KNOT THEORY
BANACH CENTER PUBLICATIONS, VOLUME 42
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

A GENERATING FUNCTION FOR SPIN NETWORK EVALUATIONS

BRUCE W. WESTBURY

Mathematics Institute, University of Warwick
Coventry CV4 7AL, England
E-mail: bww@maths.warwick.ac.uk

1. Introduction. The aim of these notes is to study the evaluation of a spin network. The evaluation of a spin network is defined in (Penrose [1971b]). The most important example is the evaluation of a tetrahedron which is closely related to the classical 6j-symbol. The aim of these notes is to combine two ideas; the first is a generating function for the classical 6j-symbols due to Schwinger, the second is the general formula for the evaluation of a spin network given in (Kauffman and Lins [1994]). The original paper by Schwinger was written in 1952 but not published until it appeared in (Schwinger [1965]). A clear account of this generating function was then given in (Bargmann [1962]). These papers compute this generating function by representing the universal enveloping algebra of $SL(2;\mathbb{C})$ by operators on a Bargmann-Fock space. The book (Kauffman and Lins [1994]) is mainly concerned with the quantum version of the evaluation of a spin network. However it does give a proof of Racah's formula for the classical tetrahedron symbol using the chromatic evaluation of the tetrahedron spin network and there is a discussion of the chromatic evaluation of a general planar spin network. In these notes we combine these two lines and give a generating function for the classical evaluation of an arbitrary planar spin network. The proof we give is purely combinatorial and is based on the chromatic evaluation; there is another proof based on the methods of Schwinger and Bargmann.

I would like to thank John Barrett for several helpful conversations.

2. Spin networks. In this section we follow (Penrose [1971b]). A spin network is a trivalent graph embedded in S^2 together with a labelling of each edge by a non-negative integer such that, for each vertex the three labels of the edges at the vertex form an admissible triple. A pair of embedded graphs are equivalent if they are isotopic. A triple

1991 *Mathematics Subject Classification*: Primary 57M.
The paper is in final form and no version of it will be published elsewhere.

(a, b, c) of non-negative integers is admissible if $a + b + c$ is even and if each of the three integers $a + b - c$, $a - b + c$ and $-a + b + c$ is non-negative. Often the trivalent graph embedded in S^2 is the 1-skeleton of the dual cell complex of a triangulation of S^2.

Let G be a trivalent graph embedded in S^2 and let $\mathcal{N}(G)$ be obtained by taking the boundary of a regular neighbourhood of G together with an edge for each edge of G. The edges of $\mathcal{N}(G)$ are called internal edges. It is conventional to draw each edge corresponding to an edge of G as a rectangle, or a bar. The diagram $\mathcal{N}(G)$ is called the strand network associated to G. The labelling of the edges of G determines a labelling of the edges of $\mathcal{N}(G)$. This is shown in Figure 1. The two sets of labels are related by the following inverse transformations:

$$\begin{aligned} a &= m + n, & m &= (a + b - c)/2, \\ b &= m + p, & n &= (a - b + c)/2, \\ c &= n + p, & p &= (-a + b + c)/2. \end{aligned}$$

The triple (a, b, c) is admissible if and only if each of m, n and p is a non-negative integer. Also a labelling of the edges of $\mathcal{N}(G)$ by non-negative integers arises from an admissible labelling of the edges of G if and only if, for each rectangle or bar, the sums of the two labels on the two sides of the bar are equal.

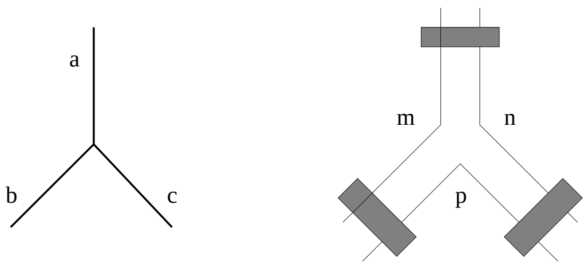

Fig. 1:

The evaluation of a spin network is given as follows: first draw each internal edge labelled by n as n parallel strands; then at each vertex connect the strand ends in pairs such that no two strands associated with the same internal edge are connected. Call a state a vertex connection at each vertex. Then the sign of a state S, $\varepsilon(S)$, is -1 to the power of the number of crossings and $|S|$ is the number of closed loops. Then the evaluation of a spin network, G, with internal edges labelled $\mathcal{T}$ is

$$Z(G; \mathcal{T}) = \sum_{S} \varepsilon(S)(-2)^{|S|}. \tag{1}$$

Note that this evaluation does not involve division by a factor $n!$ for each edge of G labelled n.

