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GENERALIZED n-COLORINGS OF LINKS

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Abstract. The notion of an (n, r)-coloring for a link diagram generalizes the idea of an n-coloring introduced by R. H. Fox. For any positive integer n the various (n, r)-colorings of a diagram for an oriented link l correspond in a natural way to the periodic points of the representation shift $\Phi_{\mathbf{Z}/n}(l)$ of the link. The number of (n, r)-colorings of a diagram for a satellite knot is determined by the colorings of its pattern and companion knots together with the winding number.

1. Introduction. Tricoloring, introduced by R. H. Fox around 1960, is an elementary technique that distinguishes a trefoil knot from a trivial knot [CrFo], [Fo1], [Fo2]. A *tricoloring* of a link diagram is an assignment of colors to the arcs of the diagram using three colors such that at any crossing either all three colors appear or only one color appears. Any diagram has a trivial, monochromatic tricoloring — in fact, three of them. It is easily checked that the number of tricolorings of a diagram is unaffected by Reidemeister moves and hence is a numerical invariant of the link. We can deduce that a trefoil knot is different from a trivial knot simply by observing that the former has a nontrivial tricoloring. Complete details of the argument can be found in [Pr].

By broadening our palette, using n colors identified with the elements of the cyclic group $\mathbf{Z}/n$, we arrive at the more general notion of *n-coloring.* An *n-coloring* of a link diagram is an assignment of colors to the arcs such that at any crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo n. The idea but not the terminology can be found in Chapter 10 of [Fo1]. (The necessary mathematics was known to Reidemeister [Re].) Again one can check that the number of n-colorings of a diagram is unchanged by Reidemeister moves. Figure 1 shows a nontrivial 5-coloring of the figure eight knot 4_1. It is known that the knot has a nontrivial n-coloring if and only if n is a multiple of 5.

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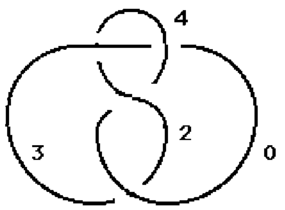

Fig. 1. 5-coloring of figure eight knot diagram

In this paper we introduce a further generalization of tricoloring called (n, r)-coloring, where n and r are positive integers and $r \geq 2$. Any n-coloring is an $(n, 2)$-coloring and conversely. For any link and positive integer n, the (n, r)-colorings for all r can be determined from a single finite graph Γ. The graph Γ describes a *representation shift* introduced in [SiWi1] using techniques of symbolic dynamical systems (see also [SiWi2]). We use the techniques to compute the number of (n, r)-colorings of a satellite knot in terms of the colorings of its pattern and companion knots.

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2. (n, r)-colorings and representations

Definition 2.1. Assume that D is a diagram of an oriented link. An (n, r)-*coloring*, for positive integers n and r with $r \geq 2$, is an assignment of $(r-1)$-tuples (*color vectors*) $C \in (\mathbf{Z}/n)^{r-1}$ to the arcs of D such that at any crossing

$$(C_i - C_k) \cdot S_r^{\epsilon} = C_j - C_k. \tag{2.1}$$

Here C_k corresponds to the overcrossing, $C_i,\ C_j$ correspond to the undercrossings, $\epsilon = \pm 1$ is the algebraic sign of the crossing (see Figure 2), and S_r is the companion matrix of the cyclotomic polynomial of degree $r - 1$; i.e.,

$$S_r = \begin{pmatrix} 0 & 0 & \dots & 0 & -1 \\ 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}.$$

When $r = 2$ condition (2.1) reduces to the familiar n-coloring condition that the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo n.

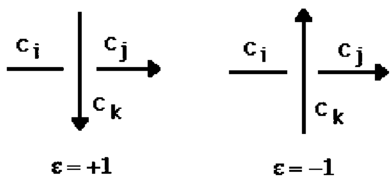

Fig. 2

Given any diagram D of an oriented link we will denote the number of its (n, r)-colorings by $col_{n,r}(D)$. The following result can be proved by elementary techniques. However, it will also follow from results in Section 4 (see Theorem 4.3).

PROPOSITION 2.2. *If D and D' are any two diagrams of an oriented link l, then $col_{n,r}(D) = col_{n,r}(D')$. Consequently, $col_{n,r}(D)$ is an invariant $col_{n,r}(l)$ of the link.*

Proof. The diagram D can be converted into D' by a finite sequence of Reidemeister moves. It suffices to check that the number of (n, r)-colorings of any diagram is unaffected by each of the three Reidemeister moves and their inverses. We leave the details to the reader. ■

Assume that D is a diagram for an oriented link l. The same diagram with reversed orientation, denoted by $r(D)$, is a diagram for a link $r(l)$. The set of (n, r)-colorings of D is in one-to-one correspondence with the set of (n, r)-colorings of $r(D)$. In fact, given an (n, r)-coloring of D we obtain an (n, r)-coloring of $r(D)$ by reversing the order of the components of each color vector. Consequently, the number $col_{n,r}(k)$ is an unoriented knot invariant. However, changing the orientation of only some of the components of a link l can change $col_{n,r}(l)$ when $r > 2$ (see Example 4.4).

PROPOSITION 2.3. *Assume that D is a diagram for an oriented link l. If D can be (n, r)-colored for some n and r, then D can be (an, br)-colored for any positive integers a and b.*

Proof. Since $\mathbf{Z}/n$ can be embedded as a subgroup in $\mathbf{Z}/an$, it is immediate that D can be (an, r)-colored. Assume that we have an (an, r)-coloring of D. Replacing each color vector $(c_1, \ldots, c_{r-1})$ by $(c_1, \ldots, c_r, c_1, \ldots, c_{r-1})$, where $c_r = -c_1 - \ldots - c_{r-1}$, results in an $(an, 2r)$-coloring of D. By induction D can be (an, br)-colored. ■

DEFINITION 2.4. Assume that D is a diagram for an oriented link with a distinguished arc δ. A *based (n, r)-coloring* of D is an (n, r)-coloring in which δ receives the trivial color vector.

