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ARC PRESENTATIONS OF KNOTS AND LINKS

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This paper presents some examples and a survey of results concerning a new way of presenting knots and links, together with the corresponding link invariant. More detailed accounts are given in [Cr, C-N, Nu1, Nu2, Nu3].

The new presentation involves placing a link in a special knot-holder. An example is illustrated in Figure 1. A trefoil is shown embedded in a sheaf of five half-planes so that it meets each half-plane in a single arc. The line where the five planes intersect is called the *axis* or *binding*. Any link L can be embedded in a sheaf of finitely many half-planes so that it meets each half-plane in a single simple arc. Such an embedding is called an *arc-presentation* of L. The minimum number of planes required to present a given link in this manner is a knot invariant called the *arc index* of L. It is denoted by $\alpha(L)$.

To be more specific, there is an open-book decomposition of the 3-sphere which has open discs as pages and an unknotted circle as the binding. We can think of the 3-sphere as $\mathbb{R}^3 \cup \{\infty\}$ and of the circle as the z-axis $\cup\{\infty\}$. Then, if we use polar coordinates in the x-y plane, the pages are half-planes H_θ at angle θ. The knot-holder is a finite subset of these pages.

It is important that the link meets each page in a single arc. If this restriction is removed then three pages always suffice for any link.

This kind of presentation was discussed very early in the development of knot theory. Brunn proved that every knot has a projection with a unique singular point of high multiplicity [Br]. His proof is to construct an arc-presentation and look along the axis. More recently, Birman and Menasco reinvented this presentation in their study of the braid index of satellite links [B-M]. The binding circle of an arc-presentation of the companion knot can be used as a braid axis for the satellite. It is hoped that the braid index of the satellite can be expressed in terms of the framing, the arc index of the companion, and some properties of the pattern. Aspects of this have been studied in [Nu1, Nu2].

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The paper is in final form and no version of it will be published elsewhere.

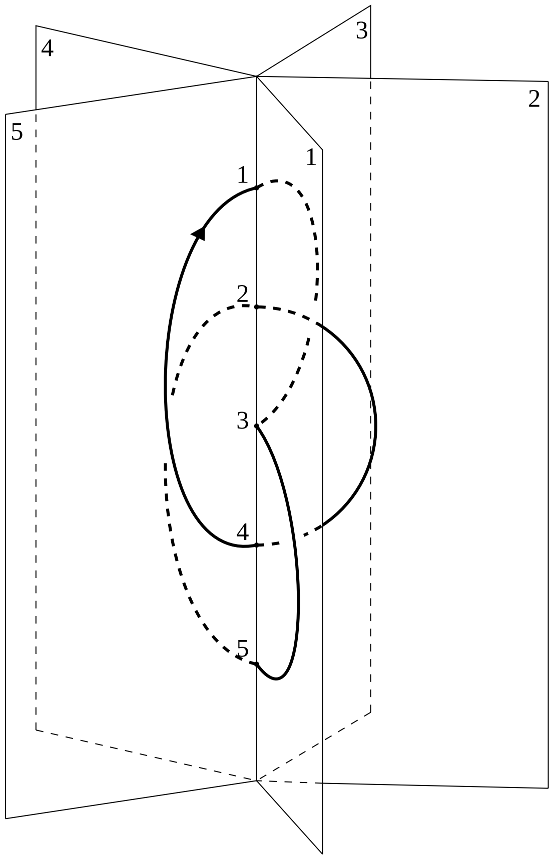

Fig. 1

However, in both these previous cases, the presentation was used only as a tool and, until now, no-one seems to have studied its properties in any detail. The first results in this area, many of which are outlined below, have been produced by Ian Nutt and myself.

1. Methods of describing arc presentations. The initial motivation for the presentation is strongly geometric and three-dimensional but the same information can be represented in many ways. Some are listed below.

1. Figure 2(a) shows a simplified version of the embedding of Figure 1. The straight line represents the binding circle and the semicircles represent the arcs. The order of the arcs around the binding circle is indicated by the numbering. (The same numbering is used in both figures.)

2. In Figure 2(c), the axis is drawn as a circle and the arcs are drawn as a layered sequence of chords. The numbering of the points along the binding circle corresponds to that in Figure 1.

