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REIDEMEISTER-TYPE MOVES FOR SURFACES IN FOUR-DIMENSIONAL SPACE

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Abstract. We consider smooth knottings of compact (not necessarily orientable) n-dimensional manifolds in $\mathbb{R}^{n+2}$ (or S^{n+2}), for the cases $n = 2$ or $n = 3$. In a previous paper we have generalized the notion of the Reidemeister moves of classical knot theory. In this paper we examine in more detail the above mentioned dimensions. Examples are given; in particular we examine projections of twist-spun knots. Knot moves are given which demonstrate the triviality of the 1-twist spun trefoil. Another application is a smooth version of a result of Homma and Nagase on a set of moves for regular homotopies of surfaces.

1. Introduction. In this paper we consider codimension two smooth embeddings of a closed n-dimensional manifold, M^n, into $\mathbb{R}^{n+2}$ (or sometimes S^{n+2}), in the case $n = 2$ or $n = 3$. (In general, $\mathbb{R}^k$ will refer to the standard k-dimensional Euclidean space and S^k will refer to the standard k-dimensional sphere; each of these manifolds with the standard differentiable structure.) Most of our results hold whether or not M^n is orientable.

Commonly, one refers to a knot as an ambient isotopy class of an embedding; however, the word "knot" is also used to refer to an embedding which represents this class—or sometimes just the image of such an embedding. We will adopt the following terminology: a smooth codimension two embedding will be called a (*smooth*) *knotting.* The corresponding ambient isotopy class will be called a *knot.* The smooth isotopy extension theorem (see, for example, [HR2]) implies that a smooth isotopy gives rise to an ambient isotopy.

The basic questions in knot theory are: When are two given knots the same? (In our terminology this translates to—When are two given knottings equivalent?) When are two given knots different? Most is known about the last question; answers are given by using a number of well-known algebraic invariants. To show two knots are the same, requires

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geometric techniques. In this paper we discuss a very general geometric technique, namely, looking at projections of knots and associated elementary changes called knot moves.

In classical knot theory, the study of knotted and linked circles in $\mathbb{R}^3$, the standard way two knottings are shown to be equivalent, is to represent the knottings by projections to a plane and then to study changes, called knot moves, one could make in such projections. Knot moves generate all equivalent knottings. In this paper we consider the higher dimensional versions of these combinatorial ideas, and discuss some examples.

In two previous papers we have developed tools to study isotopies of higher dimensional knots. In [RS3] we defined a general position for projecting higher dimensional knots. In [RS4] we applied these results to isotopies of knots. In Section 2 of this paper, we apply these results to the case of knotted surfaces in $\mathbb{R}^4$, and give a generalization of the Reidemiester knot moves to surfaces in four space, answering a question raised in [GL]. In Section 3, we apply our results to knotted 3-dimensional manifolds in $\mathbb{R}^5$. In [HM-NG] a sequence of elementary moves is given, in the piecewise linear category, for regular homotopies of surfaces in $\mathbb{R}^3$. In Section 4, we show how our techniques can lead to a different proof of the corresponding result in the smooth category. Section 5 contains some specific calculations to illustrate these ideas. We examine twist-spun knots, and in particular show explicitly how to "unknot" the one-twist spun trefoil knotting by using elementary moves.

To make this paper more self contained, we include here some of definitions of [RS3] and [RS4].

Suppose $f : P^p \to Q^{p+1}$ is a smooth map from a p-dimensional manifold to a $(p+1)$-dimensional manifold. The *branch set of* f is the subset of points $b \in P^p$ for which f fails to be an immersion. If $x \in P$ and $f^{-1}(f(x))$ consists of exactly k points, we will say that x has *multiplicity* k. The *double point set*, denoted D is the closure, in P^p, of the set of points $d \in P^p$ such that d has multiplicity at least 2.

If we are looking at a 2-dimensional manifold in $\mathbb{R}^3$, the local picture of the image of a branch point is well known and has several names: pinch point, Whitney umbrella, cone on a figure eight curve, see for example [F].

We will let B denote the set of branch points. In the generic situation, B will be a $(p-2)$-dimensional submanifold of P, with each point being a limit point of the set of points of P^p such that f is not one-to-one.

Let's take a closer look at a branch point, b, $b \in B$. We have $b \in D$. Locally, in an appropriate neighborhood of b, D looks like a $(p-1)$-dimensional submanifold and the branch set is a $(p-2)$-dimensional submanifold and the map $f|D$ can be described as "folding of D with crease along B". In particular, locally, $f|D$ is a two-to-one, except at points of B, where $\{b\} = (f|D)^{-1}(f(b))$.

If we consider the global situation, we may have points $b \in B$ with $\{b\} \neq f^{-1}(f(b))$. However, we will always have $f^{-1}(f(b)) \subseteq D$.

In our analysis which is to follow we will need to distinguish points of $d \in D$ in three situations:

1. $f^{-1}(f(d)) \subseteq D - B$ If d has multiplicity 2, we will call it a *pure double point*; if multiplicity three, a *pure triple point*, etc.

2. $f^{-1}(f(d)) \subseteq B$. If d has multiplicity 2, we will call it a *branch double point*; if multiplicity three, a *branch triple point*, etc.

3. Otherwise some points of $f^{-1}(f(d))$ are in B, and some are elsewhere in D; we call this the "mixed case". If d has multiplicity 2, we will call it a *mixed double point*; if multiplicity three, a *mixed triple point*, etc.

In this paper we will have a smooth manifold M^n where $n = 2, 3$, and a smooth isotopy which we take to be a level preserving embedding $F : M^n \times I \to \mathbb{R}^{n+2} \times I$; that is, for all $t \in I$ we will have $F(M^n \times \{t\}) \subseteq \mathbb{R}^{n+2} \times \{t\}$.

We consider $\pi \circ F : M^n \times I \to \mathbb{R}^{n+1} \times I$, where the projection $\pi : \mathbb{R}^{n+2} \times I \to \mathbb{R}^{n+1} \times I$ is defined using standard projection on the first factor, the identity map on the I factor. In either case $n = 2, 3$, we will let B denote the set of branch points, D, T, and Q the set of pure double, pure triple and pure quadruple points respectively. Note that $Q \subseteq T \subseteq D$, and $B \subseteq D$ and $B \cap T = \emptyset$.