Since the set of all (n, r)-colorings of D obviously forms a module over $\mathbf{Z}/n$, the number of based (n, r)-colorings of D is independent of the distinguished arc δ. We will denote the number of based (n, r)-colorings of D by $col^0_{n,r}(D)$. Clearly $col_{n,r}(D) = n^{r-1} \cdot col^0_{n,r}(D)$. It follows immediately from Proposition 2.2 that $col^0_{n,r}(D)$ is also an invariant $col^0_{n,r}(l)$ of the link.

EXAMPLE 2.5. No diagram for the figure eight knot can be tricolored (i.e., $(3, 2)$-colored) in a nontrivial manner. Figure 3 shows that a diagram can be nontrivially $(3, 4)$-colored.

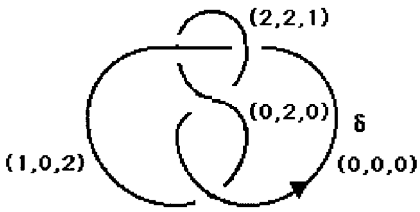

Fig. 3. Based $(3, 4)$-coloring of figure eight knot diagram

3. Representation shifts associated to links. Let $l = l_1 \cup \ldots \cup l_\mu$ be any oriented link with tubular neighborhood $N = N_1 \cup \ldots \cup N_\mu$. Let G denote the group $\pi_1(S^3 - l, *)$

of the link, where the basepoint $*$ is chosen on the boundary ∂N_1, and let x be the class of a meridian m of l_1 (with orientation induced by l_1.) The *total linking number homomorphism* $\chi : G \to \mathbf{Z}$ is the homormorphism that maps each meridian of l to $1 \in \mathbf{Z}$. We will denote the kernel of χ by K. If $\mu = 1$ then l is a knot, χ is the abelianization homomorphism and K is the commutator subgroup $[G, G]$.

Definition 3.1. Let Σ be a finite group. The *representation shift* $\Phi_\Sigma(l)$ (or simply Φ_Σ) of the link l is the set $Hom(K, \Sigma)$ of representations $\rho : K \to \Sigma$ together with the *shift map* $\sigma_x : \Phi_\Sigma \to \Phi_\Sigma$ defined by $\sigma_x\rho(a) = \rho(x^{-1}ax)$ for all $x \in K$. The set Φ_Σ has a natural topology determined by the basis sets $\mathcal{N}_{a_1,\ldots,a_s}(\rho) = \{\rho' : \rho'(a_i) = \rho(a_i),\ i = 1, \ldots, s\}$ for all $\rho \in \Phi_\Sigma$, $a_1, \ldots, a_s \in K$.

For brevity we sometimes refer to the representation shift Φ_Σ without explicit mention of the shift map σ_x. It is a straightforward matter to check that σ_x is a homeomorphism of Φ_Σ. The pair (Φ_Σ, σ_x) is an example of a *dynamical system*, by which we mean a compact topological space together with a homeomorphism. Dynamical systems (Φ_1, σ_1) and (Φ_2, σ_2) are said to be *topologically conjugate* if $h \circ \sigma_1 = \sigma_2 \circ h$ for some homeomorphism $h : \Phi_1 \to \Phi_2$. One easily checks using the uniqueness up to isotopy of tubular neighborhoods that the link type of l determines the representation shift (Φ_Σ, σ_x) up to topological conjugacy. This implies, in particular, that the set $\operatorname{Fix} \sigma_x^r = \{\rho : \sigma_x^r \rho = \rho\}$ of period r representations is an invariant of l for each $r \geq 0$.

Theorem 3.1 of [SiWi1] states that the representation shift Φ_Σ is a special sort of dynamical system known as a *shift of finite type.* Such a system can be completely described by finite directed graph Γ. The elements of Φ_Σ correspond to the bi-infinite paths in Γ in such a way that the representations with period r correspond to the closed paths of length r. We will construct Γ for a specific example and then describe the general construction.

Example 3.2. We consider the knot $k = 5_2$ oriented as in Figure 4a with Wirtinger generators indicated. The group $G = \pi_1(S^3 - k, *)$ has presentation

$$\langle x_1,\ x_2,\ x_3,\ x_4,\ x_5 \mid x_3x_2 = x_2x_1,\ x_2x_3 = x_3x_4,\ x_5x_1 = x_1x_2,\ x_4x_5 = x_5x_3\rangle.$$

We use the first three relators to eliminate x_3, x_4 and x_5 from the presentation, obtaining

$$\langle x_1,\ x_2 \mid x_2x_1^{-1}x_2x_1x_2^{-1}x_1x_2x_1^{-1}x_2x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}\rangle.$$

The Reidemeister–Schreier Theorem [LySc] enables us to find a presentation for the kernel K, which is the commutator subgroup of G. First we replace x_2 by x_1a (i.e., we introduce a new generator a and eliminate x_2 by Tietze moves). For notational convenience we will write x instead of x_1. The following presentation for G results.

$$\langle x,\ a \mid a^2 \cdot xa^{-1}x^{-1} \cdot x^2a^2x^{-2} \cdot xa^{-2}x^{-1}\rangle$$

