = 1$ if and only if i belongs to that sequence. Denote this sequence by $0 = m_1 < \dots < m_r$, and let us also define $\lambda := (\lambda_1, \dots, \lambda_r)$ where $\lambda_i := m_{i+1} - m_i$ for $i = 1, \dots, r$ (with the convention that $m_{r+1} = n$). Note that the entries of λ are all positive and their sum equals n (that is, λ forms a composition of n). The integers m_i are determined from λ by the rule $m_i = \lambda_1 + \dots + \lambda_{i-1}$ for all $i \geq 1$. With this notation, we now have $d_1 \cdots d_n = w(\lambda)$ where

$$w(\lambda) := b^+(m_1, \lambda_1) \cdots b^+(m_r, \lambda_r)$$

and where we define, for any $m \geq 0$ and $d \geq 1$,

$$b^+(m, d) := s_{m+1} \cdots s_{m+d-1}.$$

Under the isomorphism $W(A_{n-1}) \cong \mathcal{S}_n$, such an element $b^+(m, d)$ corresponds to the d -cycle $(m+1, \dots, m+d)$ which permutes the d numbers in the interval between $m+1$ and $m+d$ cyclically and leaving all other numbers unchanged. We call it a *positive block* of length d and starting at m ; the interval $\{m+1, \dots, m+d\}$ will be called the underlying index set. The braid relations imply that two positive blocks commute with each other if the underlying index sets are disjoint.

Let λ and λ' be two compositions of n which are obtained from each other by permuting the parts. Assume at first that only two consecutive parts have been permuted. Then there exists an element $y \in W(A_{n-1})$ such that $yw(\lambda)y^{-1} = w(\lambda')$ and $l(yw(\lambda)) = l(y) + l(w(\lambda))$. The element y can be written down explicitly as a reduced expression in the generators (see [GP], Proposition 2.4(a)). The relevance of the length condition will become clear later on. Applying this relation repeatedly eventually proves that every element of $W(A_{n-1})$ is conjugate to an element $w(\lambda)$ where λ is a partition of n , that is, a composition with parts ordered by increasing length. Since conjugation in \mathcal{S}_n never changes the cycle type of elements, we have thus recovered the well-known classification of conjugacy classes by partitions of n , and we have described canonically reduced expressions for a set of representatives in the various conjugacy classes. We remark that these representatives have minimal length in their classes (see [GP], (2.6)).

For example, the classes for $W(A_3) \cong \mathcal{S}_4$ are parametrized by the partitions (1111), (112), (13), (22), and (4). The corresponding representatives of minimal length are given by 1, s_3 , s_2s_3 , s_1s_3 and $s_1s_2s_3$, respectively.

3.4. Conjugacy classes for type B_n . Let us denote $W_n = W(B_n)$ for $n \geq 1$. Using Proposition 3.2, we can proceed similarly as for type A_{n-1} , with the only difference that we also have to take into account the double coset representatives t_i . Therefore we define, for any $m \geq 0$ and $d \geq 1$,

$$b^-(m, d) := t_m s_{m+1} \cdots s_{m+d-1},$$

and call this element a *negative block* of length d and starting at m , with underlying index set $\{m+1, \dots, m+d\}$. (Positive blocks are defined in exactly the same way as for type A_{n-1} .) We can now conclude that every element in W_n is conjugate to an element of the

form

$$w(\lambda, \varepsilon) := b^{\varepsilon_1}(m_1, \lambda_1) \cdots b^{\varepsilon_r}(m_r, \lambda_r),$$

where $\lambda = (\lambda_1, \dots, \lambda_r)$ is a composition of n , $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$ is a vector of signs, and m_i is the sum of the first $i-1$ parts of λ (as before). Using the description of W_n as a certain subgroup of S_{2n} described in (2.2), a positive block $b^+(m, d)$ is given by the “ d -cycle” $(m+1, \dots, m+d)((m+1)', \dots, (m+d)')$ while the corresponding negative block equals $(m+1, (m+1)')b^+(m, d)$. (Thus, multiplication by $t_m = (m+1, (m+1)')$ is considered as a “sign change” for a permutation.) Again the braid relations imply that any two blocks commute with each other if the underlying index sets are disjoint.

We conjugate our element further so as to obtain a signed block structure where the positive and negative blocks are arranged in some prescribed order. This is done as follows. In a first step, we have $w(\lambda, \varepsilon) \rightarrow w(\lambda', \varepsilon')$ where λ', ε' are obtained by rearranging λ, ε such that all negative parts are in the beginning and ordered by increasing length (see [GP], Proposition 2.4(b,c)). Thus, $\varepsilon' = (-1, \dots, -1, 1, \dots, 1)$ with m negative signs say. If the sign vector has such a form, we also denote the element $w(\lambda', \varepsilon')$ by $w(\alpha, \beta)$ where α is the sequence consisting of the first m parts of λ' (with corresponding negative sign) and β of the remaining parts. The sequence α already forms a partition, while the parts of β may have some arbitrary order. Thus, we see that for each $w \in W_n$ we have $w \rightarrow w(\alpha, \beta)$ where α is a partition and β is a composition such that the total sum of their parts equals n . Finally, the positive blocks (corresponding to the parts of β) can be arranged in increasing order in exactly the same way as this was done in type A_{n-1} .

Thus, given partitions $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_s)$ such that $\sum_i \alpha_i + \sum_j \beta_j = n$, we let $m_i := \alpha_1 + \dots + \alpha_{i-1}$ and $n_j := m_{r+1} + \beta_1 + \dots + \beta_{j-1}$ for all $i, j \geq 1$; then the corresponding representative is given by

$$w(\alpha, \beta) = b^-(m_1, \alpha_1) \cdots b^-(m_r, \alpha_r) b^+(n_1, \beta_1) \cdots b^+(n_s, \beta_s).$$

In this way we can recover the classification of conjugacy classes by pairs of partitions of n (see [Ca]). We have described reduced expressions for a set of canonical representatives in the various conjugacy classes. Again, these representatives have minimal length in their classes (see [GP], (2.7)).

For example, the classes for $W(B_3)$ are parametrized by the double partitions $(-, 111)$, $(1, 11)$, $(11, 1)$, $(111, -)$, $(-, 12)$, $(2, 1)$, $(1, 2)$, $(12, -)$, $(-, 3)$, and $(3, -)$. The corresponding representatives of minimal length are given by $1, t, tt_1, tt_1t_2, s_2, ts_1, ts_2, tt_1s_2, s_1s_2$, and ts_1s_2 , respectively.

3.5. Conjugacy classes for type D_n . We use the notation of (2.3) and consider the Coxeter group W'_n of type D_n as a subgroup of index 2 in the Coxeter group W_n of type B_n (for $n \geq 2$). The description of the classes of W'_n is then in terms of the results obtained above in (3.4). Since W'_n is a normal subgroup, it is a union of conjugacy classes of W_n . Let (α, β) be a pair of partitions such that the total sum of their parts equals n , and let $C(\alpha, \beta)$ be the corresponding class in W_n . Then $C(\alpha, \beta)$ belongs to W'_n if and only if the number of parts of α (corresponding to the negative blocks) is even. Assume that this is the case. Then $C(\alpha, \beta)$ either is one conjugacy class of W'_n or it splits up into two classes. The latter happens if and only if α is empty and all parts of β are even; in

this case, the two classes are mapped to each other under conjugation by the generator t . (For these results, see [Ca].)