In general if $X \subseteq M^2 \times I$ we will let X^* denote the image $\pi \circ F(X)$. If x has multiplicity k we will say that x^* is a *k-fold point of* $(M^n \times I)^*$. If $\pi \circ F$ is in general position, the set $(M^n \times I)^*$ is not necessarily a manifold but is a disjoint union of smooth submanifolds of the $(n+2)$-dimensional manifold $\mathbb{R}^{n+1} \times I$ as follows:

$(M^n \times I)^* - D^*$ is an $(n+1)$-dimensional submanifold,
$D^* - B^* - T^*$ is an n-dimensional submanifold,
B^* is an $(n-1)$-dimensional submanifold,
T^* is an $(n-1)$-dimensional submanifold,
Q^* is an $(n-2)$-dimensional submanifold, etc.

Roughly, *general position of an isotopy* means an isotopy in which "each point of the image of $\pi \circ F$ looks generic". In the case $n = 2$ when $\pi \circ F$ is a smooth map of a 3-manifold into a 4-manifold, this means that for each $x \in (M^2 \times I)^*$,

1. If $x \in (M^2 \times I) - D$ then $\pi \circ F$ is immersive at x, thus $(M^2 \times I)^*$ looks, locally like a 3-dimensional hyperplane.

2. If $x \in B$ then, generally, in a neighborhood of x^*, $(M^n \times I)^*$ looks like "a cone on a figure-8 crossed with I". There may be some isolated exceptional points, where x is of mixed type where there is a point of $D - B$ such that $f(d) = f(x)$. In this case, in a neighborhood of x^*, $(M^n \times I)^*$ looks like "a cone on a figure-8 crossed with I", together with a hyperplane through x^* which is transverse to "the cone point crossed with I".

3. If $x \in D - T$ then, in a neighborhood of x^*, $(M^n \times I)^*$ looks like two transversely intersecting (3-dimensional) hyperplanes in a 4-ball. (The mutual intersection points will be 2-dimensional.)

4. If $x \in T - Q$ then, in a neighborhood of x^*, $(M^n \times I)^*$ looks like three transversely intersecting (3-dimensional) hyperplanes in a 4-ball. (The mutual intersection points will be 1-dimensional.)

5. If $x \in Q$ then, in a neighborhood of x^*, $(M^n \times I)^*$ looks like four transversely intersecting (3-dimensional) hyperplanes in a 4-ball. (The mutual intersection will be a single point.)

To define our knot moves, we consider the map ϕ, $\phi : (M^n \times I) \to I$ defined as $\pi' \circ F$ where π' denotes projection onto the second factor of $\mathbb{R}^{n+1} \times I$. In addition we will want to make the map ϕ "as nice as possible" with respect to the subsets D, T, Q, etc. Our condition is that ϕ *restricted to these sets* is a Morse function. (Note that by definition of smooth isotopy, we have $\phi(\{\{x\}, \{t\}\}) = t$; it follows that ϕ itself has no non-degenerate singularities and thus is a Morse function. Also, if f is any function from a 0-dimensional manifold, X, into I, then every point of X is a critical point of f; we will say that any such f is a Morse function.)

If we are concerned with a submanifold, N^*, of D^* (for example points of $N^* = B^*$, $N^* = T^*$, etc.) then we will say $x^* \in N^*$ *is a singularity of a certain type* if $x^* = F(x)$ and $\phi|N$ has a Morse singularity of that type at x.

The basic idea of a theory of knot moves is to analyze an isotopy as a finite sequence of a changes, called "knot moves" where each change is one of a finite number of possible changes. In our analysis, a knot move will correspond to a singularity of D^*. To insure that we do not have two simultaneous changes, we will insist, as we may without loss of generality, that if x^* and y^* are distinct points of D^* then they do not project, in $\mathbb{R}^{n+1} \times I$, to the same point of I. If an isotopy has all the above properties we say that it is *prepared for moves.*

The main theorem of [RS4] then states that every isotopy can be expressed by knot moves where these knot moves which are in dimension n are denoted $\mathcal{M}^n$. Each knot move corresponds to a singularity of D^*, with different types of Morse singularities giving rise to various types of knot moves. We list our classification below. In this paper we need only consider $n = 2$ or 3.

It is simplest if we combine cases which are dual in the sense of dual Morse singularities, as will be seen—this is in accord with the usual practice in knot theory of considering a move and its inverse (sometimes called "reverse") to be of the same type, thus reducing the number of knot moves to consider. If $F : M^n \times I \to \mathbb{R}^{n+2} \times I$ represents a knot move, then the *inverse of* F is the isotopy F' where $F'(y,t) = F(y, 1-t)$. This will have an effect on the singularities so that on a k-dimensional manifold, a singularity of index i will correspond, in the inverse, to a singularity of index $k - i$. For example a local minimum, index 0, will correspond to a local maximum, index k in the inverse. We will say that a knot move and its inverse are of the same type; thus the number of types of moves is roughly half the number of knot moves of $\mathcal{M}^n$.

For high dimensional knot moves, analysis of the singularities of D^* is complicated, with many cases involving B^*. But for $n = 2$ or 3 we can simplify the general classification as pointed out below.

We consider three cases: pure multiple points, branch multiple points, and the mixed case. The first of these is the simplest—here locally, at $x^* \in \mathbb{R}^{n+1} \times I$, it looks like a certain number of mutually intersecting hyperplanes. This is the simplest generalization of the idea of a "crossing point" in a classical knot diagram. In the second case, we need to remember also that B has codimension 1 in D so that our analysis of the singularity will need to take into account this "extra" transverse direction.

1. Singularities of type $S(c,k,p)$ are all singularities, x^*, of D^* which are *not* points of B^*; the c indicates the singularity is of this type ("c" refers to the word "crossing"). Here x^* is a k-fold point. Locally, the set of such k-fold points (if non-empty) looks like a submanifold of dimension $n+2-k$. Finally p is the Morse index at x^*, when we restrict to this submanifold.

2. Singularities, x^*, of type $S(b,k,p,q)$ are all singularities of B^* which are *not* points of T; the b indicates the singularity is of this type.

Although B is a sub-manifold of $M^n \times I$, B^* is not necessarily a sub-manifold of $\mathbb{R}^{n+1} \times I$—in high enough dimensions, we will generically expect "self-crossings" of B^*. (Note we are concerned here with self-crossings *of* B^* as opposed to self-crossings *of* D^*). So if $x \in B$ we say that x^* is a k-fold point of B^* if $F \circ F^{-1}(x)$ consists of k points *of* B; k is the second argument of $S(b,k,p,q)$. This k now gives the entry in $S(b,k,p,q)$.