The kernel K is generated by the elements a_i, $i \in \mathbf{Z}$, where $a_i = x^{-i}ax^i$. Defining relations are obtained by conjugating the relation in the last presentation by powers of x and then rewriting those words in terms of the a_i:

$$K = \langle a_i \mid a_{i+2}^2a_{i+1}^{-1}a_i^2a_{i+1}^{-2},\ i \in \mathbf{Z}\rangle$$

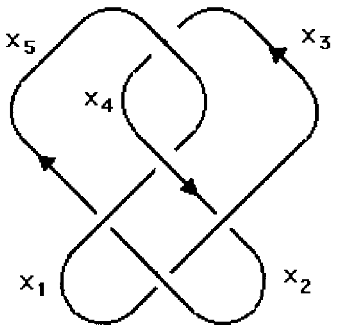

Fig. 4a. The knot $k = 5_2$

$\circlearrowleft(0,0)$ $\quad (1,6) \rightleftarrows (6,1)$ $\quad (2,5) \rightleftarrows (5,2)$ $\quad (3,4) \rightleftarrows (4,3)$

$(1,0) \to (0,6) \to (6,2) \to (2,4) \to (4,4) \to (4,2) \to (2,6)$
$(5,1) \to (3,5) \leftarrow (3,3) \leftarrow (5,3) \leftarrow (1,5) \leftarrow (0,1) \leftarrow (6,0)$

$(2,0) \to (0,5) \to (5,4) \to (4,1) \to (1,1) \to (1,4) \to (4,5)$
$(3,2) \leftarrow (6,3) \leftarrow (6,6) \leftarrow (3,6) \leftarrow (2,3) \leftarrow (0,2) \leftarrow (5,0)$

$(3,0) \to (0,4) \to (4,6) \to (6,5) \to (5,5) \to (5,6) \to (6,4)$
$(1,3) \leftarrow (2,1) \leftarrow (2,2) \leftarrow (1,2) \leftarrow (3,1) \leftarrow (0,3) \leftarrow (4,0)$

Fig. 4b. Graph Γ for $k = 5_2$

We regard the relation $a_{i+2}^2 a_{i+1}^{-1} a_i^2 a_{i+1}^{-2}$ as a word $r = r(a_i, a_{i+1}, a_{i+2})$. A representation $\rho : K \to \Sigma$ is a function ρ from the set of generators a_i into Σ such that for every $i \in \mathbf{Z}$ the element $r(\rho(a_i), \rho(a_{i+1}), \rho(a_{i+2}))$ is trivial in Σ. Any such function can be constructed as follows, beginning with Step 0 and proceeding to Steps $\pm 1, \pm 2$, etc.:

$\vdots$

(Step -2) Choose $\rho(a_{-2})$ if possible such that $r(\rho(a_{-2}), \rho(a_{-1}), \rho(a_0)) = e$.
(Step -1) Choose $\rho(a_{-1})$ if possible such that $r(\rho(a_{-1}), \rho(a_0), \rho(a_1)) = e$.
(Step 0) Choose values $\rho(a_0)$ and $\rho(a_1)$.
(Step $+1$) Choose $\rho(a_2)$ if possible such that $r(\rho(a_0), \rho(a_1), \rho(a_2)) = e$.
(Step $+2$) Choose $\rho(a_3)$ if possible such that $r(\rho(a_1), \rho(a_2), \rho(a_3)) = e$.

$\vdots$

The process of selecting values $\rho(a_i)$ is accomplished by following any bi-infinite path on a directed graph Γ. The vertices of Γ are maps $\rho_0 : \{a_0, a_1\} \to \Sigma$, each of which can be regarded as an ordered pair $(\rho_0(a_0), \rho_0(a_1))$. There is a directed edge from ρ_0 to ρ_0' if and only if (1) $\rho_0(a_1) = \rho_0'(a_0)$ and (2) $r(\rho_0(a_0), \rho_0(a_1), \rho_0'(a_1)) = e$. Conditions

(1) and (2) enable us to extend the function $\rho_0 : \{a_0, a_1\} \to \Sigma$ by defining $\rho_0(a_2)$ to be equal to $\rho_0'(a_1)$. Now if there is an edge from ρ_0' to ρ_0'' we can likewise extend ρ_0 by defining $\rho_0(a_3)$ to be $\rho_0''(a_1)$. In fact, a bi-infinite path in the graph corresponds to a map from the generating set of K to Σ which sends all relators to the identity element, and hence corresponds to a representation of K. When $\Sigma = \mathbf{Z}/7$, condition (2) becomes $\rho_0'(a_1) \equiv 5\rho_0(a_1) - \rho_0(a_0) \pmod 7$. The graph Γ (see Figure 4b) consists of seven disjoint cycles: a single cycle of length 1; three cycles of length 2; three cycles of length 14. Later we will show how the graph can be used to construct the 6 nontrivial $(7, 2)$-colorings of a diagram for the knot 5_2.

We now describe the graph Γ corresponding to an arbitrary link. The associated group K has a presentation of the form

$$K = \langle a_{i,j} \mid r_{k,j} \rangle,$$

where $1 \le i \le n$, $1 \le k \le m$, $j \in \mathbf{Z}$. Each relator $r_{k,j}$ is a word in the generators $a_{i,j}$ such that $r_{k,q+t}$ is obtained from $r_{k,q}$ by adding t to the second subscript of every symbol in $r_{k,q}$. (Presentations of this type were studied in [HaKe].) Without loss of generality we can assume that if any word $r_{1,0}, \ldots, r_{n,0}$ contains $a_{i,j}$ then it contains $a_{i,0}$ but does not contain any $a_{i,j}$ with $j < 0$. Let M_i be the largest positive value of j such that $a_{i,j}$ occurs in $r_{1,0}, \ldots, r_{n,0}$, or 1 if there is no such j. Let A_0 be the set of generators obtained from $A = \{a_{1,0}, \ldots, a_{1,M_1}, a_{2,0}, \ldots, a_{n,M_n}\}$ by deleting $a_{1,M_1}, \ldots, a_{n,M_n}$. The vertex set of Γ consists of all functions $\rho_0 : A_0 \to \Sigma$; that is, all M-tuples ($M = M_1 + \ldots + M_n$) of elements in Σ. As in the previous example we regard ρ_0 as a partial assignment of elements of Σ to the generators $a_{i,j}$. There is an edge from ρ_0 to ρ_0' if and only if (1) $\rho_0(a_{i,j+1}) = \rho_0'(a_{i,j})$ whenever $a_{i,j}, a_{i,j+1} \in A_0$ and (2) the images of $r_{1,0}, \ldots, r_{n,M_n}$ under the partial assignment are trivial.