Taking the elements $w(\alpha, \beta)$ obtained in (3.4) and rewriting them in terms of the generators of W'_n will yield canonical representatives of minimal length in the classes of W'_n . This is done as follows. We define a negative block of length $d \geq 1$ and starting at $m \geq 0$ by

$$b^-(m, d)' := u_m s_{m+1} \cdots s_{m+d-1} \in W'_n,$$

with the convention that $u_0 = 1$. (Positive blocks are defined in exactly the same way as before.) Given compositions α and β such that the total sum of their parts equals n we define the element $w(\alpha, \beta)'$ in a similar way as in (3.4) as the product of consecutive negative and positive blocks corresponding to the sequence of parts in α and β . The relation $u = ts_1t$ and the definition of u_i show that $u_i = tt_i$ for all $i \geq 0$. Inserting this into the above expressions for negative blocks we conclude that $w(\alpha, \beta)'$ is either equal to $w(\alpha, \beta)$ (defined above for type B_n) or to $tw(\alpha, \beta)t$.

Now, if α is empty and all parts of β are even then $w(\alpha, \beta)'$ is a product of positive blocks, starting with $s_1 \cdots s_{d-1}$, for some $d \geq 2$. Conjugation by t then leads to an expression of the same form where s_1 has been replaced by u . These two elements are representatives for the two classes into which $C(\alpha, \beta)$ splits up in W'_n . For example, if n is even, then the two elements $s_1 s_2 \cdots s_{n-1}$ and $u s_2 \cdots s_{n-1}$ are transformed into each other by conjugation with t but they are not conjugate in W'_n . Note that if α is not empty, then $w(\alpha, \beta)$ and $tw(\alpha, \beta)t$ are in the same conjugacy class in W'_n .

Using these relations and a careful comparison of conjugation in W_n and in W'_n we arrive at the following conclusion (see [GP], (2.8), and [Pf], Section 4.1, for details). For each $w \in W'_n$ we either have $w \rightarrow w(\alpha, \beta)'$ where α is a partition with an even number of parts and β a composition such that the total sum of their parts equals n or, possibly, $w \rightarrow tw(\alpha, \beta)'t$ if α is empty and all parts of β are even. In both cases, the positive blocks can be arranged in increasing order by the same reasoning as in type A_{n-1} . Thus, we obtain canonical representatives of minimal length in the conjugacy classes of W'_n .

For example, in $W(D_4)$, the above classification yields that there are 9 classes which are invariant under conjugation by t ; these are parametrized by the double partitions $(-, 1111)$, $(11, 11)$, $(1111, -)$, $(-, 112)$, $(12, 1)$, $(11, 2)$, $(22, -)$, $(-, 13)$, and $(13, -)$. The corresponding representatives of minimal length are given by 1 , u_1 , $u_1 u_2 u_3$, s_3 , $u_1 s_2$, $u_1 s_3$, $s_1 u_2 s_3$, $s_2 s_3$, and $u_1 s_2 s_3$, respectively. Moreover, we have the two double partitions $(-, 22)$ and $(-, 4)$ for which we have two classes each, with representatives of minimal length given by $s_1 s_3$, $u s_3$ and $s_1 s_2 s_3$, $u s_2 s_3$, respectively.

In summary, the above arguments were used in [GP] to prove the following basic results for Coxeter groups of classical type.

THEOREM 3.6 (See [GP], Theorem 1.1). *Let W be a finite Coxeter group, C a conjugacy class in W , and C_{min} be the set of elements of minimal length in C .*

(a) *For each $w \in C$ there exists some $w' \in C_{min}$ such that $w \rightarrow w'$, that is, there exists a sequence of elements $w = w_1, \dots, w_m = w'$ (for some $m \geq 1$) and indices i_1, \dots, i_{m-1} such that $w_{k+1} = s_{i_k} w_k s_{i_k}$ and $l(w_{k+1}) \leq l(w_k)$ for all k .*

(b) If $w, w' \in C_{\min}$ then $w \sim w'$, that is, there exists a sequence $w = w_0, \dots, w_m = w'$ (for some $m \geq 1$) and elements $x_1, \dots, x_m \in W(X_n)$ such that $l(w_k) = l(w_{k+1})$, $w_{k+1} = x_k w_k x_k^{-1}$, and $l(x_k w_k) = l(x_k) + l(w_k)$ or $l(w_k x_k^{-1}) = l(w_k) + l(x_k^{-1})$ for all k .

4. Iwahori–Hecke algebras. Let W be a finite Coxeter group with generating set $S = \{s_i \mid i \in I\}$ as introduced in Section 2. Let A be a commutative ring with identity and fix a collection of invertible elements $q_i \in A$ ($i \in I$) such that $q_i = q_j$ whenever s_i and s_j are conjugate in W . (The latter condition holds for s_i and s_j which are joined on the Dynkin diagram if and only if the integer $m(i, j)$ is odd.) The associated Iwahori–Hecke algebra H is a free A -module with a basis labelled by the elements of W , say $\{g_w \mid w \in W\}$, and where the multiplication of two basis elements is determined by “deforming” the multiplication in the Coxeter group using the parameters q_i ($i \in I$). To simplify notation, let us write $g_i = g_{s_i}$ for $i \in I$. A convenient way to describe the multiplication is then given by the following two rules.

If $w \in W$ and $w = s_{i_1} \cdots s_{i_k}$ is any reduced expression then $g_w = g_{i_1} \cdots g_{i_k}$.

If $i \in I$ then $g_i^2 = q_i \cdot 1_H + (q_i - 1)g_i$.

It is a non-trivial fact that this indeed is a well-defined multiplication and gives H the structure of an associative algebra over A . The identity element 1_H is the basis element corresponding to $w = 1$. (A proof is sketched in [Bo], Ch. IV, §2, Exercise 23; for full details see [CR], Proposition 68.1. Note that this works even if W is not finite.) If the q_i are all equal to 1 then the second rule simply becomes $g_i^2 = 1$ for all $i \in I$; in this case, H is nothing but the group algebra of W over A .

The above relations show that H is generated by the elements g_i ($i \in I$). Moreover, for $i \neq j$, we have $g_i g_j g_i \cdots = g_j g_i g_j \cdots$ (with $m(i, j)$ factors on both sides) since $s_i s_j s_i \cdots$ and $s_j s_i s_j \cdots$ (with $m(i, j)$ factors) are reduced expressions and equal in W . It is then not difficult to see (cf. [CR], Proposition 68.8) that these homogeneous relations together with the above quadratic relations for the generators g_i form in fact a set of defining relations for H (as an associative A -algebra with identity).