We are interested in $n=2$ or 3 and thus the dimension of $M^n \times I$ is 3 or 4. In the first case, B is a 1-dimensional manifold mapped to a 4-dimensional manifold and, generically we may avoid self-intersections of B^*. Similarly, in the second case, B is a 2-dimensional manifold mapped to a 5-dimensional manifold and, generically we may avoid self-intersections of B^*. Thus, in this paper we need only consider singularities of type $S(b,1,p,q)$.

Finally p is the Morse index at x of $\phi|B$. Since B has codimension 1 in D, we need to consider a transverse Morse index which is our q. In the case $k=1$ there is, of course, only one such direction. Here $q=0$ means that we have a local minimum at x in this transverse direction, and $q=1$ means that we have a local maximum at x. Another way of expressing this is:

$$q = (\text{Morse index at } x \text{ of } \phi|D) - (\text{Morse index at } x \text{ of } \phi|B).$$

For the record, if $1 < k$, then we define:

$$\sum_{i=1}^{i=k}((\text{Morse index at } x_i \text{ of } \phi|D) - (\text{Morse index at } x_i \text{ of } \phi|B)).$$

3. Singularities x^* of type $S(m,(i,j),p,q)$ occur at points of B which are of mixed type and the m indicates this. Here $F \circ F^{-1}(x)$ consists of $i+j$ points—i points of B and j points of T.

In the first case, B is a 1-dimensional manifold and $(M^2 \times I)^*$ is a 3-dimensional manifold both mapped to a 4-dimensional manifold. Generically we may expect a 0-dimensional intersection. However, in the second case, B is a 2-dimensional manifold and $(M^3 \times I)^*$ is a 4-dimensional manifold mapped to a 5-dimensional manifold and, generically we will not have self intersection points of B^*. Thus, generically, for a knotting of a 3-manifold, we may expect either a 1-dimensional set of mixed double points or 0-dimensional set of mixed triple points. Thus we need only consider singularities for knotted 3-manifolds of types $S(m,(1,1),p,q)$ and $S(m,(1,2),p,q)$.)

As in the previous case, p is the Morse index at x of $\phi|B$ and q is defined:

$$q = (\text{Morse index at } x \text{ of } \phi|D) - (\text{Morse index at } x \text{ of } \phi|B).$$

Also for the general case, not used in this paper, if $1 < k$, then we define:

$$\sum_{i=1}^{i=k}((\text{Morse index at } x_i \text{ of } \phi|D) - (\text{Morse index at } x_i \text{ of } \phi|B)).$$

One final comment on the practicality of higher dimensional knot moves. Our techniques are rather tedious for hand calculation, but quite tractable for computer calculations. We have already begun such a project and hope to discuss this in the near future.

2. Knot moves for surfaces in $\mathbb{R}^4$. We now wish to consider the knot moves for surfaces in $\mathbb{R}^4$.

THEOREM 1. *If $F : M^2 \times I \to \mathbb{R}^4 \times I$ is an isotopy of a closed 2-dimensional manifold and if $F|M^2 \times \{0\}$ and $F|M^2 \times \{1\}$ are in general position with respect to projection, then F is equivalent to an isotopy by elementary moves where these moves are of seven types which we list below by the corresponding singularities*:

(a) *Local maximum or local minimum of the crossing set D^*.*

(b) *Saddle point of D^*.*

(c) *Local maximum or local minimum of the branch set B^* which is a local maximum or local minimum (respectively) of D^* in direction transverse to B^*.*

(d) *Local maximum or local minimum of the branch set B^* which is a local minimum or local maximum (respectively) of D^* in direction transverse to B^*.*

(e) *Local maximum or local minimum of the crossing set of D^*.*

(f) *A point where B^* meets the crossing set of D^*.*

(g) *A triple point of D^*.*

Proof. We replace F by an isotopy, F', arranged for moves as described in Section 1.

We need to examine the map $\pi' \circ F : M^2 \times I \to \mathbb{R}^3 \times I$. Let D^* and B^* be the crossing set of this map and image of the branch point set, respectively. We are considering a map from a 3-manifold, $M^2 \times I$ into a 4-manifold $\mathbb{R}^3 \times I$, here D^* is an immersed 2-manifold and B^*, being 1-dimensional, may be taken to be embedded. In addition, we will not have points of multiplicity more than 4. In the paragraphs below we will refer, parenthetically, to the singularity notation of [RS4], also described in Section 1.

Since D^* is 2-dimensional, we can expect to have singularities of $D^* - B^* - T^*$ index 0, 1, or 2. Index 0 and 2 correspond to the same type and give rise to (a) (singularities $S(c, 2, 0)$, $S(c, 2, 2)$); index 1 gives rise to (b) (singularity $S(c, 2, 1)$).

Since B^* is 1-dimensional, and generically would not have self-intersections we expect two possibilities (c) and (d), depending on the behavior in the direction in D transverse to B. Either the tangential and the transverse singularities are of the same index or they differ. The first case gives us (c) (and corresponds to $S(b, 1, 0, 0)$ and $S(b, 1, 1, 1)$). The second case gives us (d) (and corresponds to $S(b, 1, 0, 1)$ and $S(b, 1, 1, 0)$).

The crossing set of D^*, where we have pure triple points of $(M^3 \times I)^*$, is 1-dimensional, giving rise to (e) (singularities $S(c, 3, 0)$ and $S(c, 3, 1)$).

Finally the set of mixed singularities, where B^* intersects D^* transversely, as well as the set of mixed triple points of D^* (pure quadruple points of $(M^2 \times I)^*$) are

0-dimensional. This gives us, respectively, move (f) (singularity $S(m,(1,2),0,0)$ or $S(m,(1,2),0,1)$, according to whether or not we have, transversely to B in D, a local minimum or local maximum) and (g) (singularity $S(c,4,0)$).

We have now considered all possible singularities. ■

We want to draw pictures of these elementary local knot moves in two ways. The first, Figure 1 shows the local changes in the projection; the second shows the corresponding changes in what we term, below, the projection link—this describes how D looks in M^n.