For each internal edge e introduce a formal variable z_e and adopt multi-index notation, so that

$$\mathcal{T}! = \prod_{e \in E(G)} \mathcal{T}(e)! \quad \text{and} \quad z^{\mathcal{T}} = \prod_{e \in E(G)} z_e^{\mathcal{T}(e)}$$

where $E(G)$ is the set of internal edges of G. Then the object of these notes is to give, for each G, a formula for the exponential generating function

$$Z(G) = \sum_{\mathcal{T}} Z(G;\mathcal{T})\left(\frac{z^{\mathcal{T}}}{\mathcal{T}!}\right).$$

EXAMPLE 1. Take $G = O$, the graph with one vertex and one edge. This is a degenerate spin network but still has an evaluation. Then the usual evaluation of O labelled by n is $(-1)^n(n+1)$; however since we have not divided by $n!$ the evaluation is $Z(O,n) = (-1)^n(n+1)!$. This gives

$$Z(O) = \sum_{n\geq 0} Z(O;n)\left(\frac{z^n}{n!}\right) = \sum_{n\geq 0}(-1)^n(n+1)z^n = (1+z)^{-2}. \tag{2}$$

A multi-cycle on G is a non-empty subset of the set of edges of G such that any vertex of G is a vertex of either zero or two elements of the subset. This is usually called a 2-regular subgraph. Note that this subgraph is not required to be connected. Denote the set of multi-cycles on G by $\Lambda(G)$. Now each multi-cycle, l, on G can be drawn canonically as a multi-cycle on $\mathcal{N}(G)$; this is illustrated in Figure 2. Hence each $l \in \Lambda(G)$ gives a subset, $p(l)$, of the set of internal edges. The rule for determining this set of internal edges is to look at each vertex of G which contains some edge of l: then this vertex contains precisely two edges of l and these two edges determine one of the three internal edges associated with the vertex. This set of internal edges is $p(l)$.

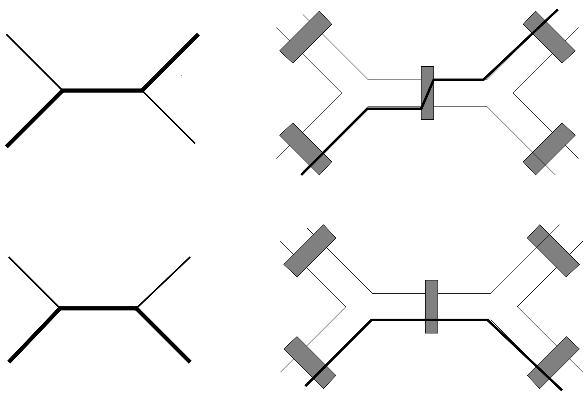

Fig. 2:

A more systematic way of producing these sets of internal edges is as follows. Each component of the complement of $G \subset S^2$ gives a multi-cycle by taking the edges of G which lie on the boundary of the component. More generally, each subset of the set of components of the complement of $G \subset S^2$ gives a multi-cycle by taking the edges of G which lie on the boundary of an odd number of elements of the subset. Then two subsets determine the same multi-cycle if and only if they are complementary subsets. It is clear that if l is a cycle corresponding to a component of the complement of $G \subset S^2$ then the set of internal edges, $p(l)$, makes up the corresponding boundary component of the regular neighbourhood of G. Now consider a subset of the set of components of the complement of $G \subset S^2$ and take the union of the corresponding sets of internal edges. Modify this set of internal edges by the following two rules: if all three internal edges at a vertex are included then remove all three of these internal edges: if two of the internal edges at a vertex are included then replace these two internal edges by the third remaining internal edge at the vertex. These rules at different vertices do not interact and so they can be applied at all vertices in any order. The resulting set of internal edges is $p(l)$.