It may be worthwhile to be a bit more explicit about the multiplication in H . Let us multiply two arbitrary basis elements g_v and g_w for $v, w \in W$. For this purpose, we take a reduced expression for one of them, say $v = s_{i_1} \cdots s_{i_k}$. By the first rule above we have $g_v = g_{i_1} \cdots g_{i_k}$. Thus, we are reduced to the case where $v = s_i$ for some $i \in I$. If $l(s_i w) = l(w) + 1$ then $s_i w$ also is reduced and we have, again by the first rule above, that $g_i g_w = g_{s_i w}$. If $l(s_i w) = l(w) - 1$ we take a reduced expression for $s_i w$, say $s_i w = s_{j_1} \cdots s_{j_t}$. Then $w = s_i(s_i w) = s_i s_{j_1} \cdots s_{j_t}$ also is reduced and we can compute that

$$g_w = g_{s_i(s_i w)} = g_{s_i s_{j_1} \cdots s_{j_t}} = g_i(g_{j_1} \cdots g_{j_t}) = g_i g_{s_i w}.$$

We conclude that

$$g_i g_w = g_i^2 g_{s_i w} = q_i g_{s_i w} + (q_i - 1)g_i g_{s_i w} = q_i g_{s_i w} + (q_i - 1)g_w.$$

Thus, in order to compute the product of basis elements we essentially need to be able to compute reduced expressions of products of elements in the underlying Coxeter group. We have described algorithms for computing such reduced expressions for the classical types A_{n-1} , B_n and D_n in Section 2.

DEFINITION. A trace function on H is an A -linear map $\lambda : H \rightarrow A$ such that $\lambda(hh') = \lambda(h'h)$ for all $h, h' \in H$.

This definition is in fact valid for any associative algebra over a commutative ground ring. In the case of a group algebra it is clear that every trace function is constant on the conjugacy classes of the underlying group. Since our Iwahori–Hecke algebra H is defined as a deformation of the group algebra of the corresponding Coxeter group W , we can hope for similar “deformed” relations between the values of trace functions on H . This is indeed true. At first note that the fact that the parameters q_i are invertible in A implies that the basis elements g_i are invertible in H ; we have

$$g_i^{-1} = (1/q_i)g_i + (1/q_i - 1)1_H \quad \text{for all } i \in I.$$

Consequently, every basis element g_w (for $w \in W$) is invertible in H .

In order to understand how conjugation in W translates into H , we first note that it is not really conjugation that counts in an algebra. Instead we have to look at the quotient space of H by the commutators. For any $h, h' \in H$ we define their commutator by $[h, h'] := hh' - h'h$. Let $[H, H] \subseteq H$ be the A -subspace of H generated by all commutators. Note that, if two elements h, h' are conjugate by some unit in H then $h \equiv h' \pmod{[H, H]}$. Thus, “conjugation” translates to “equivalence modulo commutators”.

It is clear that $[H, H]$ lies in the kernel of every trace function on H . Conversely, if $\lambda : H \rightarrow A$ is any A -linear map which is identically zero on the subspace $[H, H]$ then λ is a trace function. We conclude that there is a canonical bijective correspondence between the space of trace functions on H and the dual space of the quotient module $H/[H, H]$. Thus, we have to find out how the images of basis elements g_w in $H/[H, H]$ look like. Let us first consider the following two special cases which are in fact the elementary steps in the relations \longrightarrow and \sim defined on W (cf. [GP], Section 1).

Let $w, w' \in W$ and $i \in I$ such that $w' = s_i w s_i$ and $l(w') = l(w)$. Then g_i conjugates g_w to $g_{w'}$. More generally, this holds if $w, w' \in W$ and $x \in W$ such that $l(w) = l(w')$, $w' = x w x^{-1}$ and $l(xw) = l(x) + l(w)$. For, we then have $g_{w'} g_x = g_{w'x} = g_{xw} = g_x g_w$, and so g_w and $g_{w'}$ are conjugate in H . (The same conclusion holds if the length condition is $l(w x^{-1}) = l(w) + l(x^{-1})$.) Hence, in these cases, we have

$$g_w \equiv g_{w'} \pmod{[H, H]}.$$

Now let $w, w' \in W$ and $i \in I$ such that $w' = s_i w s_i$ and $l(w') < l(w)$. Then g_i conjugates g_w to $q_i g_{w'} + (q_i - 1)g_{s_i w}$, hence we have

$$g_w \equiv q_i g_{w'} + (q_i - 1)g_{s_i w} \pmod{[H, H]}.$$

(Note that both w' and $s_i w$ have length strictly less than $l(w)$.)

Theorem 3.6(a) implies that, for each $w \in W$ there exists some $w' \in W$ of minimal length in its conjugacy class such that $w \longrightarrow w'$. Applying repeatedly the above relations shows that, modulo $[H, H]$, the basis element g_w can be expressed as an A -linear combination of basis elements $g_{w'}$ where w' runs over a set of elements of minimal length in various conjugacy classes.

Theorem 3.6(b) implies that, if $w, w' \in W$ are conjugate and of minimal length in their conjugacy class then g_w and $g_{w'}$ will be conjugate in H . In particular, g_w and $g_{w'}$ are equal modulo $[H, H]$.

If we take these two statements together we arrive at the following conclusion.

LEMMA 4.1. *Let $\{C\}$ be the conjugacy classes of W . In each class C , we choose once and for all a representative $w_C \in C$ of minimal length. Then the following holds. For each $w \in W$ there exist constants $f_{w,C} \in A$ such that*

$$g_w \equiv \sum_C f_{w,C} g_{w_C} \pmod{[H, H]}.$$

We have $f_{w,C} = 0$ unless $l(w_C) \leq l(w)$. The constants $f_{w,C}$ are called class polynomials in [GP]; they can be computed recursively using the above rules.

Note that, at this stage, it is conceivable that the class polynomials might not only depend on w and C but also on the sequences of generators which are used to conjugate arbitrary elements in W to those of minimal length in their classes. We will see below that this is not the case. For this purpose, we have to show that the basis elements g_{w_C} are linearly independent modulo $[H, H]$. In the case where A is a suitably chosen field, this was already established in [GP]. The general case can be easily deduced from this, using an argument of [GR]. Since this is essential in the existence proof of Markov traces (in fact, of any kind of trace functions) we will give a proof for that general case here.

THEOREM 4.2 (Existence and uniqueness of trace functions). *The quotient space $H/[H, H]$ is a free A -module. A canonical basis is given by the images (under the natural map $H \rightarrow H/[H, H]$) of the elements g_{w_C} where C runs over the conjugacy classes of W and w_C is some fixed element of minimal length in C .*

(The term canonical means that the basis does not depend on the choice of elements w_C .)

Proof. Lemma 4.1 already implies that the images of the elements $\{g_{w_C}\}$ form a generating set for the A -module $H/[H, H]$. It remains to prove their linear independence. For this purpose, it is sufficient to construct a collection of trace functions $\{f_C\}$ on H , one for each conjugacy class C of W , such that

$$f_C(g_w) = \delta_{C,C'} \quad \text{if } w \in C'_{min}.$$

The following arguments are a typical application of the technique of specialization: At first, the desired result is proved in a “generic” situation, for a sufficiently general choice of the ground ring A where the parameters q_i are in fact indeterminates. Then the conclusion for any other choice of A and the q_i is achieved by “specializing” those indeterminates to the given values.

Step 1: The “generic” situation. Let \mathbf{A} be the ring of Laurent polynomials (over the integers) in indeterminates u_i ($i \in I$) such that $u_i = u_j$ whenever s_i, s_j are conjugate in W . Let \mathbf{H} be the “generic” Iwahori–Hecke algebra associated with W and defined over \mathbf{A} (with parameters u_i). Let \mathbf{K} be an algebraic closure of the field of fractions of \mathbf{A} and \mathbf{KH} the \mathbf{K} -algebra obtained from \mathbf{H} by extending scalars from \mathbf{A} to \mathbf{K} . Then \mathbf{KH} is a split semisimple \mathbf{K} -algebra (see [CR], Corollary 68.12). Hence every trace function on \mathbf{KH} is a linear combination of the characters of the irreducible representations of \mathbf{KH} .