Example 3.3. Consider the trivial link l of two components. The group $G = \pi_1(S^3 - l, *)$ is a free group on meridian generators x and y corresponding to the two components l_1 and l_2, respectively. We replace y by xa and apply the Reidemeister–Schreier Theorem as we did in Example 3.2 in order to see that the kernel K is the free group generated by a_i, $i \in \mathbf{Z}$. Since there are no nontrivial relations, the directed graph Γ describing $\Phi_\Sigma(l)$ is the complete graph on Σ. The resulting representation shift consists of all bi-infinite paths in Γ, and it is also known as the *full shift* on Σ.

Example 3.4. Consider the Borromean rings $l = 6_2^3$ oriented as in Figure 5a with Wirtinger generators indicated. The group $G = \pi_1(S^3 - l, *)$ has presentation

$$\langle x,\ x_1,\ y,\ y_1,\ z,\ z_1 \mid zx = x_1 z,\ xy = y_1 x,\ yz = z_1 y,\ z_1 y_1 = y_1 z,\ z_1 x = x_1 z_1 \rangle.$$

We use the first three relators to eliminate x_1, y_1 and z_1 from the presentation, obtaining

$$\langle x,\ y,\ z \mid yzy^{-1}xyx^{-1} = xyx^{-1}z,\ zxz^{-1}yzy^{-1} = yzy^{-1}x \rangle.$$

We replace y by xa and z by xb, and apply the Reidemeister–Schreier method to produce the following presentation for the kernel K:

$$K = \langle a_i,\ b_i \mid a_{i+2}b_{i+1}a_{i+1}^{-1}a_i b_{i+1}^{-1}a_{i+1}^{-1},\ b_{i+2}b_{i+1}^{-1}a_{i+1}b_i a_i^{-1}a_{i+1}b_{i+1}^{-1}a_{i+2}^{-1},\ i \in \mathbf{Z} \rangle$$

When Σ is abelian, any representation $\rho : K \to \Sigma$ factors through the quotient map

$K \to K/[K,K]$. It is clear from the presentation for K that the quotient $K/[K,K]$ decomposes as

$$\langle a_i \mid a_{i+2} - 2a_{i+1} + a_i,\ i \in \mathbf{Z}\rangle \oplus \langle b_i \mid b_{i+2} - 2b_{i+1} + b_i,\ i \in \mathbf{Z}\rangle.$$

Hence when Σ is abelian the representation shift $\Phi_\Sigma(l)$ is a Cartesian product $\Psi \times \Psi$. For example, when $\Sigma = \mathbf{Z}/4$ the graph for Ψ computed from its presentation has eight disjoint cycles: 4 cycles of length 1; 2 cycles of length 2; 2 cycles of length 4. (See Figure 5b.)

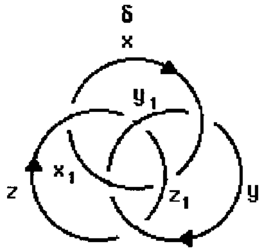

Fig. 5a. The link $l = 6^2_3$

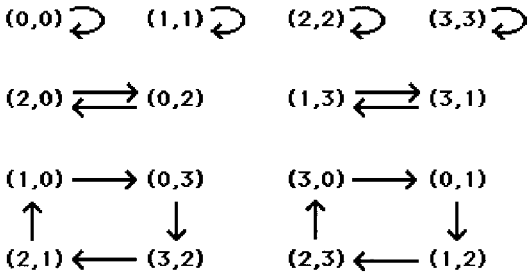

Fig. 5b. Graph of Ψ

In Example 3.2 the only fixed point is the trivial representation, and the representation shift is finite. In general, if k is any knot and Σ is arbitrary, then the the only fixed point of $\Phi_\Sigma(k)$ will be the trivial representation; if Σ is abelian, then the shift will also be finite (see [SiWi2]). These statements need not be true for links, as Examples 3.3 and 3.4 reveal.

4. Color representations. Assume that D is a diagram for a knot k. It is well known that the n-colorings of D correspond to representations of $G = \pi_1(S^3 - k)$ onto the dihedral group $D_n = \langle \alpha,\ \tau \mid \alpha^n = \tau^2 = e,\ \tau\alpha\tau^{-1} = \alpha^{-1}\rangle$. More precisely, given any n-coloring of D, we obtain a representation $\overline{\rho} : G \to D_n$ by mapping the Wirtinger generator x_i corresponding to the ith arc of D to the element $\tau\alpha^{c_i}$, where c_i is the color of the ith arc. Conversely, any representation $\overline{\rho}$ of G onto D_n must map each Wirtinger generator x_i to one of the elements $\tau, \tau\alpha, \ldots, \tau\alpha^{n-1}$ and hence determines an n-coloring of D. The restriction of such a representation $\overline{\rho}$ to the commutator subgroup K produces an element $\rho \in \Phi_{\mathbf{Z}/n}(k)$ with the property that $\rho + \sigma_x\rho$ is trivial.