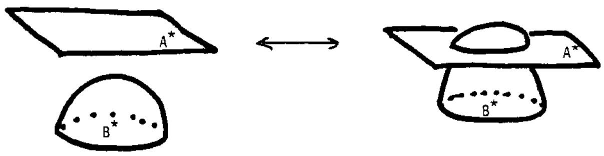


Fig. 1(a)

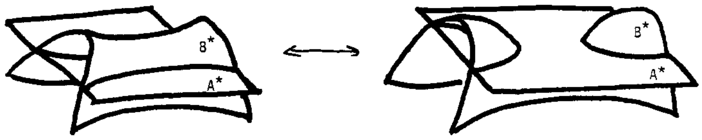


Fig. 1(b)

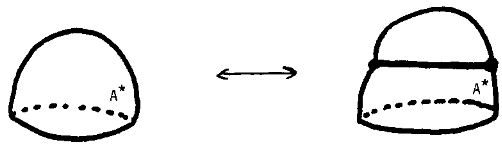


Fig. 1(c)

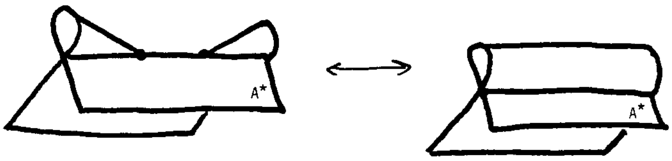


Fig. 1(d)

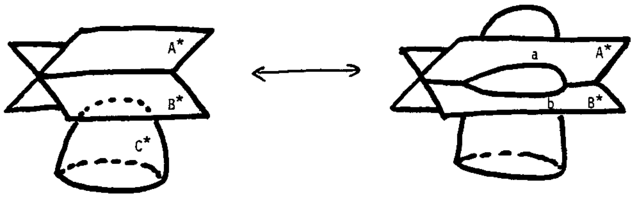


Fig. 1(e)

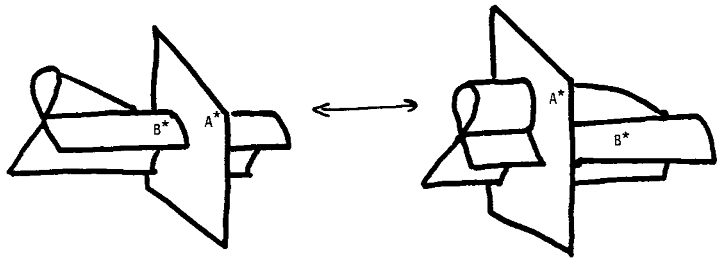


Fig. 1(f)

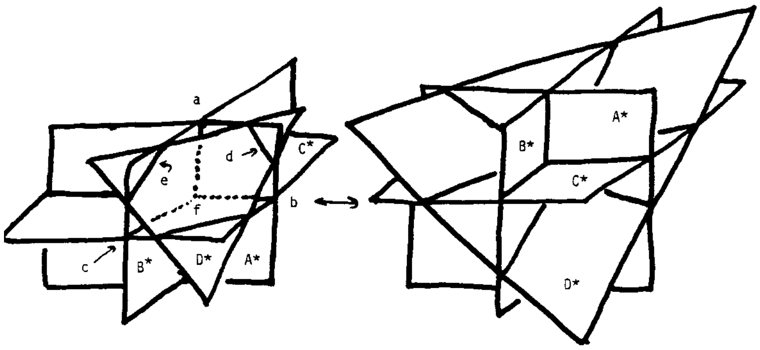


Fig. 1(g)

When looking at the double point set D the following idea yields a helpful graphical device. Consider a canonical triple point, x^*, obtained by projecting, into $\mathbb{R}^3$, an embedding of M^2 into $\mathbb{R}^4$. Locally we have three subdisks of M (say A, B, and C) whose image in $\mathbb{R}^3$ is as shown in Figure 3. Let a, b, and c be the points of A, B, and C (respectively) which are mapped onto x^*. Consider the intersection of the double point set of the map and the disk A. It will look like the letter "x" with a at the crossing point; part of this is from the intersection of A^* and B^*, and part from the intersection of A^* and C^*. Now in $\mathbb{R}^4$, the images of A, B, and C are disjoint. Let us suppose, as we may without loss of generality, that A lies over B and B lies over C. Replace the "x" in A by an overpass diagram (that is, "break the line corresponding to C"). You may also think about this as a drawing on the surface M. Now do the same sort of thing for B and C as shown in Figure 3. If we do this for each triple point we will get drawing of the double point set as collection of curves drawn on M with "overs and unders" at the crossing points. We call such a drawing a *projection link.* We will discuss further examples in Section 4. In our classification of knot moves, as is done with the classical Reidemeister moves, we ignore the particularities of height relations, so what is pictured in Figure 2 is the local projection link for a choice of height relations.

In the first, Figure 1, we illustrate these moves by showing a typical picture in $\mathbb{R}^3 \times \{a\}$, shown on the left, and $\mathbb{R}^3 \times \{b\}$, shown on the right. Here, $[a, b]$ is a small interval in I containing a single critical point of the type in question. In Figures 1(a) and (b) our move involves two disks. These are shown as two separate disks (drawn as rectangles) thought of as lying somewhere in M^2 in Figures 2(a) and (b). However, Figure 1(c) involves only one disk. Thus we have only one in Figure 2(c), etc. The single heavy dots indicate branch points. We next consider Figures 1 and 2 in detail.

A move of type (a), in the projection, (Figure 1(a)), looks like a plane passing through the vertex of a paraboloid in a direction perpendicular to the tangent plane at that vertex. It corresponds to an introduction (or elimination) of a pair of circles in the double point set, each of these circles bounding disks which contain no other double points. This move is analogous to Ω_2 of Reidemeister.