Moreover, by Tits' Deformation Theorem (see [CR], Theorem 68.21), the algebra \mathbf{KH} is in fact isomorphic to the group algebra of W over \mathbf{K} . It follows that the number of irreducible characters of \mathbf{KH} equals the number of conjugacy classes of W , and that the (square) matrix of values $(\chi(\mathbf{g}_{w_C}))$, where χ runs over the irreducible characters of \mathbf{KH} and C runs over the conjugacy classes of W , is invertible. This immediately implies that the basis elements \mathbf{g}_{w_C} are linearly independent modulo $[\mathbf{KH}, \mathbf{KH}]$, and hence the class polynomials $\mathbf{f}_{w,C}$ of Lemma 4.1 are uniquely determined. The definition of the class polynomials shows that $\mathbf{f}_{w,C} = \delta_{C,C'}$ for $w \in C'_{min}$.

The following arguments are taken from [GR], Section 5.1. For each class C , we can define a linear map $\mathbf{f}_C : \mathbf{KH} \rightarrow \mathbf{K}$ by

$$\mathbf{f}_C(\mathbf{g}_w) := \mathbf{f}_{w,C} \quad \text{for all } w \in W.$$

We check that \mathbf{f}_C is a trace function on \mathbf{KH} . Indeed, for each $w \in W$, we have equations

$$\chi(\mathbf{g}_w) = \sum_C \mathbf{f}_{w,C} \chi(\mathbf{g}_{w_C}) \quad \text{for all irreducible characters } \chi.$$

We can invert these equations and hence obtain expressions of the functions \mathbf{f}_C as linear combinations of the irreducible characters of \mathbf{KH} . Hence the functions \mathbf{f}_C must be trace functions themselves.

Finally, we note that the class polynomials of \mathbf{H} and \mathbf{KH} clearly are the same, and that they lie in \mathbf{A} . Hence the functions \mathbf{f}_C restrict to trace functions on \mathbf{H} with values in \mathbf{A} . This collection of trace functions on \mathbf{H} has the required properties.

Step 2: "Specialization". Given any other choice for A and the parameters $\{q_i\}$, we have a unique ring homomorphism $\varphi : \mathbf{A} \rightarrow A$ such that $\varphi(u_i) = q_i$ for all i . Then the Iwahori–Hecke algebra H (over A and with parameters $\{q_i\}$) can also be described as the tensor product $A \otimes_{\mathbf{A}} \mathbf{H}$ where A becomes an \mathbf{A} -module via φ . This operation of extending scalars from \mathbf{A} to A also induces a canonical map from the space of trace functions on \mathbf{H} to the space of trace functions on H . (In more concrete terms, the following happens. Let $\mathbf{a}(w, w', w'') \in \mathbf{A}$ be the structure constants for the multiplication of basis elements in the generic algebra \mathbf{H} , that is, we have

$$\mathbf{g}_w \cdot \mathbf{g}_{w'} = \sum_{w'' \in W} \mathbf{a}(w, w', w'') \mathbf{g}_{w''} \quad \text{for } w, w' \in W.$$

Then the elements $\varphi(\mathbf{a}(w, w', w'')) \in A$ are the structure constants for the multiplication of basis elements in the algebra H over A . If $\mathbf{f} : \mathbf{H} \rightarrow \mathbf{A}$ is a trace function on \mathbf{H} then the map $\mathbf{f}_\varphi : H \rightarrow A$ defined by $g_w \mapsto \varphi(\mathbf{f}(\mathbf{g}_w))$, is a trace function on H .) The images of the functions \mathbf{f}_C then have the required properties. This completes the proof of the Theorem.

Using the duality between trace functions and the quotient space modulo the commutators, we can immediately translate the last two results to trace functions.

Lemma 4.1 implies that every trace function on H is uniquely determined by its values on the basis elements g_{w_C} , where C runs over the conjugacy classes of W .

Theorem 4.2 implies that, conversely, if $\{a_C\}$ is a collection of elements in A , one for each conjugacy class C of W , then there exists a unique trace function λ on H such that $\lambda(g_w) = a_C$ if the element $w \in W$ has minimal length in the class C .

5. Definition and basic properties of Markov traces. Let again X_n (for $n \geq 1$) be one of the classical types A_{n-1} , B_n or D_n , and $W(X_n)$ the corresponding finite Coxeter group. We have natural embeddings $W(X_1) \subset W(X_2) \subset \dots$, and we let

$$W_\infty := \bigcup_{n \geq 1} W(X_n).$$

Let $H(X_n)$ be the associated Iwahori–Hecke algebra defined over a commutative ring A as in Section 4. Since all generators s_i are conjugate, the corresponding parameters q_i must be equal, and we denote them by q . In type B_n , the parameter corresponding to the generator t will be denoted by Q . In type D_n , the parameter corresponding to the generator u also equals q (since u is conjugate to s_1 , as soon as $n \geq 3$). We also have natural embeddings $H(X_1) \subset H(X_2) \subset \dots$, and we let

$$H_\infty := \bigcup_{n \geq 1} H(X_n).$$

The algebra H_∞ is a free A -algebra of infinite rank, generated by the elements g_1, g_2, \dots (corresponding to the generators s_1, s_2, \dots), and an additional generator which we denote by T in type B_n , and by U in type D_n .

DEFINITION (cf. [Bi], p. 264) Let $z \in A$ and $\tau : H_\infty \rightarrow A$ be A -linear. Then τ is called a Markov trace (with parameter z) if the following conditions are satisfied.

- (1) τ is a trace function.
- (2) $\tau(1) = 1$ (normalization).
- (3) $\tau(hg_n) = z\tau(h)$ for all $n \geq 1$ and $h \in H(X_n)$ (Markov property).

We note that all generators g_i (for $i = 1, 2, \dots$) are conjugate in H_∞ . In particular, any trace function on H_∞ must have the same value on these elements. This explains why the parameter z is independent of n in rule (3) of this definition.

The crucial step in the classification of Markov traces is the observation that the values of such a trace on basis elements of H_∞ corresponding to elements in signed block form in W_∞ only depend on the *number* of positive and negative blocks *if* we replace some generators g_i by their inverses. For this purpose, recall the definition of the sets $\mathcal{D}(X_n)$ and the double coset representatives t_{n-1} (in type B_n) and u_{n-1} (in type D_n) from Section 3. We define corresponding elements in the Iwahori–Hecke algebras as follows.

$$T'_{n-1} := g_{n-1} \cdots g_1 T g_1^{-1} \cdots g_{n-1}^{-1} \quad \text{for all } n \geq 1 \text{ in type } B,$$

and

$$U'_{n-1} := g_{n-1} \cdots g_2 U g_1^{-1} g_2^{-1} \cdots g_{n-1}^{-1} \quad \text{for all } n \geq 2 \text{ in type } D.$$

The braid relations in $H(X_n)$ imply that the generators g_1, \dots, g_{n-2} commute with T'_{n-1} in type B_n , and that the generators g_2, \dots, g_{n-2} commute with U'_{n-1} in type D_n , while $UU'_{n-1} = U'_{n-1}g_1$ but $g_1U'_{n-1} \neq U'_{n-1}U$ for all $n \geq 3$!