3. In Figure 2(b) the arc-presentation has been projected onto a torus which has then been cut open and unrolled.

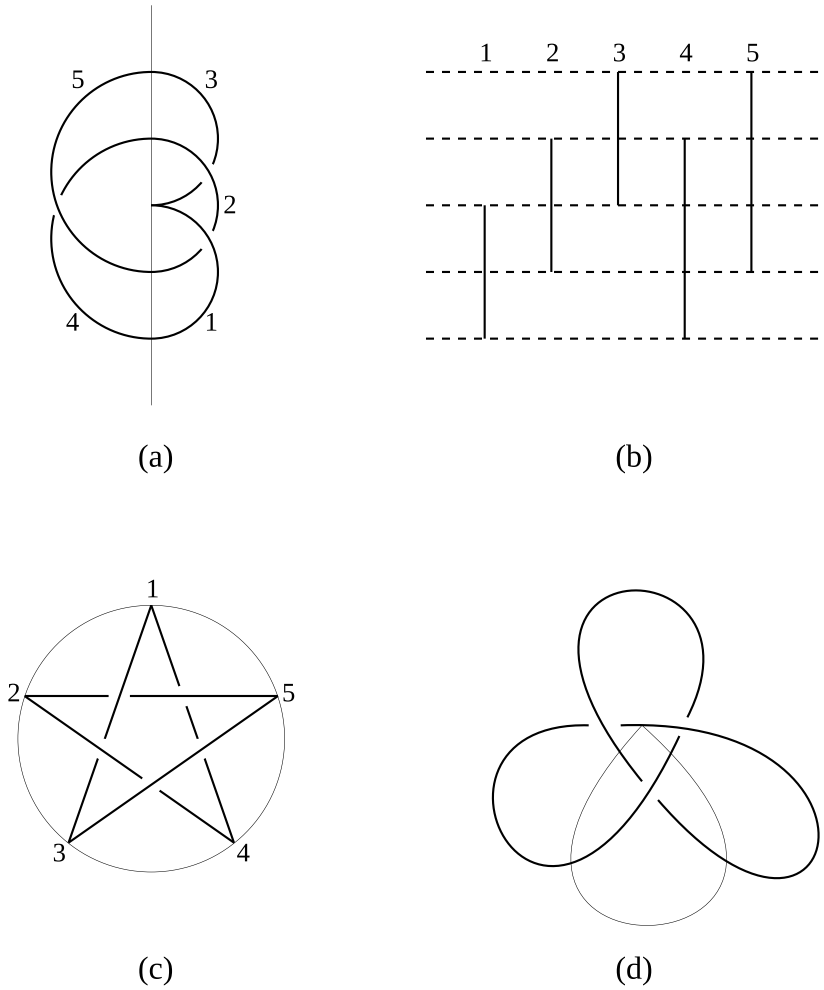

Fig. 2

To do this for an arc presentation of a link L we take a small solid tubular neighbourhood N of the axis such that ∂N intersects all the half-planes transversely in a single line and each component of $L \cap N$ meets the axis in a single point. Let $P = \{p_1, \cdots, p_n\}$ be the set of points where L meets the axis. We can define a height function h in N by projection onto the axis. Isotop L so that h is constant on each component of $L \cap N$. Replace the component of $L \cap N$ containing p_i with a loop λ_i in ∂N such that $h(\lambda_i) = h(p_i)$. Suppose that the arc in half-plane H_θ connects p_i to p_j. Draw an arc in $H_\theta \cap \partial N$ connecting the loops λ_i and λ_j.

In Figure 2(b) the solid vertical lines are the arcs. When the left and right edges of the diagram are identified the broken horizontal lines become the loops λ_i. In [Cr] this is called a *loops and lines diagram*.

4. In Figure 2(d) a binding circle has been added to an ordinary knot diagram. A procedure for doing this in general is given in [C-N].

5. Suppose that a link L meets the binding circle in n points. Number these points from 1 to n in order around the binding circle. There are also n half-planes containing an arc of L. Number these from 1 to n in order of increasing polar angle. Now take an $n \times n$ matrix and index the columns by arcs, and the rows by points of $L \cap A$. Set the (i, j) entry to 1 if arc i ends at point j, and to zero otherwise. This produces a matrix having exactly two 1's in each row and each column. A link orientation can be encoded by setting an entry to -1 when an arc departs from the axis and to $+1$ when it arrives. The matrix for the oriented trefoil of Figure 1 (starting from the arrowhead) is

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \end{pmatrix}$$

6. The embedding can also be described by a pair of permutations. Choose a start point on the knot and follow it round noting the order in which you visit both the half-planes and the axis. Starting from the arrow in Figure 1 we get (53142) for the order of the half-planes, and (13524) for the order of the points on the axis.

2. Properties of arc index. Some of the basic properties of arc presentations were discussed in [Cr]. For example, a set of moves is described which allow one arc presentation of a link to be converted into any other arc presentation of the same link. The behaviour of arc index under the knot operations of distant union and connected sum was also investigated.

Theorem [Cr].

$$\alpha(L_1 \sqcup L_2) = \alpha(L_1) + \alpha(L_2), \qquad \alpha(L_1 \# L_2) = \alpha(L_1) + \alpha(L_2) - 2.$$

Some simple observations from loops and lines diagrams lead to rather crude lower bounds on arc index. The notation $\text{breadth}_t(X)$ denotes the difference between the maximum and minimum degrees in the variable t in the knot polynomial X.