Definition 4.1. Let l be an oriented link and let n and r be positive integers with $r \geq 2$. An (n,r)*-color representation* of l is a representation $\rho \in \Phi_{\mathbf{Z}/n}(l)$ such that $\rho + \sigma_x\rho + \ldots + \sigma_x^{r-1}\rho$ is trivial. A *color representation* is an (n,r)-color representation for some n and r.

Lemma 4.2. *Every (n,r)-color representation is periodic with period r. If k is an oriented knot then, conversely, every period r representation is an (n,r)-color representation.*

Proof. If $\rho + \sigma_x\rho + \ldots + \sigma_x^{r-1}\rho$ is trivial then so is $\sigma_x(\rho + \sigma_x\rho + \ldots + \sigma_x^{r-1}\rho) = \sigma_x\rho + \ldots + \sigma_x^{r}\rho$. Hence $\sigma_x^r\rho = \rho$.

Conversely, suppose that $\rho \in \Phi_{\mathbf{Z}/n}(l)$ is a representation such that $\sigma_x^r\rho = \rho$. Since the representation $\rho + \sigma_x\rho + \ldots + \sigma_x^{r-1}\rho$ is fixed by σ_x, it must be trivial [SiWi2]. Hence ρ is an (n,r)-color representation. ∎

For a link l a period r representation need not be an (n,r)-color representation, as Examples 3.3 and 3.4 show.

The significance of color representations is contained in the next result.

Theorem 4.3. *Let D be a diagram for an oriented link l with a distinguished arc δ. For each n, the based (n,r)-colorings of D are in one-to-one correspondence with the (n,r)-color representations $\rho \in \Phi_{\mathbf{Z}/n}(l)$.*

Proof. Suppose that $\rho : K \to \mathbf{Z}/n$ is a color representation of l. Let r be the smallest integer ≥ 2 such that $\rho+\sigma_x\rho+\ldots+\sigma_x^{r-1}\rho$ is trivial. We obtain a based (n,r)-coloring of D as follows. Assign the trivial color vector $(0,\ldots,0) \in (\mathbf{Z}/n)^{r-1}$ to the arc δ corresponding to the distinguished generator x. Any other arc determines a Wirtinger generator x_i of the group G of the link, and the product $a = x^{-1}x_i$ is contained in K. Assign the color vector $(\rho(a),\ldots,\sigma_x^{r-1}\rho(a))$ to the arc. Using the Wirtinger relations, it is not difficult to check that our assignment satisfies condition (2.1).

Conversely, suppose we have a based (n,r)-coloring of D that assigns the vector $(c_{i,0},\ldots,c_{i,r-2})$ to the ith arc. By the Reidemeister–Schreier Theorem, K is generated by the elements $x^{-\nu}(x^{-1}x_j)x^{\nu}$ subject to families of relations corresponding to each crossing of the diagram: a positive crossing (see Figure 2) introduces the family of relations

$$x^{-\nu}(x_k^{-1}x_i)x^{\nu} = x^{-\nu}(x_jx_k^{-1})x^{\nu},$$

or equivalently,

$$x^{-\nu-1}(x^{-1}x_k)^{-1}x^{\nu+1} \cdot x^{-\nu-1}(x^{-1}x_i)x^{\nu+1} = x^{-\nu}(x^{-1}x_j)x^{\nu} \cdot x^{-\nu}(x^{-1}x_k)^{-1}x^{\nu},$$

while a negative crossing introduces

$$x^{-\nu-1}(x^{-1}x_k)^{-1}x^{\nu+1} \cdot x^{-\nu-1}(x^{-1}x_j)x^{\nu+1} = x^{-\nu}(x^{-1}x_i)x^{\nu} \cdot x^{-\nu}(x^{-1}x_k)^{-1}x^{\nu}.$$

Condition (2.1) ensures that the mapping

$$\rho(x^{-\nu}(x^{-1}x_i)x^{\nu}) = c_{i,\nu} \text{ if } \nu \equiv 0,1,\ldots,r-2 \bmod r,$$
$$\rho(x^{-\nu}(x^{-1}x_i)x^{\nu}) = -c_{i,0} - \ldots - c_{i,r-2} \text{ if } \nu \equiv r-1 \bmod r$$

determines a color representation $\rho \in \Phi_{\mathbf{Z}/n}(l)$. If we assume that the (n,r)-coloring with which we began is not an extension in the sense of Proposition 2.3 of any (n,r')-coloring with r' a proper factor of r, then by applying the procedure in the first half of the proof we recover that coloring. Hence the theorem is proved. ∎

Just as n-colorings of a diagram for a knot correspond to dihedral representations of the group G of the knot, the more general (n,r)-colorings correspond to certain metabelian representations of G. We have chosen the symbolic dynamical approach in favor of the more algebraic one for two reasons. First, as R. Hartley has noted in [Ha], the complicated structure of the Alexander module forbids a complete algebraic analysis. Second, the dynamical approach is constructive and often comparatively simple.