We use these elements to define the analogues of the sets $\mathcal{D}(X_n)$ as subsets of H_∞ . If $n = 1$, we let $\mathcal{D}'(X_1) := \{g_w \mid w \in W(X_1)\}$. For $n \geq 2$ we give the definition case by case.

Type A_{n-1} : $\mathcal{D}'(X_n) := \{1, g_{n-1}\}$.

Type B_n : $\mathcal{D}'(X_n) := \{1, g_{n-1}, T'_{n-1}\}$

Type D_n : $\mathcal{D}'(X_n) := \{1, g_{n-1}, U'_{n-1}\}$ for $n \geq 3$ and $\mathcal{D}'(X_2) := \{1, g_1, U, U_1'\}$.

Now recall from Section 4 that a trace function on an Iwahori-Hecke algebra is uniquely determined by its values on basis elements corresponding to representatives of minimal length in the various conjugacy classes of the underlying Coxeter group. Also recall from Section 3 that representatives of minimal length in the classes of Coxeter groups of classical types are of the form $d_1 \cdots d_n$, where d_i is a distinguished double coset representative of $W(X_i)$ with respect to $W(X_{i-1})$. If we are also willing to consider elements in signed block form which have not necessarily minimal length in their class we arrive at the following result (which already appeared in [GL], Proposition 3.4 and Corollary 3.5, for type B). It translates Proposition 3.2 from the level of Coxeter groups to the level of Iwahori-Hecke algebras, taking into account the modified sets $\mathcal{D}'(X_n)$.

LEMMA 5.1. *Let $h \in H(X_n)$ for some $n \geq 1$. Then there exists a function $a_h : \mathcal{D}'(X_1) \times \cdots \times \mathcal{D}'(X_n) \rightarrow A$ such that*

$$\lambda(h) = \sum_{(d'_1, \dots, d'_n)} a_h(d'_1, \dots, d'_n) \lambda(d'_1 \cdots d'_n)$$

and

$$\lambda(hg_n) = \sum_{(d'_1, \dots, d'_n)} a_h(d'_1, \dots, d'_n) \lambda(d'_1 \cdots d'_n g_n)$$

for all trace functions $\lambda : H \rightarrow A$ (where $d'_i \in \mathcal{D}'(X_i)$ for all i).

Proof. We may clearly assume that $h = g_w$ for some $w \in W(X_n)$. We proceed by induction on $l(w)$. If $l(w) = 0$ or $l(w) = 1$, there is nothing to prove. Now let $l(w) \geq 2$ (and hence $n \geq 2$). Recall from Proposition 3.2 that for each $w \in W(X_n)$ there exists a sequence of generators in $W(X_{n-1})$ which conjugates w step by step to an element of the form $d_1 \cdots d_n$ (with $d_i \in \mathcal{D}(X_i)$ for all i) in such a way that the length either remains the same or decreases at each step. (Note that we don't necessarily reach representatives of minimal length in the classes of $W(X_n)$.) Since the generators in $W(X_{n-1})$ commute with s_n we thus have

$$w \longrightarrow d_1 \cdots d_n \quad \text{and} \quad ws_n \longrightarrow d_1 \cdots d_n s_n,$$

where in both cases we can use the same sequence of generators.

If we apply the analogous conjugations to the basis elements in $H(X_n)$ and use the relations in the beginning of Section 4, we find the following. For each $w \in W(X_n)$ there exists a function $b_w : \mathcal{D}(X_1) \times \cdots \times \mathcal{D}(X_n) \rightarrow A$ such that

$$g_w \equiv \sum_{(d_1, \dots, d_n)} b_w(d_1, \dots, d_n) g_{d_1 \cdots d_n} \pmod{[H(X_n), H(X_n)]}$$

and

$$g_w g_n \equiv \sum_{(d_1, \dots, d_n)} b_w(d_1, \dots, d_n) g_{d_1 \cdots d_n g_n} \pmod{[H(X_n), H(X_n)]}.$$

Now we have to replace the basis elements corresponding to the elements d_i by the modified elements d'_i . If $d_i = g_{i-1}$ there is nothing to be changed. So let us assume that $d_i = t_{i-1}$ (in type B_n) or $d_i = u_{i-1}$ (in type D_n). Consider the corresponding product of the generators in $H(X_i)$. Replacing the appropriate generators g_j by their inverses and using the inversion formula for g_j shows that

$$g_{d_i} = q^{l(d_i)} d'_i + A\text{-linear combination of } g_v \text{ with } v \in W(X_i) \text{ and } l(v) < l(d_i)$$

where d'_i is the element in $\mathcal{D}'(X_i)$ analogous to d_i . We insert these expressions in the above relations and conclude that

$$g_w \equiv \tilde{h} + \sum_{(d_1, \dots, d_n)} b_w(d_1, \dots, d_n) d'_1 \cdots d'_n \pmod{[H(X_n), H(X_n)]}$$

and

$$g_w g_n \equiv \tilde{h} g_n + \sum_{(d_1, \dots, d_n)} b_w(d_1, \dots, d_n) d'_1 \cdots d'_n g_n \pmod{[H(X_n), H(X_n)]},$$

where $\tilde{h} \in H(X_n)$ is a linear combination of basis elements corresponding to elements in $W(X_n)$ of strictly smaller length than w . We can then apply induction to complete the proof.

The above Lemma allows us to reduce the computation of the value of any trace function on H_∞ to the computation of the values on elements of the form $d'_1 \cdots d'_n$, with $d'_i \in \mathcal{D}'(X_i)$ for all i . The next result is the key to the evaluation of Markov traces on such elements. For type B , this already appeared in [GL], Lemma 4.2. It shows that we can always apply a Markov type rule to factors g_n , and shift the remaining factors T'_n or U'_n to the left as far as possible.

LEMMA 5.2. *Let τ be a Markov trace on H_∞ with parameter z . To abbreviate notation, we let Y_i denote the element T'_i (in type B) respectively U'_i (in type D).*

(a) *Let $n \geq 1$ in type B and $n \geq 2$ in type D . If $h \in H(X_n)$ and $m \geq 0$, then*

$$\tau(h g_n Y_{n+1} \cdots Y_{n+m}) = z \tau(h Y_n \cdots Y_{n+m-1}).$$

(b) *Let $n \geq 1$. If $h \in H(X_n)$ and $m \geq 0$, then*

$$\tau(h Y_{n+1} \cdots Y_{n+m}) = \tau(h Y_n \cdots Y_{n+m-1}).$$

Proof. To prove these relations we will proceed by induction on m . If $m = 0$ then there is nothing to prove in (b), while in (a) we can apply directly rule (3) in the definition of Markov traces. Now let us assume that $m > 0$. Let $x = g_n$ in (a) and $x = 1$ in (b). We have to evaluate the expression

$$\tau(h x Y_{n+1} \cdots Y_{n+m}).$$

Since $n \geq 1$ we can write $Y_{n+1} = g_{n+1} Y_n g_{n+1}^{-1}$. The braid relations in H imply that g_{n+1}^{-1} commutes with Y_{n+2}, \dots, Y_{n+m} . Since τ is a trace our expression is equal to

$$\tau(g_{n+1}^{-1} h x g_{n+1} Y_n Y_{n+2} \cdots Y_{n+m}).$$

Now h lies in $H(X_n)$, that is, h only involves the generators t, g_1, \dots, g_{n-1} . It follows that h commutes with g_{n+1}^{-1} . Hence, if $x = 1$ we find that our expression equals

$$\tau(hY_n Y_{n+2} \cdots Y_{n+m}).$$

We can apply induction with $h' := hY_n$, and we are done.