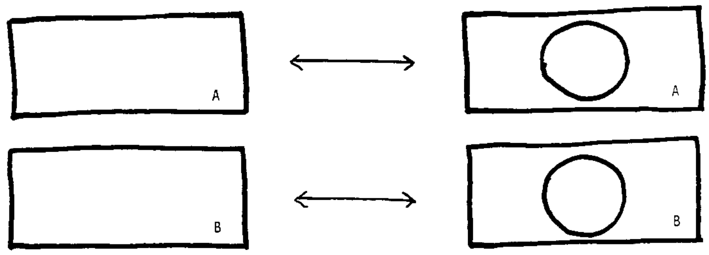


Fig. 2(a)

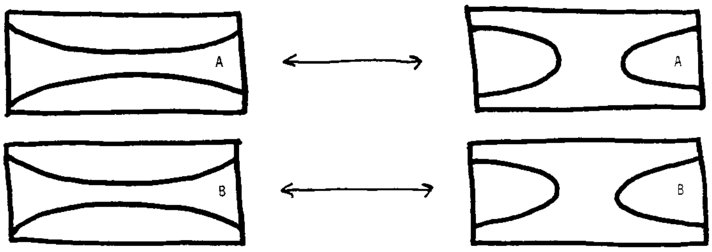


Fig. 2(b)

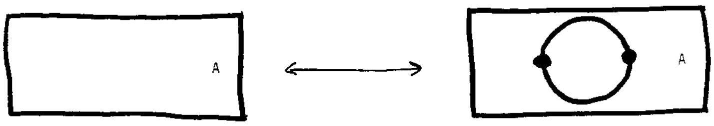


Fig. 2(c)

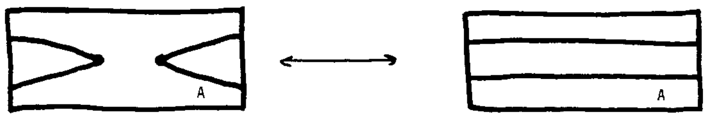


Fig. 2(d)

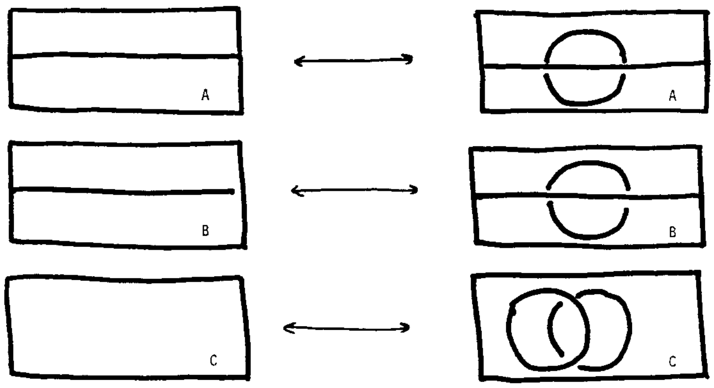


Fig. 2(e)

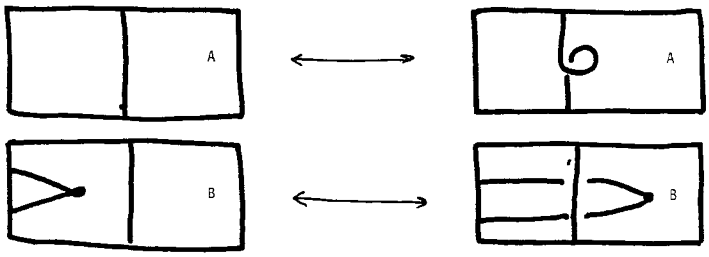


Fig. 2(f)

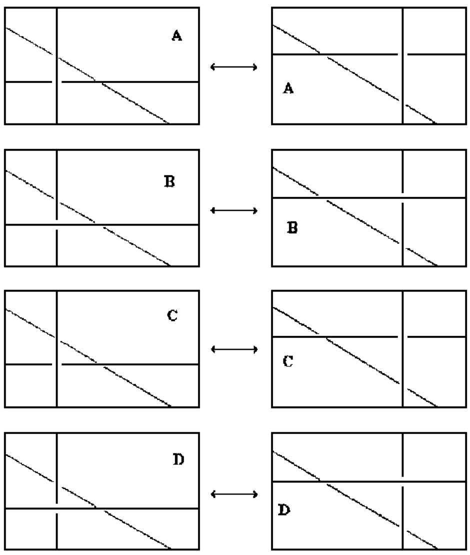


Fig. 2(g)

A move of type (b), in the projection, looks like a plane passing through the saddle point of a hyperbolic paraboloid in a direction perpendicular to the tangent plane at the saddle point. In the double point set we see a pair of hyperbolas of one kind changing into a pair of the other kind. To use the terminology frequently employed in knot theory this

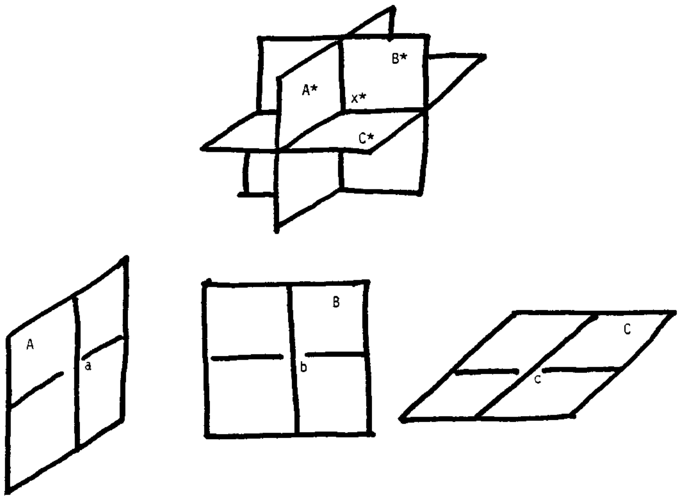


Fig. 3

corresponds to a pair of band moves. That is, given, say, the left hand side of Figure 2(b) we imagine two bands B_1 and B_2 as shown in Figure 4(a). These bands are rectangles situated as shown, the band move then consists of erasing the sides of B_1 and B_2 which touched the double point set and replacing these with those sides which didn't, thus obtaining Figure 4(b).

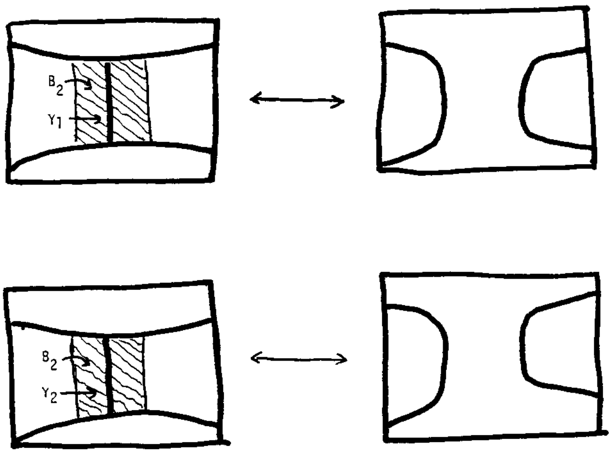


Fig. 4(a) Fig.4(b)

A move of type (c), in its projection might be described as a suspension of the Reidemeister Move Ω_1. In the double point set this introduces a single circle which contains exactly two branch points; this circle bounds a disk, B_1, which contains no other double points.