Then the main result of these notes is:

Theorem 2. *For any planar trivalent graph, G, $Z(G)$ is given by*

$$Z(G) = \Big(1 + \sum_{l\in\Lambda(G)} \prod_{e\in p(l)} z_e\Big)^{-2}. \tag{3}$$

This theorem is proved in the next section. Here we give two examples. Although we give these as separate examples, in fact the θ-net is a special case of the tetrahedron as in can be obtained from the tetrahedron by putting one of the external edge labels equal to zero.

Example 3. The first example of this is the evaluation of the θ-net with external edges labelled by (a, b, c), where (a, b, c) is an admissible triple and internal edges labelled (m, n, p). The associated strand network is drawn in Figure 3. The left hand diagram shows the labels on the internal edges; any other labelling evaluates to zero. The right hand diagram shows the formal variables associated to each internal edge; the formal variable associated to the internal edge labelled i, j is z_{ij}, for $i = 1, 2$ and $j = 1, 2, 3$.

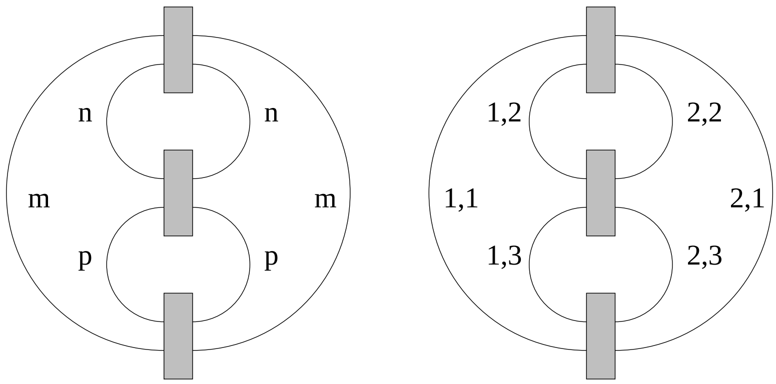

Fig. 3:

The evaluation is given by

$$\theta(m+n, n+p, p+m) = (-1)^{(m+n+p)}(m+n+p+1)!m!n!p!.$$

Then the generating function is

$$Z(\theta) = (1 + z_{11}z_{21} + z_{12}z_{22} + z_{13}z_{23})^{-2}.$$

Expanding this generating function gives

$$\begin{aligned} Z(\theta) &= (1 + z_{11}z_{21} + z_{12}z_{22} + z_{13}z_{23})^{-2} \\ &= \sum_{N\geq 0} (-1)^N (N+1)(z_{11}z_{21} + z_{12}z_{22} + z_{13}z_{23})^N \end{aligned}$$

$$= \sum_{N\geq 0}(-1)^N(N+1)\sum_{m+n+p=N}\left(\frac{N!}{m!n!p!}\right)(z_{11}z_{21})^m(z_{12}z_{22})^n(z_{13}z_{23})^p$$
$$= \sum_{m+n+p\geq 0}(-1)^{m+n+p}\frac{(m+n+p+1)!}{m!n!p!}(z_{11}z_{21})^m(z_{12}z_{22})^n(z_{13}z_{23})^p$$
$$= \sum_{m+n+p\geq 0}(-1)^{m+n+p}(m+n+p+1)m!n!p!\frac{(z_{11}z_{21})^m(z_{12}z_{22})^n(z_{13}z_{23})^p}{(m!n!p!)^2}.$$

The next example is the tetrahedron symbol. The original references are (Schwinger [1965]) (where the signs are different) and (Bargmann [1962]). The following calculation is given in (Biedenharn and Louck [1981, 5.9 Appendix D]). This calculation shows that the generating function for the tetrahedron symbol gives a formula equivalent to Racah's well-known formula for the $6j$-symbol.

EXAMPLE 4. The strand network associated to the tetrahedron is drawn in Figure 4. This diagram shows the formal variables associated to each internal edge; the formal variable associated to the internal edge labelled i,j is z_{ij}, for $i = 1,2,3$ and $j = 1,2,3,4$.