EXAMPLE 4.4. Consider the 3-component link $l = 6^3_1$ oriented as in Figure 6a with Wirtinger generators indicated. The group G of the link has presentation

$$\langle x,\ x_1\ ,y\ ,y_1,\ z,\ z_1 \mid y_1z = zy,\ z_1x = xz,\ x_1y = yx,\ x_1y_1 = yx_1,\ zy_1 = y_1z_1\rangle.$$

Using the first three relators we can eliminate the generators $x_1,\ y_1$ and z_1 from the presentation, obtaining

$$\langle x,\ a,\ b \mid xy^{-1}zyz^{-1}yx^{-1}y^{-1},\ zyz^{-1}xz^{-1}x^{-1}zy^{-1}\rangle.$$

We apply the same steps as in Example 3.2 in order to present the kernel K of the total linking homomorphism.

$$K = \langle a_i,\ b_i\ | a_ia_{i+1}^{-2}b_{i+1}a_ib_i^{-1},\ b_i^{-2}b_{i+1}a_{i+1}^{-1}b_{i+1}a_i,\ i \in \mathbf{Z}\rangle.$$

If we are interested in the $(3,3)$-colorings of a diagram for l, then we can allow the generators $a_i,\ b_i$ to commute and reduce all coefficients modulo 3 (i.e., replace K by its abelianization tensored with $\mathbf{Z}/3$). When we do this the two families of relations become

$$-a_i + a_{i+1} - b_i + b_{i+1}, \qquad a_i - a_{i+1} + b_i - b_{i+1}.$$

Clearly the second family is a consequence of the first. Moreover, the first relations can be rewritten as $b_{i+1} = a_i - a_{i+1} + b_i$. We can construct all homomorphisms ρ from K to $\mathbf{Z}/3$ by mapping the generators $a_i,\ b_0$ arbitrarily; the images of the remaining generators $b_i,\ i \neq 0$, are then determined by the relations. (The graph Γ that describes $\Phi_{\mathbf{Z}/3}$ consists of three disjoint complete directed graphs – each component corresponding to a choice for the image of b_0. See [SiWi2]). In order to determine the based $(3,3)$-coloring corresponding to any $(3,3)$-color representation, we must first express $x^{-1}y,\ x^{-1}z,\ x^{-1}x_1,\ x^{-1}y_1$, and $x^{-1}z_1$ in terms of the generators $a_i,\ b_i$. An easy computation reveals

$$x^{-1}y = a_0, \qquad x^{-1}z = b_0, \qquad x^{-1}x_1 = a_0 - a_{-1},$$
$$x^{-1}y_1 = a_{-1} - b_{-1} + b_0, \qquad x^{-1}z_1 = b_{-1}.$$

If α and β are elements of $\mathbf{Z}/3$, then the mapping $\rho : K \to \mathbf{Z}/3$ that sends each generator a_i to α and each b_i to β is a $(3,3)$-color representation, a fixed point in the shift $\Phi_{\mathbf{Z}/3}(l)$. From our computation we see that ρ corresponds to a based $(3,3)$-coloring of our diagram in which the y-arc is colored by $(\rho(a_0), \rho(a_1)) = (\alpha, \alpha)$ while the z-arc is colored by $(\rho(b_{-1}), \rho(b_0)) = (\beta, \beta)$, etc. Figure 6b contains the based coloring.

The shift $\Phi_{\mathbf{Z}/3}(l)$ also contains $(3,3)$-color representations that are not fixed points. One such representation is determined by

$$a_{3i} \mapsto 1, \qquad a_{3i+1} \mapsto 0, \qquad a_{3i+2} \mapsto 2,$$

$$b_{3i} \mapsto 2, \quad b_{3i+1} \mapsto 0, \quad b_{3i+2} \mapsto 1.$$

The corresponding based $(3,3)$-coloring is shown in Figure 6c.

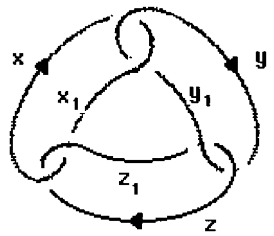

Fig. 6a. The link $l = 6_1^3$

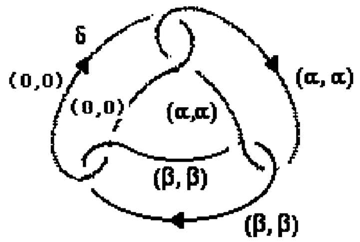

Fig. 6b. Based $(3,3)$-coloring of 6_1^3 diagram determined by a fixed point

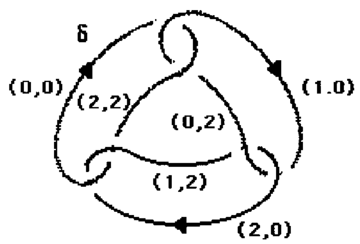

Fig. 6c. Based $(3,3)$-coloring of 6_1^3 diagram determined by a nonfixed point

Now consider the oriented link l' obtained from l by reversing the orientation of the component containing the arc δ. Repeating the steps above we discover that the kernel K abelianized has a new presentation

$$\langle a_i,\ b_i \mid -a_i + 2a_{i+1} - a_{i+2} - b_{i+1} + b_{i+2},\ -a_i + a_{i+1} + b_i - 2b_{i+1} + b_{i+2} \rangle,$$

where i ranges over the integers. Reducing the coefficients modulo 3 produces the relations

(R1) $$a_i + a_{i+1} + a_{i+2} + b_{i+1} - b_{i+2},$$

(R2) $$-a_i + a_{i+1} + b_i + b_{i+1} + b_{i+2}.$$

Any $(3,3)$-color representation ρ must vanish on $a_i + a_{i+1} + a_{i+2}$ and $b_i + b_{i+1} + b_{i+2}$. However, from relations (R1) and (R2) we see that ρ must also vanish on $-a_i + a_{i+1}$ and $b_{i+1} - b_{i+2}$. Consequently, ρ must be a fixed point of the shift $\Phi_{\mathbf{Z}/3}(l')$. This means that $col_{3,3}(l') \neq col_{3,3}(l)$.