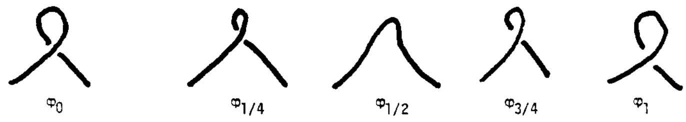


Fig. 5(a)

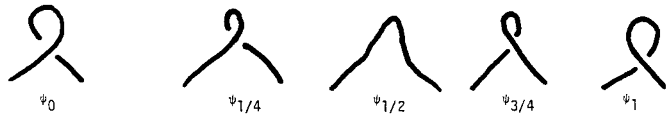


Fig. 5(b)

A move of type (d) is a cancellation (or introduction) of a pair of branch points. The drawing on the left in Figure 1(d) may be thought of as the projection of the trace of the isotopy shown in Figure 5(a) (or perhaps the isotopy of the mirror images). (Warning: This is not the same as using the isotopy shown in 5(b)). A projection of a knotting of a 2-disk into $\mathbb{R}^3$ whose image is as shown on the left in Figure 1(d) (with height relations as shown or reversed) will be called a "*c*-disk" since it can be used "to cancel branch points." In the double point set we see, left side of Figure 2(d), two double point arcs,

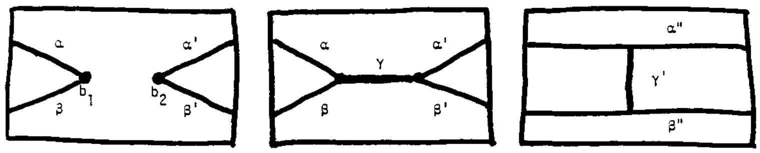


Fig. 6(a) Fig.6(b) Fig.6(c)

each containing a branch point. In Figure 6(a), we label these branch points b_1 and b_2 and these divide these arcs into subarcs α, β, α', and β' as shown, where we will assume α is part of the overset paired with β and similarly α' paired with β'. Then α and α' are seen to merge to form an arc α'' which is paired with and lies over the merging of β and β', called β''. The double point set of the projection of the isotopy of Figure 5(b) with notations as above would be as shown in Figure 6(a), with α' and β' interchanged. Given the picture on the left of Figure 2(d), and using notations as in Figure 6(b) we may think of the one on the right as obtained by merging the branch points b_1 and b_2 along an arc γ which connects them and intersects no other points of the double point set. Given two paired arcs of double points, we could view the inverse operation as obtained by taking the drawing on the right in Figure 2, drawing an arc γ' connecting corresponding points as shown and intersecting no other points of the double point set and "shrinking and splitting along γ'", see Figure 6(c).

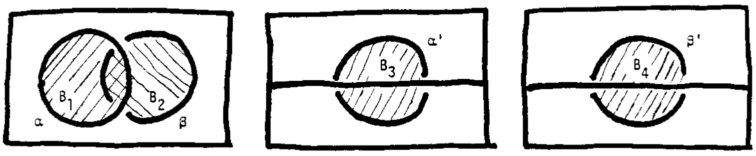


Fig. 7(a) Fig.7(b) Fig.7(c)

A move of type (e) may be described, in the projection, as looking like a paraboloid passing through the intersection of two intersecting planes as shown n Figure 1(e). If the height relations happen to be that plane A passes over B and that the paraboloid passes under both A and B then using the conventions outlined above, the local projection link may be drawn as in Figure 2(e). Details are shown in Figure 7 where the circles α and α' are mapped to the circle a, and β and β' are paired and sent to b. We may describe the passage from right to left in Figure 2(e) as follows. If we have two circles of double points and which intersect in exactly two points which bound disks B_1 and B_2, which intersect the double point set only as indicated in Figure 7(a) and if the corresponding circles also intersect paired arcs as shown, these circles bound disks, B_3 and B_4, intersecting the double point set only as shown in Figure 7(b) and (c), then we can eliminate these circles.

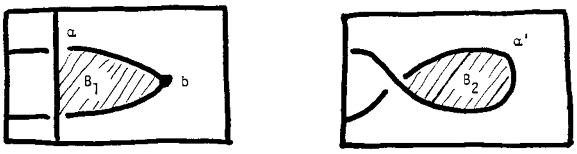


Fig. 8

In the projection, a move of type (f) may be described: a branch point is pushed through a plane. In the double point set, if we have the plane A higher than B, then we will see the branch point pass under the arc and a loop introduced as shown in Figure 2(f). Given the figures on the right in Figure 2(f) we may describe the change in the double point set as follows. Suppose there is a branch point b and an arc of double points α and a disk B_1 which intersects the double point set as shown in Figure 8(a) and suppose α is paired with an arc α' which is a simple loop bounding a disk B_2 which intersects the double point set exactly as shown in Figure 8(b), then we can "pull b under α and shrink the loop α'."

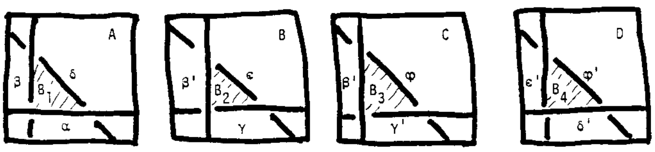


Fig. 9

Finally a move of type (g), in the projection, may be described as follows: there are three planes A, B, C, intersecting transversely in a triple point and a fourth plane, D, moves across that triple point. This is a higher dimensional analogue of Ω_3 of Reidemeister. In the double point set the effect of this can be described as "four simultaneous Ω_3 moves" ; here we use the example where A is above B, B above C, and C above D and the arcs of double points are paired α to α', etc. We may describe this as finding, in the double point set, triangles B_1, B_2, B_3 and B_4 which intersect the double point set exactly as shown in Figure 9, with the parings and height relations as shown, then we perform simultaneously the Reidemeister moves of type Ω_3.

3. Immersions of surfaces. In [HM-NG], Homma and Nagase have defined a sequence of elementary moves which describe a homotopy of a surface in a three-manifold. Locally these moves look like the projections of the knot moves described above. Their proof is in the piecewise linear category. We point out that the techniques we have used above can furnish a proof in the differentiable category. In outline such a proof would proceed as follows. A smooth homotopy of a surface M^2 in a 3-submanifold N^3 can be viewed as a map $F : M^2 \times I \to N^3 \times I$ and since this is to be level preserving, the co-rank of the map is at most one, thus the nature of the branch set and double point set will be as in the case of the projection of an isotopy. We can then define a notion of "arranged for moves" (as before, this will be a "generic" situation) by using Morse functions, on the I-coordinate of a point in $N^3 \times I$. Continuing we would obtain moves according to the elementary singularities.