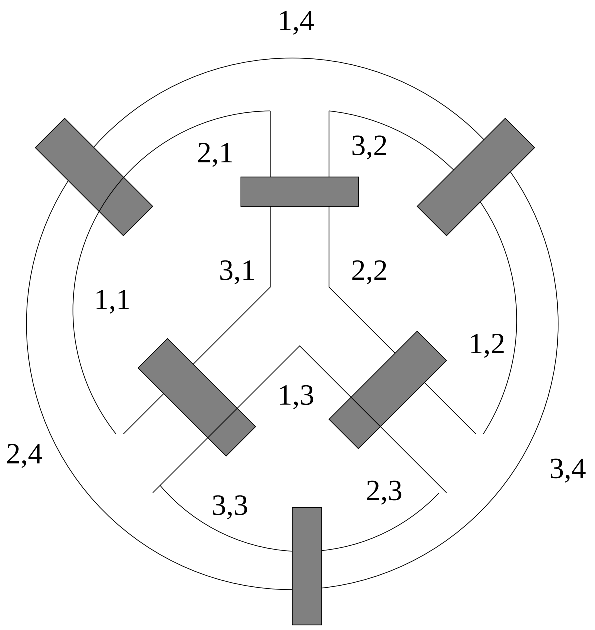

Fig. 4:

For $j = 1,2,3,4$ put $R_j = z_{1j}z_{2j}z_{3j}$ and for $i = 1,2,3$ put $C_i = z_{i1}z_{i2}z_{i3}z_{i4}$. Then the generating function is

$$Z(T) = \Big(1 + \sum_{j=1,2,3,4} R_j + \sum_{i=1,2,3} C_i\Big)^{-2}.$$

Expanding this generating function gives

$$\begin{aligned} Z(T) &= \Big(1 + \sum_{j=1,2,3,4} R_j + \sum_{i=1,2,3} C_i\Big)^{-2} \\ &= \sum_{N\geq 0}(-1)^N(N+1)\Big(\sum_{j=1,2,3,4} R_j + \sum_{i=1,2,3} C_i\Big)^N \\ &= \sum_{N\geq 0}(-1)^N(N+1)\sum_{|\alpha|+|\beta|=N}\Big(\frac{N!}{\alpha!\beta!}\Big)\Big(\prod_{j=1,2,3,4} R_j^{\alpha_j}\Big)\Big(\prod_{i=1,2,3} C_i^{\beta_i}\Big) \\ &= \sum_{\alpha,\beta}(-1)^{(|\alpha|+|\beta|)}\frac{(|\alpha|+|\beta|+1)!}{\alpha!\beta!}\prod_{i,j} z_{ij}^{\alpha_j+\beta_i} \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are multi-indices. The coefficient of $\prod_{i,j} z_{ij}^{k_{ij}}/k_{ij}!$ is

$$\Big(\prod_{i,j} k_{ij}!\Big)\sum_{\alpha,\beta:\alpha_j+\beta_i=k_{ij}}(-1)^{(|\alpha|+|\beta|)}\frac{(|\alpha|+|\beta|+1)!}{\alpha!\beta!}.$$

If the set of equations $\alpha_j+\beta_i = k_{ij}$ has no solution then the coefficient of $\prod_{i,j} z_{ij}^{k_{ij}}/k_{ij}!$ is zero. Otherwise there is a unique solution to the equations

$$q_i - p_j = k_{ij} \quad \text{and} \quad \sum_j p_j = \sum_i q_i.$$

This can be seen as follows. First sum $q_i - p_j = k_{ij}$ over both i and j. This gives

$$\sum_j p_j = \sum_i q_i = \sum_{i,j} k_{ij}.$$

Now sum $q_i - p_j = k_{ij}$ over i with j fixed and also over j with i fixed. This gives

$$3p_j = \sum_{i,j} k_{ij} - \sum_i k_{ij} \quad \text{and} \quad 4q_i = \sum_{i,j} k_{ij} + \sum_j k_{ij}.$$

Then putting $(z - p_j)$ for α_j and $(q_i - z)$ for β_i, the coefficient of $\prod_{i,j} z_{ij}^{k_{ij}}/k_{ij}!$ is

$$\Big(\prod_{i,j} k_{ij}!\Big)\sum_z \frac{(-1)^z(z+1)!}{(\prod_i(q_i-z)!)(\prod_j(z-p_j)!)}.$$