Theorem [C-N]. *Let L be a link with arc index $\alpha(L)$ and crossing number $c(L)$. Then*

$$\begin{aligned} \alpha(L) &\geq 1 + \sqrt{1 + 4\,c(L)} \quad \text{when } \alpha(L) \text{ is even.} \\ \alpha(L) &\geq 1 + \sqrt{4\,c(L)} \quad \text{when } \alpha(L) \text{ is odd.} \end{aligned}$$

Theorem [Cr]. *Let $s(L)$ denote the braid index of L. Then $\alpha(L) \geq 2s(L)$.*

Theorem [Cr] *Let $P_L(v, z)$ be the Homfly polynomial of a link L. Then*

$$\alpha(L) \geq \text{breadth}_v(P_L) + 2.$$

A much better lower bound is obtained with a corresponding increase in effort.

THEOREM [Nu1, Nu3]. *Let K be a knot and let $G_K(a,x) = F_K(a,x) \bmod 2$, that is, the Kauffman polynomial of K with coefficients reduced modulo 2. Then*

$$\alpha(K) \geq \mathrm{breadth}_a(G_K) + 2.$$

3. Arc index of 2-bridge knots. The 2-bridge knots and links have been studied in many contexts. Each member of the family has a diagram of the form shown in Figure 3. Each box contains a sequence of half-twists and the way in which the diagram is closed depends on whether there are an even or odd number of these twist-boxes. The narrow line drawn on each diagram represents a binding circle. The full arc presentation is obtained as shown in Figure 4. Where the binding circle runs adjacent to a twist-box the path of the knot diagram 'bounces' along it. The figure shows how a sequence of five half-twists can be represented by four such 'bounces'. In general, the link meets the binding circle $c(D) + 2$ times where $c(D)$ is the number of crossings in the diagram.

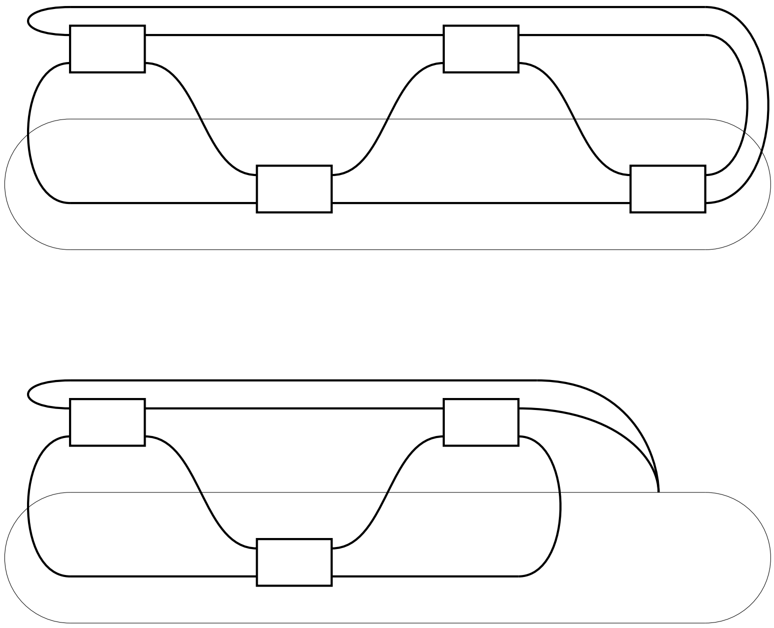

Fig. 3

Fig. 4

Proposition. *The construction just described produces an arc presentation of a 2-bridge link with* $c(D)+2$ *arcs.*

A more general construction of this type is described in [C-N]. A knot diagram is split up into twist-boxes and a binding circle is then threaded round the diagram so that it runs adjacent to each twist-box. The algorithm also leads to the upper bound $\alpha(L) \le c(L)+2$ on the class of diagrams to which it can be applied. The method applies to all but a specified set of diagrams. However, we do not know of any example where this bound fails.

To show that this arc presentation of 2-bridge knots cannot be improved we use Nutt's connection between arc index and the Kauffman polynomial. Lickorish has found a formula for the Kauffman polynomial of a 2-bridge link [Li]. Properties of the coefficients of the Kauffman polynomial of more general links have been studied by Thistlethwaite [Th1, Th2]. Using his results we can prove the following proposition (the proof is deferred to §5).

Proposition. *If* L *is an alternating link then* $\mathrm{breadth}_a(G_L) \ge c(L)$.

It is known that 2-bridge links are alternating so we can derive the following corollary.