Moreover, one can generalize the Homma-Nagase results, to homotopies of 3-manifolds in 4-manifolds or even more generally to homotopies of n-manifolds in $(n+1)$-manifolds; the principal complication would be that branch singularities would have more complicated types. However, Morin's Theorem does give a local model for many different types of singularities, and would apply in this case. We also note in passing that Homma and Nagase show in their paper relations of moves which imply that some of our elementary moves may be obtained from combinations of others.

4. Moves for knotted three-dimensional manifolds. We conclude this section with a brief discussion of knot moves for 3-dimensional manifolds embedded in $\mathbb{R}^5$. (Note that all closed 3-dimensional manifolds do embed in $\mathbb{R}^5$, see for example [HR1], with the non-orientable case covered in [W] and [RH].)

We will use D, B, T and Q as before. In addition, because of increase in dimensions, we will need to also consider 5-fold generic points, pure "quintuple points" which we will denote Q'. Our analysis is similar to that in Theorem 1. We have a map from a four-manifold $M^3 \times I$ into a five-manifold $\mathbb{R}^4 \times I$; here D^* is an immersed 3-manifold and B^* is 2-dimensional, and may be taken to be embedded. Also the dimensions of T, Q, Q' are 2, 1 and 0, respectively. Examining the possible knot moves in these dimensions we obtain the following.

Theorem 2. *If $F : M^3 \times I \to \mathbb{R}^5 \times I$ is an isotopy of a closed 3-dimensional manifold and if $F|M^3 \times \{0\}$ and $F|M^3 \times \{1\}$ are in general position with respect to the the* *projection, then F is equivalent to an isotopy by moves where these moves are of twelve types, which we list below by the corresponding singularities:*

(a) *Local maximum or a local minimum of pure double points.*

(b) *A critical point of pure double points of index* 1 *or of index* 2.

(c) *Local maximum or a local minimum of the crossing set of T^*.*

(d) *A saddle point of the crossing set of T^*.*

(e) *Local maximum or local minimum of Q^*.*

(f) *Quintuple points, $(Q')^*$.*

(g) *Local maximum or local minimum of the branch set B^* which is a local maximum or a local minimum (respectively) of D^* in direction transverse to B^*.*

(h) *Local maximum or local minimum of B^* which is a local minimum or a local maximum (respectively) of D^* in direction transverse to B^*.*

(i) *Saddlepoint of B^*.*

(j) *Local maximum or local minimum of the intersection of B^* and the crossing set of D^* which is local maximum or local minimum (respectively) of D^* in a direction transverse to B^*.*

(k) *Local maximum or local minimum of the intersection of B^* and the crossing set of D^* which is a local minimum or local maximum (respectively) of D^* in a direction transverse to B^*; here we have a* 2*-fold point of mixed type.*

(l) *A point of B^* which is a triple point of D^*; here we have a* 3*-fold point of mixed type.*

Proof. Since D^* is 3-dimensional, we can expect to have singularities of index 0, 1, 2, or 3. Index 0 and 3 correspond to the same type, $S(p, 2, 0)$ and $S(p, 2, 3)$ and give rise to (a) above; index 1 and 2 give rise to (b) for types $S(p, 2, 1)$ and $S(p, 2, 2)$. The crossing set of D^*, where we have pure triple points of $(M^3 \times I)^*$, is 2-dimensional. The singularities of index 0 and 2, $S(p, 3, 0)$ and $S(p, 3, 2)$, correspond to (c); index 1 corresponds to (d) and type $S(p, 3, 1)$. The triple point set of D^* (quadruple points of $(M^3 \times I)^*$) is 1-dimensional; the associated singularities, $S(p, 4, 0)$ and $S(p, 4, 1)$, are local maxima and local minima giving rise to one type which corresponds to (e). Lastly we have isolated pure five-fold points; these are accounted for in (f), type $S(p, 5, 0)$.

Since B^* is 2-dimensional, we need to consider six possibilities since a branch point could have index 0, 1, or 2, and we could have either a local minimum or local maximum in the direction tangent to the double point set, transverse to the branch set. The case of index 0 on B and local minimum transversely, and the case of index 2 on B and local maximum transversely are of the same type, $S(b, 1, 0, 0)$ and $S(b, 1, 2, 1)$, and give us (g). The case of index 0 on B and local maximum transversely, and index 2 on B and a local minimum transversely are of the same type and give rise to (h). If we find index 1 singularity on B then the two possibilities for transverse singularity give rise to the same type, $S(b, 1, 1, 1)$ and $S(b, 1, 1, 0)$, and correspond to (i).

Finally we consider mixed types of singularities. Note that B^* could meet disks of $(M^3 \times I)^*$ in a 1-dimensional set giving rise to four possibilities since we could have a local minimum or a local maximum on that set as well as local minimum or local maximum transversely. If both of these are local minima or both are local maxima, then we have a type, $S(m,(1,1),0,0)$ and $S(m,(1,1),1,1)$, corresponding to (k). The other two possibilities, types $S(m,(1,1),0,1)$ and $S(m,(1,1),1,0)$, correspond to (j). Now it is possible that B^* meets D^* transversely; if so, it is in a collection of points, and whether we have a transverse local maximum or minimum, we have the same type, $S(m,(1,2),0,0)$ and $S(m,(1,2),0,1)$; this corresponds to (l). ■

Although we cannot directly show these moves of Theorem 2 as we did in Figure 1 for Theorem 1, we can draw the corresponding changes in the double point sets as we did in Figure 2.

We could adopt an "under-over" convention exactly as we did in the lower dimension and define a higher dimensional projection link. We might then use a convention, such as the one used in [RS1], to show this graphically. One could also define non-elementary knot moves for 3-manifolds as we will do for $T\Omega_2$ as described in Section 5. However, for simplicity, we will not do these here.

The potential use for all this is the following. Suppose for simplicity we consider isotopies of S^3 in $\mathbb{R}^5$. Given a projection of a particular knotting we would look at the projection link, which is now a collection of immersed manifolds in S^3 (which we could actually draw in $\mathbb{R}^3$). We might analyze and discover isotopies by means of our moves. (But we must keep in mind the "realization problem"—we discuss this problem in the case of knotted surfaces in Section 5.1.)