These two examples have the property that every multi-cycle has one component. The simplest example which shows that we need to include multi-cycles with more than one component is the following:

Example 5. Let $G = OO$, the disjoint union of two copies of O. It is clear that $Z(G)$ is multiplicative under disjoint union and so

$$Z(OO) = (1+x)^{-2}(1+y)^{-2} = (1+x+y+xy)^{-2}.$$

3. Chromatic evaluation. The chromatic evaluation of a spin network is defined by a tensorial calculation. This evaluation is mentioned in (Penrose [1971a]) and is discussed in (Moussouris [1979]) and (Kauffman and Lins [1994, Chapter 8]). Let V be a finite dimensional vector space. Then there is a standard action of the symmetric group $S(n)$

on $\otimes^n V$, for each $n > 0$. Put a copy of V on each strand in the strand network and, for each external edge labelled n, put the following anti-symmetric quasi-idempotent in the corresponding rectangle on the strand network,

$$\sum_{\sigma \in S(n)} \varepsilon(\sigma)\sigma$$

where $\varepsilon(\sigma)$ is the sign of the permutation σ. This is a quasi-idempotent and not an idempotent as we have not divided by $n!$. The tensor associated to each strand of the strand network is the identity map. Inserting these tensors and contracting in the usual way gives the chromatic evaluation of the spin network. The first observation is that this evaluation is also given by

$$Z_N(G;\mathcal{T}) = \sum_S \varepsilon(S)(N)^{|S|}$$

where the notation is the same as in (1) and N is the dimension of V. This observation shows that the original evaluation is obtained from the chromatic evaluation by substituting -2 for N, which is (Kauffman and Lins [1994, 8.2 Theorem 3]).

Definition 6. The *chromatic generating function* is defined by

$$Z_N(G) = \sum_{\mathcal{T}} Z_N(G;\mathcal{T})\left(\frac{z^{\mathcal{T}}}{\mathcal{T}!}\right).$$

The aim of this section is to prove the following theorem:

Theorem 7. *For any planar trivalent graph, G, the chromatic generating function $Z_N(G)$ is given by*

$$Z_N(G) = \Big(1 + \sum_{l \in \Lambda(G)} \prod_{e \in p(l)} z_e\Big)^N.$$

Then theorem (2) is obtained as an immediate corollary by substituting -2 for N. The proof of this theorem we give here is purely combinatorial and is based on the proof of (Kauffman and Lins [1994, 8.4 Theorem 4]).

Example 8. The simplest illustration of this theorem is the case $G = O$. If the label is n then the chromatic evaluation is

$$Z_N(O, n) = \binom{N}{n} n!.$$

This gives the following chromatic generating function

$$Z_N(O) = \sum_{n \geq 0} Z_N(O;n)\left(\frac{z^n}{n!}\right) = \sum_{n \geq 0} \binom{N}{n} z^n = (1+z)^N.$$

Putting -2 for N gives (2) as expected.

Proof. An alternative description of the chromatic evaluation is that it is the number of diagrams which satisfy the two conditions

1. no loop can go through a given bar more than once
2. loops that share an internal edge receive different colours

Note that with this description of the chromatic evaluation the special case of the theorem with $N = 1$ is clear. For the general case: assume we are given a coloured diagram satisfying these two conditions. Let I be the subset of the set of colours that are actually used. Construct a function $\mathcal{L}$ on multi-cycles as follows. For each colour, $n \in I$, the set of all loops coloured n is a multi-cycle. This gives a set of $|I|$ multi-cycles and $\mathcal{L}(l)$ is the number of times l occurs in this set. Note that $|\mathcal{L}| = |I|$ and that $\mathcal{L}$ determines $\mathcal{T}$. Now count the number of diagrams for a given $\mathcal{L}$. The number of these diagrams is

$$\mathcal{T}(\mathcal{L})!\binom{N}{|\mathcal{L}|}\binom{|\mathcal{L}|}{\mathcal{L}}.$$

This is because there are $\binom{N}{|\mathcal{L}|}$ ways to choose the colours I, $\binom{|\mathcal{L}|}{\mathcal{L}}$ ways to colour the diagram once the colours have been chosen and it is shown in the proof of (Kauffman and Lins [1994, 8.4 Theorem 4]) that there are $\mathcal{T}(\mathcal{L})!$ diagrams for a given $\mathcal{L}$. We can insert an arbitrary permutation on the strands running through any internal edge; the idea of the proof is that any two diagrams are related by a unique insertion of a permutation on each strand.