We let $x = g_n$ from now on. Using the braid relation $g_{n+1}^{-1}g_n g_{n+1} = g_n g_{n+1} g_n^{-1}$, the above expression can be rewritten as

$$\tau(hg_n g_{n+1} g_n^{-1} Y_n Y_{n+2} \cdots Y_{n+m}).$$

If we are in type B with any n , respectively in type D with $n \geq 2$, we can write $Y_n = g_n Y_{n-1} g_n^{-1}$. Then the left hand term g_n will cancel and then g_{n+1} commutes with Y_{n-1} . Now our expression reads

$$\tau(hg_n Y_{n-1} g_{n+1} g_n^{-1} Y_{n+2} \cdots Y_{n+m}).$$

The element g_n^{-1} commutes with all terms to the right of it. Hence our expression equals

$$\tau(g_n^{-1} h g_n Y_{n-1} g_{n+1} Y_{n+2} \cdots Y_{n+m}).$$

We write $h' := g_n^{-1} h g_n Y_{n-1}$ and observe that this element lies in H_{n+1} . So we can apply the induction and obtain that

$$\tau(hg_n Y_{n+1} \cdots Y_{n+m}) = \tau(h' g_{n+1} Y_{n+2} \cdots Y_{n+m}) = z \tau(h' Y_{n+1} \cdots Y_{n+m-1}).$$

We insert the expression for h' again, note that g_n^{-1} commutes with $Y_{n+1} \cdots Y_{n+m-1}$, and conclude that

$$\tau(h' Y_{n+1} \cdots Y_{n+m}) = \tau(hg_n Y_{n-1} g_{n+1} g_n^{-1} Y_{n+1} \cdots Y_{n+m}) = z \tau(h Y_n Y_{n+1} \cdots Y_{n+m-1}).$$

Putting things together we see that this completes the proof.

5.3. Markov traces for type A . Let τ be a Markov trace with parameter z , and let $d'_i \in \mathcal{D}'(A_{i-1})$ for $i = 1, \dots, n$. Then each d'_i either equals 1 or g_{i-1} , hence the Markov property directly implies that

$$\tau(d'_1 \cdots d'_n) = z^a,$$

where a is the number of factors equal to g_{i-1} . Thus, τ is uniquely determined by its parameter z .

To prove the existence of a Markov trace with any given parameter $z \in A$, we refer to [Jo], Theorem 5.1, or [GP], (4.1). In the framework of the theory developed here, the argument is as follows. We define a trace that has the Markov property on the elements $d'_1 \cdots d'_n$ corresponding to elements of minimal length in the conjugacy classes of W_∞ , and we have to show that the Markov property then holds for all elements of H_∞ .

In particular for type A we have, by (3.3), that each element of the form $d_1 \cdots d_n$, with $n \geq 1$ and $d_i \in \mathcal{D}(A_{i-1})$ for all i , has minimal length in its class in $W(A_{n-1})$. Note that the corresponding elements $d'_i \in \mathcal{D}'(A_{i-1})$ are just g_{d_i} . By Theorem 4.2 we can define a trace function $\tau_n : H(A_{n-1}) \rightarrow A$ by specifying its values on elements of the form $d'_1 \cdots d'_n$, and the Markov property forces us to let $\tau(d'_1 \cdots d'_n) = z^a$, with a as above. These traces are compatible with the embeddings $H(A_1) \subset H(A_2) \subset \dots$, hence they define a unique trace function on the infinite union of these algebras. It remains to

show that the Markov property holds for all elements of H_∞ . This follows by combining the two equations in Lemma 5.1.

5.4. Markov traces for type B. Let τ be a Markov trace with parameter z , and let $d'_i \in \mathcal{D}'(B_i)$ for $i=1, \dots, n$. Then each d'_i either equals 1 or g_{i-1} or T'_{i-1} , hence Lemma 5.2 implies that

$$\tau(d'_1 \cdots d'_n) = z^a \tau(T'_0 \cdots T'_{b-1}),$$

where a is the number of factors d'_i which are equal to g_{i-1} and b is the number of factors d'_i which are equal to T'_{i-1} . Thus, τ is uniquely determined by its parameter z and the values on the elements in the set

$$\{T'_0 \cdots T'_{k-1} \mid k = 1, 2, \dots\}.$$

Conversely, given $z, y_1, y_2, \dots \in A$ there does exist a Markov trace τ with parameter z and such that $\tau(T'_0 T'_1 \cdots T'_{k-1}) = y_k$ for all $k \geq 1$. For the details of this existence result, we refer to [GL], Theorem 4.3. The argument is very roughly analogous to type A but complications arise from the fact that one also has to consider conjugates of T where the conjugating element is of the form $g_1^{\pm 1} \cdots g_i^{\pm 1}$, with *any* choice of signs.

6. Markov traces for type D. The aim of this section is to classify Markov traces for Iwahori–Hecke algebras of type D . The best way to achieve this seems to be by embedding these algebras into those for type B , and then to use the known results for type B . To fix notation we let W_n (for $n \geq 2$) denote the Coxeter group of type B_n , with generators $\{t, s_1, \dots, s_{n-1}\}$ and $W'_n \subseteq W_n$ the Coxeter group of type D_n , with generators $\{u = ts_1t, s_1, \dots, s_{n-1}\}$ (see Section 2.3). We let

$$W'_\infty := \bigcup_{n \geq 2} W'_n \quad \subset \quad W_\infty := \bigcup_{n \geq 2} W_n.$$

We denote the Iwahori–Hecke algebras of type B_n and D_n by H_n and H'_n , respectively. In order to have an embedding of H'_n into H_n , we have to set the parameter Q equal to 1. Indeed, in this case, we have $T^2 = 1$ in H_n and can compute that

$$U^2 = (Tg_1T)^2 = Tg_1^2T = q \cdot 1_H + (q-1)Tg_1T = q \cdot 1_H + (q-1)U.$$

Furthermore, we have a decomposition of A -modules $H_n = H'_n \oplus H'_n T$ where $H'_n T = T H'_n$. Thus, H'_n is the subalgebra of H_n generated by the elements $U = Tg_1T, g_1, \dots, g_{n-1}$. We let

$$H'_\infty := \bigcup_{n \geq 2} H'_n \quad \subset \quad H_\infty := \bigcup_{n \geq 2} H_n$$

and consider the restriction of trace functions from H_∞ to H'_∞ . It is clear that the restriction of a Markov trace on H_∞ is a Markov trace on H'_∞ (and both have the same parameter). Our main result will show that *every* Markov trace on H'_∞ can be obtained in this way.