Example 3.2 revisited. The diagram for the knot $k = 5_2$ has 6 nontrivial based $(7, 2)$-colorings corresponding to the 6 nontrivial representations of period 2. It has 48 nontrivial based $(7, 14)$-colorings corresponding to the 48 nontrivial representations of period 14. Figure 7 displays the based $(7, 2)$-coloring that corresponds to the representa-

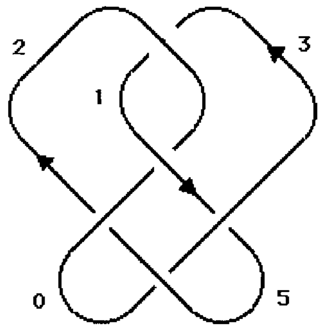

Fig. 7. Based $(7, 2)$-coloring of 5_2

tion ρ such that $\rho(a_{2i}) \equiv 5 \bmod 7$, $\rho(a_{2i+1}) \equiv 2 \bmod 7$. We discover this coloring by the same steps we used in Example 4.4. First we express $x^{-1}x_2, \cdots, x^{-1}x_5$ in terms of the generators a_i. An easy calculation shows

$$x^{-1}x_2 = a_0, \quad x^{-1}x_3 = a_0 - a_{-1},$$
$$x^{-1}x_4 = 2a_0 - a_{-1}, \quad x^{-1}x_5 = a_{-1}.$$

From this we see that if δ is colored 0, then the arc corresponding to x_2 is colored $\rho(a_0) = 5$. Likewise, the arc corresponding to x_3 is colored 3, etc.

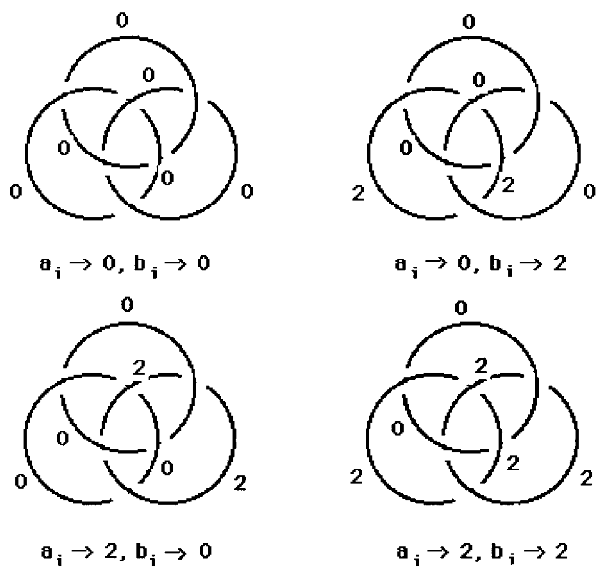

Fig. 8. Based $(4, 2)$-colorings of 6^3_2

EXAMPLE 3.4 REVISITED. $\Phi_{\mathbf{Z}/4}(l)$ has 256 representations: 16 representations with period 1 (fixed points); 48 representations of least period 2; 192 representations of least period 4. For example, the representations with least period 2 have the form $(\psi_1, \psi_2) \in \Psi \times \Psi = \Phi_{\mathbf{Z}/4}(l)$ where ψ_1, ψ_2 have periods 1 or 2, but they do not both have period 1. None of the representations of least period 2 is a $(4,2)$-color representation. However, fixed points also have period 2 (although not least period 2), and $\Phi_{\mathbf{Z}/4}(l)$ contains 4 fixed points that are $(4,2)$-color representations. These representations have the form (ψ_1, ψ_2) where ψ_1, ψ_2 correspond to the 1-cycles $(0,0) \to (0,0)$ and $(2,2) \to (2,2)$ in Γ. The resulting based $(4,2)$-colorings are shown in Figure 8.

5. (n,r)**-colorings of satellite knots.** If $\tilde{k}$ is a knot that is contained in a solid torus then knotting the solid torus will convert $\tilde{k}$ to a more complicated knot k called a satellite knot. The idea was introduced by H. Schubert [Sc]. More precisely, assume that $\tilde{k}$ is contained in a standard solid torus $\tilde{V}$ in S^3, but not contained in any 3-ball in $\tilde{V}$. Assume that $\hat{k}$ is a nontrivial second knot, and let $f : \tilde{V} \to V(\hat{k})$ be a diffeomorphism from $\tilde{V}$ onto a closed tubular neighborhood of $\hat{k}$, mapping a longitude of $\tilde{V}$ onto a longitude of the knot $\hat{k}$. (A *longitude* of $\hat{k}$ is an essential simple closed curve in the boundary of $V(\hat{k})$ that is nullhomologous in the complement of $\hat{k}$.) The image $k = f(\tilde{k})$ is a nontrivial knot, a *satellite knot* with *companion knot* $\hat{k}$ and *pattern* $(\tilde{V}, \tilde{k})$. The solid torus $\tilde{V}$ has infinite cyclic first homology, and the class of $\tilde{k}$ generates a subgroup $d \cdot H_1(\tilde{V})$ for some nonnegative integer d. We call d the *winding number* of the satellite knot. (See [BuZi] or [Ro].) In the special case that $\tilde{k}$ is a torus knot in the boundary of a smaller solid torus $\tilde{V}_1 \subset \tilde{V}$ sharing a common core circle with $\tilde{V}$ the satellite knot k is also called a *cable* of $\hat{k}$. If $\tilde{k}$ is the result of tying a knot in the core circle of $\tilde{V}$ locally (i.e., in a small 3-ball in $\tilde{V}$), then the satellite knot k is just the connected sum of $\tilde{k}$ and $\hat{k}$.