Hence the exponential generating function is

$$Z_N(G) = \sum_{|\mathcal{L}|\le N} \mathcal{T}(\mathcal{L})!\binom{N}{|\mathcal{L}|}\binom{|\mathcal{L}|}{\mathcal{L}}\frac{z^{\mathcal{T}(\mathcal{L})}}{\mathcal{T}(\mathcal{L})!} = \sum_{|\mathcal{L}|\le N}\binom{N}{|\mathcal{L}|}\binom{|\mathcal{L}|}{\mathcal{L}} z^{\mathcal{T}(\mathcal{L})}.$$

On the other hand,

$$\begin{aligned}\Big(1+\sum_{l\in\Lambda(G)}\prod_{e\in p(l)} z_e\Big)^N &= \sum_{i=0}^{N}\binom{N}{i}\Big(\sum_{l\in\Lambda(G)}\prod_{e\in p(l)} z_e\Big)^i \\ &= \sum_{i=0}^{N}\binom{N}{i}\sum_{|\mathcal{L}|=i}\binom{i}{\mathcal{L}} z^{\mathcal{T}(\mathcal{L})} = \sum_{|\mathcal{L}|\le N}\binom{N}{|\mathcal{L}|}\binom{|\mathcal{L}|}{\mathcal{L}} z^{\mathcal{T}(\mathcal{L})}. \quad\blacksquare\end{aligned}$$

4. Quantum evaluation. In this article we have only been concerned with the classical evaluation of a spin network. There is also quantum evaluation, originally given in terms of the representation theory of the quantised enveloping algebra of $SU(2)$. This approach also gives a tensorial description of this evaluation. An alternative description of quantum evaluation is the main topic of the first nine chapters of (Kauffman and Lins [1994]).

The usual classical evaluation involves division by a factor $n!$ for each edge of G with label n but this factor has not been included in the evaluation we have been using. Similarly the quantum evaluation includes division by $[\mathcal{E}]!$ where $\mathcal{E}$ is the multi-index associated to the labels on the edges of G. If this factor is left out the quantum evaluation gives a Laurent polynomial with integer coefficients and so it is natural to ask if this has a combinatorial interpretation.

It is natural to ask whether theorem (2) has a q-analogue. Although the formula we have arrived at has an obvious q-analogue this particular q-analogue is incorrect. (This was pointed out to me by Rick Litherland.) Here we consider two particular spin networks, the tetrahedral network and the double θ-network shown in Figure 5.

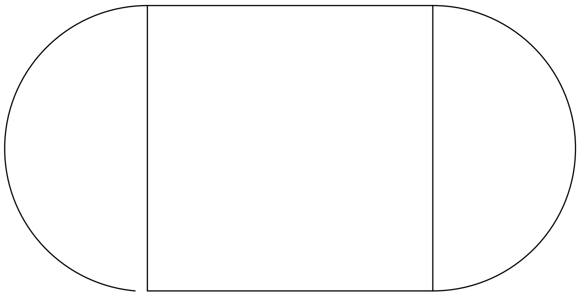

Fig. 5:

For these two examples, there is an explicit formula for the quantum evaluation for a general labelling. I do not know a formula for a general spin network which generalises theorem (2) and which gives the correct answer for these two examples. Consequently, the problem of finding a q-analogue of theorem (2) is unsolved.