At first we will establish a uniqueness result for Markov traces on H'_∞ , similarly to that for H_∞ . Recall that we have $\mathcal{D}'(D_2) = \{1, g_1, U, U_1'\}$ and $\mathcal{D}'(D_i) = \{1, g_{i-1}, U'_{i-1}\}$ for $i \geq 3$, where

$$U'_i = g_i \cdots g_2 U g_1^{-1} g_2^{-1} \cdots g_i^{-1} \quad \text{for all } i \geq 1.$$

The main difference to type B lies in the set $\mathcal{D}'(D_2)$ which does not just contain the elements g_1 and U'_1 (as would be analogous to type B) but also the extra generator U . The main complication arising from this is that g_1 respectively U do not commute with U'_i for $i \geq 2$. On the level of W'_n we have $uu_i = u_i s_1$ and $u_i u = s_1 u_i$, but on the level of H'_n we only have $UU'_i = U'_i g_1$ but $U'_i U \neq g_1 U'_i$ for all $i \geq 2$. Nevertheless, we have the following relations:

$$\begin{aligned} U^{\pm 1} U'_i &= U'_i g_1^{\pm 1} + A\text{-linear combination of } g_w \text{ with } w \in W'_n \text{ and } l(w) < l(uu_i), \\ g_1^{\pm 1} U'_i &= U'_i U^{\pm 1} + A\text{-linear combination of } g_w \text{ with } w \in W'_n \text{ and } l(w) < l(uu_i). \end{aligned}$$

(Indeed, we know that $uu_i = u_i s_1$ and $s_1 u_i = u_i u$. So we also have $UU_i = U_i g_1$ and $g_1 U_i = U_i U$ where $U_i = g_i \cdots g_2 U g_1 g_2 \cdots g_i$. The above formulas then follow by inserting the expressions $g_j = qg_j^{-1} + (q-1)1_H$ or $U = qU^{-1} + (q-1)1_H$ at the appropriate places, and expanding.)

We will argue carefully by induction on the length of elements, and then the above two rules will be sufficient.

PROPOSITION 6.1 (Uniqueness of Markov traces for type D). *Let $z \in A$ and $\tau : H'_\infty \rightarrow A$ be a Markov trace with parameter z . Then τ is uniquely determined by its values on the elements in the set*

$$\{U'_1 \cdots U'_{2k-1} \mid k = 1, 2, \dots\}.$$

PROOF. For each $N \geq 0$ let $H'(\leq N)$ be the A -subspace of H' generated by all elements g_w with $w \in W'_n$ for some $n \geq 2$ and $l(w) \leq N$. The subspace $H'(< N)$ is defined in the analogous way. We claim that, for all N , the values of τ on elements in $H'(\leq N)$ are determined by the values on elements of the form $U'_1 \cdots U'_{2k-1}$, for $k = 1, 2, \dots$. If $N = 0$ there is nothing to prove. Now assume that our claim holds for $N \geq 0$ and that $w \in W'_n$ for some n such that $l(w) = N + 1$. Since the W_n form an increasing chain we may assume that n is large enough. (Actually, we shall use that $n \geq 3$.) We must show that $\tau(g_w)$ is determined as required.

By Lemma 4.1, $\tau(g_w)$ can be expressed as a linear combination of values $\tau(g_{w_C})$ where C runs over a set of conjugacy classes of W'_n such that $l(w_C) \leq l(w)$. Let us fix such a class C , labelled by the pair of partitions (α, β) say. The discussion in (3.5) shows that two different types of classes can occur.

Type I. The elements w_C and $tw_C t$ are not conjugate in W'_n . Then w_C must have a very special form: Up to possibly interchanging the roles of w_C and $tw_C t$, we must have $w_C = s_1 s_{i_1} \cdots s_{i_r}$ and hence $tw_C t = u s_{i_1} \cdots s_{i_r}$, for some $2 \leq i_1 < \cdots < i_r$. Using the Markov property, we can therefore compute that

$$\tau(g_{w_C}) = \tau(g_1 g_{i_1} \cdots g_{i_r}) = \tau(g_1) z^r \quad \text{and} \quad \tau(g_{tw_C t}) = \tau(U g_{i_1} \cdots g_{i_r}) = \tau(U) z^r.$$

Now we observe that u and s_1 are two generators of W'_n , and that all generators are conjugate (since $n \geq 3$). Clearly, they are of minimal length in their conjugacy class. Using Theorem 3.6(b) and the Markov property, we conclude that $\tau(U) = \tau(g_1) = z$. Hence, the values of τ on g_{w_C} and $g_{tw_C t}$ are already determined (they are equal to z^{r+1}).

Type II. The elements w_C and $tw_C t$ are conjugate in W'_n . If the partition α is empty, then we can assume that w_C is a product of generators s_i , hence the Markov property

alone determines the value of τ on g_w . From now on, let us assume that α is not empty. We can assume that $w_C = d_1 \cdots d_n$ with $d_i \in \mathcal{D}(D_i)$ for all i . As in the proof of Lemma 5.1 we see that, up to a power of q , the element $g_{w_C} = g_{d_1 \cdots d_n}$ is congruent to $d'_1 \cdots d'_n$ modulo $H'(\leq l(w_C)) \subseteq H'(\leq N)$ where $d'_i \in \mathcal{D}'(D_i)$ is the element of the analogous form as d_i . By induction, it is therefore sufficient to show that the value of τ on $d'_1 \cdots d'_n$ is determined as required.

Recall from (3.5) that $w_C = w(\alpha, \beta)'$ where α has an even number of parts and elements of the form $u_i s_{i+1} \cdots s_{i+d-1}$ are considered as negative blocks (with the convention that $u_0 = 1$). Hence the total number of “proper” factors $u_i \neq 1$ in this expression for w_C is odd!

If the first negative block has length 1 it just consists of $u_0 = 1$ hence in our expression $d'_1 \cdots d'_n$ we have $d'_1 = 1, d'_2 = U'_1$. We can directly apply Lemma 5.2 and conclude that

$$\tau(d'_1 \cdots d'_n) = z^a \tau(U'_1 \cdots U'_{2b-1})$$

where a is the number of factors d'_i equal to g_{i-1} and $2b$ is the number of parts of α .

If the first negative block has length > 1 it consists of a product of factors $s_1 \cdots s_{m-1}$ for some $m \geq 2$ hence in our expression $d'_1 \cdots d'_n$ we have $d'_1 = 1, d'_2 = g_1, \dots, d'_m = g_{m-1}$. Using Lemma 5.2, we conclude that

$$\tau(d'_1 \cdots d'_n) = z^a \tau(g_1 U'_2 \cdots U'_{2b}),$$

where a is the number of factors d'_i which are equal to g_{i-1} and $2b$ is the number of parts of α . We write $U'_2 = g_2 U g_1^{-1} g_2^{-1}$ and obtain that

$$\tau(g_1 U'_2 \cdots U'_{2b}) = \tau(g_1 g_2 U g_1^{-1} g_2^{-1} U'_3 \cdots U'_{2b}) = \tau(g_2^{-1} g_1 g_2 U g_1^{-1} U'_3 \cdots U'_{2b}),$$

since g_2 and hence also g_2^{-1} commutes with U'_i for $i \geq 3$. Now we use the braid relation $g_2^{-1} g_1 g_2 = g_1 g_2 g_1^{-1}$ and rewrite the above expression as $\tau(g_1 g_2 U g_1^{-2} U'_3 \cdots U'_{2b})$. We apply the above commutation rules for g_1^{-2}, U with $U'_3 \cdots U'_{2b}$ and conclude that

$$g_1 g_2 U g_1^{-2} U'_3 \cdots U'_{2b} \equiv U g_1^{-2} g_1 g_2 U'_3 \cdots U'_{2b} = U'_1 g_2 U'_3 \cdots U'_{2b} \pmod{H(\leq N)'}$$

Now Lemma 5.2 implies that

$$\tau(U'_1 g_2 U'_3 \cdots U'_{2b}) = z \tau(U'_1 U'_2 \cdots U'_{2b-1}).$$

This completes the proof.