Theorem. *If* K *is a 2-bridge knot then* $\alpha(K) = c(K)+2$.

4. Arc index of small knots. The knots with arc index at most ten have been determined in a computer assisted enumeration [Nu1]. The arc presentations were represented as pairs of permutations (method 6 above). Generating all combinations clearly covers every possibility but the notation has much redundancy. Applying some simple sieves removes many duplications and reduces the search space substantially making the task computationally reasonable. As far as possible, the resulting knots were identified by their Homfly polynomials.

Homfly polynomials of all the ten-arc presentations were generated but not all of them corresponded to easily identifiable knots. Knots with crossing numbers outside the range of easily accessible tables can have arc index 10. However, data on prime knots with up to ten crossings is fairly widespread; diagrams can be found in [Ro].

Empirical evidence had led us to conjecture [C-N] that $\alpha(L) = c(L)+2$ in the case of alternating links. Just as in the example of 2-bridge knots discussed above, these small knots provide further evidence in favour of this conjecture. We can obtain an upper bound on their arc index by construction [C-N] and a lower bound can be derived from the Kauffman polynomial. Combining these we get the following.

Theorem. *If* K *is an alternating knot with at most 10 crossings then* $\alpha(K) = c(K)+2$.

All the non-alternating knots with at most 10 crossings appear in the enumeration. The results are shown in the table below. The 10-crossing non-alternating knots that are not included have arc index 10. The notation used is that in [Ro].

5. The Kauffman polynomial and alternating links. The proof that $\mathrm{breadth}_a(G_L) \ge c(L)$ when L is an alternating link can be deduced as a special case

knot K	$\alpha(K)$	knot K	$\alpha(K)$
8_{19}	7	10_{124}	8
8_{20}	8	10_{128}	9
8_{21}	8	10_{132}	9
9_{42}	8	10_{136}	9
9_{43}	9	10_{139}	9
9_{44}	9	10_{140}	9
9_{45}	9	10_{142}	9
9_{46}	8	10_{145}	9
9_{47}	9	10_{160}	9
9_{48}	9	10_{161}	9
9_{49}	9		

of a more general result on graphs [Th2]. First, we need to establish some graph theoretic notation.

Let G be a graph with $V(G)$ vertices and $E(G)$ edges. Suppose the edges are labelled '+' or '−'. Then G_+ denotes the subgraph of G formed by deleting all the edges labelled '−', and $\overline{G_+}$ denotes the quotient graph formed by collapsing all the edges labelled '−'. The graphs G_- and $\overline{G_-}$ are defined similarly.

Now take G to be the graph derived from the chessboard shading of a diagram D by placing a vertex in each shaded region and connecting them through the crossings in the usual way. The edges of such a graph can be labelled '+' or '−' according to the sense of the crossings but for an alternating diagram all the labels are the same. Assume they are '+'.

The Kauffman polynomial $F_L(a,x)$ also comes in a framed version which is usually denoted $\Lambda_D(a,x)$. Since we are interested in the breadth in a of the polynomial, and the two versions differ only by multiplication by a power of a, we can consider the framed version. Now

$$\Lambda_D(a,x) \;=\; \sum_{i,j} \alpha_{i,j}\, a^i\, x^j$$

and $\alpha_{i,j}$ is non-zero only when $|i| + j \leq c(D)$. Thistlethwaite calls the coefficients for which $|i|+j = c(D)$ *outermost* and encapsulates them in two single variable polynomials:

$$\phi^+_D(t) \;=\; \sum_i \alpha_{i,n-i}\, t^i \quad \text{and} \quad \phi^-_D(t) \;=\; \sum_i \alpha_{-i,n-i}\, t^i$$

From Corollary 1.1 of [Th2] we get

Proposition. *Suppose that D is an irreducible connected alternating link diagram, and that G is its associated signed chessboard graph. Then the highest degree terms in ϕ^+_D and ϕ^-_D are monic and have degrees*

$$\begin{aligned}\text{degree}(\phi^+_D) &= \text{rank}(G_+) + V(\overline{G_-}) - 1\\ \text{degree}(\phi^-_D) &= \text{rank}(G_-) + V(\overline{G_+}) - 1.\end{aligned}$$

We can now easily establish the desired bound. Since all the edge labels in our graph are '+', we have $G_+ = G = \overline{G_+}$, also G_- has the same vertices as G but no edges, and

$\overline{G_-}$ is a single vertex. Therefore

$$\text{degree}(\phi_D^+) = \text{rank}(G) + 1 - 1, \qquad \text{degree}(\phi_D^-) = 0 + V(G) - 1.$$

So

$$\text{breadth}_a(G_L) \geq \text{rank}(G) + V(G) - 1 = \Big(E(G) - V(G) + 1\Big) + V(G) - 1 = E(G) = c(D).$$

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