The naive q-analogue of theorem (2) for the evaluation of a graph G with internal edges labelled $\mathcal{T}$ is

$$[\mathcal{T}]! \sum_{\mathcal{L}:\mathcal{T}(\mathcal{L})=\mathcal{T}} \frac{(-1)^{|\mathcal{L}|}[|\mathcal{L}|+1]!}{[\mathcal{L}]!} \tag{4}$$

where for any integer n, $[n]!$ is defined by $[n]! = 0$ if $n < 0$ and otherwise

$$[n]! = \prod_{k=1}^{n} \left(\frac{q^k - q^{-k}}{q - q^{-1}} \right)$$

and the notation $[\mathcal{T}]!$ and $[\mathcal{L}]!$ is defined by

$$[\mathcal{T}]! = \prod_{e \in E(G)} [\mathcal{T}(e)]! \qquad \text{and} \qquad [\mathcal{L}]! = \prod_{l \in \Lambda(G)} [\mathcal{L}(l)]!.$$

The formula (4) is shown to give the correct answer for the tetrahedral spin network in (Masbaum and Vogel [1994]). This is proved by an inductive argument.

Next we discuss the double θ-network. This is the simplest spin network which is 2-connected and which has a multi-cycle with more than one component. The associated strand network together with notation for the labels on the strands is shown in Figure 6. If the labelling is not of this form then the evaluation is zero. Denote the common value of $a_1 + b_1 = a_2 + b_2$ by n.

The quantum evaluation is given by using (Kauffman and Lins [1994, 5.1 Lemma 7 & 6.3 Corollary 2]). The result is

$$(-1)^{n+c_1+c_2} \frac{[n+c_1+1]![n+c_2+1]![a_1]![b_1]![c_1]![a_2]![b_2]![c_2]!}{[n+1]}.$$

In particular, if all the labels are taken to be 1, this gives

$$[4]^2[3][2]^2. \tag{5}$$

On the other hand, by theorem (2), the classical evaluation is given by the coefficient

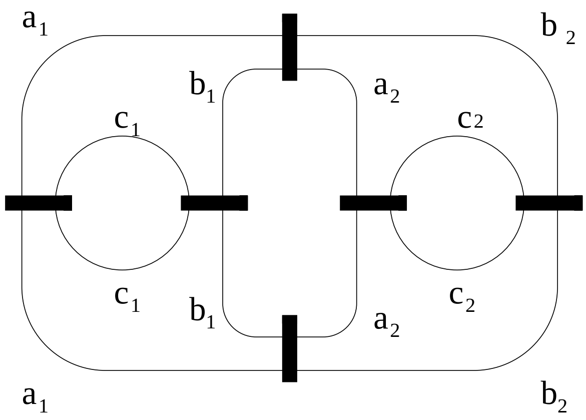

Fig. 6:

of

$$x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2} z_1^{c_1} z_2^{c_2}$$

in the formal power series expansion of

$$[(1+z_1)(1+z_2)+(x_1+x_2)(y_1+y_2)]^{-2}.$$

Then, again taking all labels to be 1, the prediction given by (4) is

$$[5]!+[5]!-[4]!-[4]!.$$

This prediction does not agree with (5) since $2([5]-1) \neq [4][2]$ and so formula (4) is not correct.

References

V. Bargmann [1962], *On the representations of the rotation group*, Rev. Modern Phys. 34, 829.

L. C. Biedenharn and J. D. Louck [1981], “Angular momentum in quantum physics”, Encyclopedia of Mathematics and its applications, vol. 8, Addison-Wesley Publishing Company.

L. H. Kauffman and S. L. Lins [1994], “Temperley-Lieb recoupling theory and invariants of 3-manifolds”, Annals of Mathematics Studies, vol. 134, Princeton University Press.

G. Masbaum and P. Vogel [1994], *3-valent graphs and the Kauffman bracket*, Pacific J. Math. 164, 361–381.

J. P. Moussouris [1979], *The chromatic evaluation of strand networks*, “Advances in twistor theory”, Huston and Ward, eds., Research Notes in Mathematics, Pitman Publ., 308–312.

R. Penrose [1971a], *Applications of negative dimensional tensors*, “Combinatorial mathematics and its applications”, D. J. A. Welsh, ed., Academic Press.

— [1971b], *Angular momentum: an approach to combinatorial space-time*, in “Quantum theory and beyond”, T. Bastin, ed., Cambridge Univ. Press, New York, NY, 151–180.

J. Schwinger [1965], *On angular momentum*, in “Quantum theory of angular momentum”, L. C. Biedenharn and H. V. Dam, eds., Perspectives in Physics, Academic Press.