THEOREM 6.2 (Classification of Markov traces for type D). *Let $z, y_2, y_4, \dots \in A$. Then there exists a unique Markov trace τ on H'_∞ with parameter z such that*

$$\tau(U'_1 \cdots U'_{2k-1}) = y_{2k} \quad \text{for all } k \geq 1.$$

Every Markov trace on H'_∞ is the restriction of a Markov trace on H_∞ .

Proof. Uniqueness was already proved in Proposition 6.1. In order to prove existence, choose any elements $y_1, y_3, \dots \in A$, in addition to the given elements $z, y_2, y_4, \dots \in A$. By the classification of Markov traces for type B (see (5.4) and [GL], Theorem 4.3), there exists a (unique) Markov trace τ on H_∞ with parameter z such that $\tau(T'_0 T'_1 \cdots T'_{k-1}) = y_k$ for all $k \geq 1$. We restrict this Markov trace to H'_∞ and it remains to check that the value of τ on $U'_1 \cdots U'_{2k-1}$ indeed equals y_{2k} . This is done as follows. We rewrite the elements

U'_i in terms of the generators of H_∞ and obtain that

$$U'_i = g_i \cdots g_2 T g_1 T g_1^{-1} g_2^{-1} \cdots g_i^{-1} = T T'_i \quad \text{for all } i \geq 1.$$

Hence we must compute the value of τ on $(T T'_1) \cdots (T T'_{2k-1})$. While, on the level of W_∞ , the elements t and t_i commute with each other, this is no longer true on the level of H_∞ for the elements T and T'_i . By [GL], Lemma 2.4(c), we have the commutation rule $T T'_{2i-1} = T'_{2i-1,0} T$ where $T'_{2i-1,0} = g_{2i-1} \cdots g_2 g_1^{-1} T g_1 g_2^{-1} \cdots g_{2i-1}^{-1}$. We apply this rule to every factor with an odd index; this will produce, for each i , a factor $T T$ between the $(2i-1)$ -th and the $2i$ -th term. Since $Q = 1$ we have $T T = 1_H$, and hence our desired trace value equals

$$\tau(T'_{1,0} T'_2 \cdots T'_{2k-2} T'_{2k-1,0} T) = \tau(T'_0 T'_{1,0} T'_2 \cdots T'_{2k-2} T'_{2k-1,0}).$$

The argument on the right hand side now has a form to which we can apply the results [GL], (4.4), on the evaluation of Markov traces for type B . By [*loc. cit.*], we can replace each $T'_{i,0}$ by an arbitrary element of the form $x T x^{-1}$ with $x = g_i^{\pm 1} \cdots g_1^{\pm 1}$, without affecting the trace value. Applying this to the factors $T'_{2i-1,0}$ yields that the above value in fact equals

$$\tau(T'_0 T'_1 \cdots T'_{2k-1}) = y_{2k}.$$

This completes the proof.

Finally, we can establish the following improvement to Proposition 6.1.

COROLLARY 6.3. *Let $\tau : H_\infty \rightarrow A$ be a Markov trace with parameter z . Let $n \geq 1$ and $d'_i \in \mathcal{D}'(D_i)$ for $1 \leq i \leq n$. Then*

$$\tau(d'_1 \cdots d'_n) = z^a \tau(U'_1 \cdots U'_{2b-1}),$$

where a is the number of factors d'_i which are equal to g_{i-1} or U , and $2b-1$ is the smallest odd number less than or equal to the number of factors d'_i which are equal to U'_{i-1} .

Proof. We only sketch this. By Theorem 6.2, the trace τ is the restriction of a Markov trace on H_∞ , which we shall denote by the same symbol. We can rewrite each U'_i as $T T'_i$ and obtain an expression for $d'_1 \cdots d'_n$ in terms of the generators of H_n . If there is a factor U we also rewrite it as $T g_1 T$. Now we use the commutation rule for T and T'_i already mentioned in the proof of Theorem 6.2 to cancel intermediate factors T . The resulting expression either has signed block form in H_n or is conjugate to such an expression by T (but possibly with factors $T'_{i,0}$ instead of T'_i). Performing this conjugation if necessary (which does not effect the trace value!) and using once more the evaluation rule in [GL], (4.4), we conclude that the value of τ on $d'_1 \cdots d'_n$ is equal to the value on an element of the form $d''_1 \cdots d''_n$ where each d''_i either equals g_{i-1} or T'_{i-1} , and the number of factors T'_{i-1} is even. Now Lemma 5.2 implies that this value equals $z^a \tau(T'_0 \cdots T'_{2b-1})$ for some a, b . We can argue as above in the proof of Theorem 6.2 to compare the values of τ on products $T'_0 T'_1 \cdots$ with those on products $U'_1 U'_2 \cdots$, and hence get the desired conclusion.

7. Concluding remarks and open questions

7.1. Relations with knot theory. Iwahori–Hecke algebras of type A can also be described as quotients of the group algebras of Artin’s braid groups. The motivation for

introducing Markov traces in type A is then given by Markov's Theorem in classical knot theory (see [Jo], Section 4). Now braid groups exist for any given type of finite Coxeter groups (cf. [BS]), and the corresponding Iwahori–Hecke algebras again are quotients of their group algebras. Does there exist a knot theory in some suitable topological space such that the Markov traces for type B and D yield invariants for isotopy classes of oriented knots and links in an analogous way as Ocneanu's trace leads to the Jones polynomial? S. Lambropoulou [La] has shown that this is indeed the case for type B . The same question for type D seems to be open.

7.2. Relations with R -matrices. Turaev has given an interpretation in terms of R -matrices for the construction of generalized Markov traces on classical braid groups (see [Bi], Section 3). In this way, invariants like the Jones polynomial and the Kauffman polynomial can all be derived in a uniform way. Does there exist such an R -matrix interpretation for the Markov traces discussed in this paper? (I wish to thank J. Birman for mentioning this problem to me.) In type B and in the special case where there exists some $y \in A$ such that $\tau(T'_0 T'_1 \cdots T'_{k-1}) = y^k$ for all $k \geq 1$ (notation of (5.4)), such an interpretation has been given by T. tom Dieck in [Di], §4 and 6. The question seems to be open for general choices of the y_k 's, as well as for type D .

7.3. The problem of weights. Let \mathbf{KH} be the “generic” Iwahori–Hecke algebra of type A_{n-1} , B_n or D_n , defined over an algebraically closed field as in the proof of Theorem 4.2. Then every trace function on \mathbf{KH} is a linear combination of the characters corresponding to the irreducible representations of \mathbf{KH} . (These irreducible characters are labelled by partitions or pairs of partitions in a similar way as this was the case for the conjugacy classes of the underlying Coxeter groups.) In particular, it must be possible to express the Markov traces discussed in this paper as linear combinations of the irreducible characters, where the coefficients are called “weights”. For Ocneanu's original trace function in type A these weights are known and given in [Jo], pp.345. The weights in type B and D do not seem to be known.

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