Let $K, \tilde{K}$ and $\hat{K}$ be the respective commutator subgroups of the groups of $k, \tilde{k}$ and $\hat{k}$. It can be seen from work of Seifert [Se] that the abelianization $K/[K,K]$ is isomorphic to the direct sum of $\tilde{K}/[\tilde{K},\tilde{K}]$ and d copies of $\hat{K}/[\hat{K},\hat{K}]$. Moreover, if $x, \tilde{x}$ and $\hat{x}$ denote respective classes of meridians of $k, \tilde{k}$ and $\hat{k}$, then conjugation by x in the group of k induces an automorphism of $K/[K,K]$ that maps $a \in \tilde{K}/[\tilde{K},\tilde{K}]$ to $\tilde{x}^{-1} a \tilde{x}$ and maps $(a_0, \ldots, a_{d-1}) \in K/[K,K] \oplus \ldots \oplus K/[K,K]$ to $(a_1, \ldots, a_{d-1}, \hat{x}^{-1} a_0 \hat{x})$. Explicit proofs of these statements can be found in [LvMe].

We conclude with a theorem that demonstrates the power of symbolic dynamical techniques.

THEOREM 5.1. *Assume that k is a satellite knot with companion knot $\hat{k}$, pattern knot $\tilde{k}$ and winding number d. Let n and r be positive integers with $r \geq 2$.*

If $d = 0$, then $col^0_{n,r}(k) = col^0_{n,r}(\tilde{k})$.

If $d \neq 0$, then $col^0_{n,r}(k) = col^0_{n,r}(\tilde{k}) \cdot [col^0_{n,r/q}(\hat{k})]^q$, where $q = \gcd(d,r)$.

Theorem 5.1 follows from the above comments and a general result about dynamical systems that we describe now. Assume that $(\tilde{\Phi}, \tilde{\sigma})$ and $(\hat{\Phi}, \hat{\sigma})$ are two dynamical systems. Given any positive integer d we define a *satellite dynamical system* (Φ, σ) such that $\Phi = \tilde{\Phi} \times \hat{\Phi} \times \ldots \times \hat{\Phi}$ (d copies of $\hat{\Phi}$), and $\sigma(\rho, \tau_0, \ldots, \tau_{d-1}) = (\tilde{\sigma}\rho, \tau_1, \ldots, \tau_{d-1}, \hat{\sigma}\tau_0)$. Recall that $\operatorname{Fix} f$ denotes the set of fixed points of the automorphism f.

PROPOSITION 5.2. *Assume that* $(\Phi,\ \sigma)$ *is the satellite dynamical system determined by* $(\tilde{\Phi},\ \tilde{\sigma})$, $(\hat{\Phi},\ \hat{\sigma})$ *and positive integer* d. *Then for any positive integer* r,

$$|\mathrm{Fix}\,\sigma^r| = |\mathrm{Fix}\,\tilde{\sigma}^r|\cdot|\mathrm{Fix}\,\hat{\sigma}^{r/q}|^q,$$

where $q=\gcd(d,r)$.

Proof. Let $\phi=(\rho,\tau_0,\ldots,\tau_{d-1})\in\mathrm{Fix}\,\sigma^r$. Clearly $\rho\in\mathrm{Fix}\,\tilde{\sigma}^r$. Also,

$$\phi=\sigma^{rd/q}\phi=(\tilde{\sigma}^{rd/q}\rho,\hat{\sigma}^{r/q}\tau_0,\ldots,\hat{\sigma}^{r/q}\tau_{d-1}),$$

so $\tau_i\in\mathrm{Fix}\,\hat{\sigma}^{r/q}$ for all i. We can write $mr=nd+q$ for some positive integers m and n, so $\phi=\sigma^{mr}\phi=\sigma^{nd+q}\phi$. This gives

$$\tau_i=\hat{\sigma}^n\tau_{i+q},\qquad 0\le i\le d-q-1. \tag{5.1}$$

Thus $\tau_0,\ldots,\ \tau_{q-1}$ uniquely determine $\tau_q,\ldots,\ \tau_{2q-1}$, which in turn uniquely determine $\tau_{2q},\ldots,\ \tau_{3q-1}$ and so on. Conversely, given ρ in $\mathrm{Fix}\,\tilde{\sigma}^r$ and $\tau_0,\ldots,\ \tau_{q-1}$ in $\mathrm{Fix}\,\hat{\sigma}^{r/q}$, we can use (5.1) to define $\tau_q,\ldots,\tau_{d-1}$, so that $\phi=(\rho,\tau_0,\ldots,\tau_{d-1})$ is in $\mathrm{Fix}\,\sigma^r$. ∎

Let D be a diagram for an oriented knot k. We have remarked previously that for any positive integers n, r with $r\ge 2$ the based (n,r)-colorings of D form a module over $\mathbf{Z}/n$. In [SiWi2] we showed that $\mathrm{Fix}\,\sigma_x^r$ is isomorphic to $H_1(M_r(k);\mathbf{Z}/n)$, where $M_r(k)$ is the r-fold branched cyclic cover of k (see [BuZi] or [Ro]). The following reformulation of Theorem 5.1 is a consequence.

THEOREM 5.3 [Li], [LvMe]. *Assume that* k *is a satellite knot with companion knot* $\hat{k}$, *pattern knot* $\tilde{k}$ *and winding number* d. *Let* n *and* r *be positive integers with* $r\ge 2$, *and* $q=\gcd(d,r)$.

If $d=0$, *then* $H_1(M_r(k);\mathbf{Z}/n)\cong H_1(M_r(\tilde{k});\mathbf{Z}/n)$.
If $d\ne 0$, $H_1(M_r(k);\mathbf{Z}/n)\cong H_1(M_r(\tilde{k});\mathbf{Z}/n)\oplus[H_1(M_{r/q}(\hat{k});\mathbf{Z}/n)]^q$.

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