Referring to Figure 10, we now show the changes in our double point sets. Move (a) introduces or eliminates a pair of spheres. Move (b) corresponds to two changes from a hyperboloid of two sheets to a hyperboloid of one sheet. In move (c) we see a pair of paraboloids and two planes moving so as to introduce (or eliminate) a circle of crossing points of the double point set (i.e., a circle of triple points of the projection of the 3-manifold). Move (d) has two paired planes of double points passing through the saddle points of paired hyperbolic paraboloids of double points; this looks like two copies of Figure 1(b). However in Figure 1 we show portions of the projection of the knotting; in Figure 10 we show portions of the double point set (a subset of the manifold we are knotting).

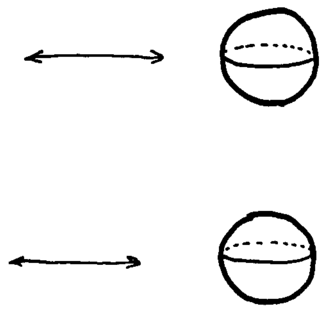


Fig. 10(a)

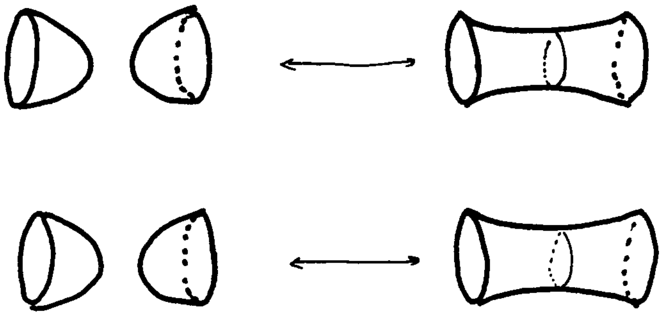


Fig. 10(b)

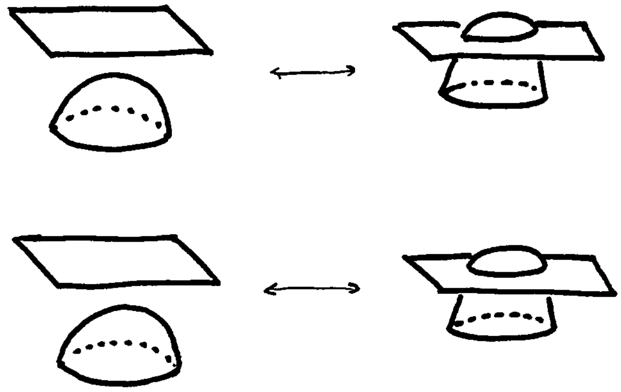


Fig. 10(c)

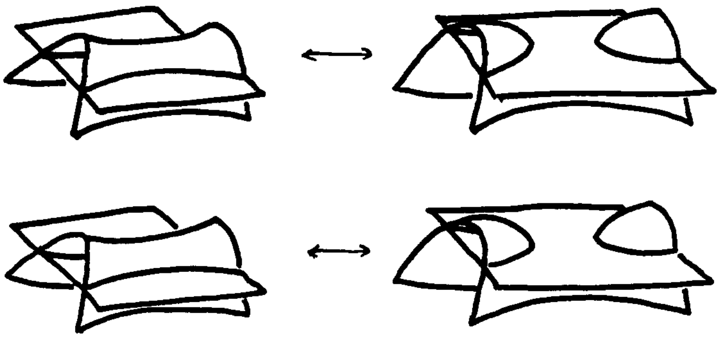


Fig. 10(d)

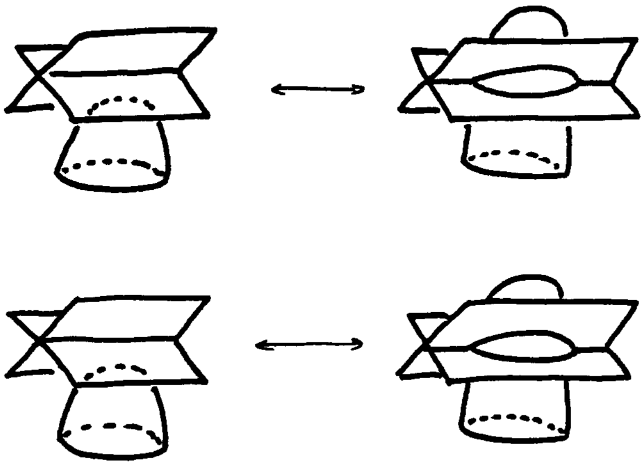


Fig. 10(e)

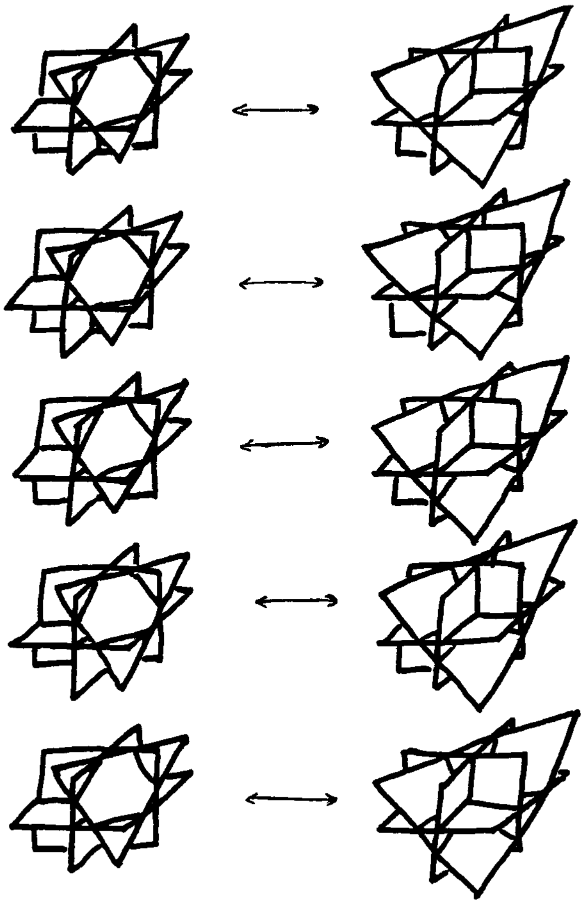


Fig. 10(f)

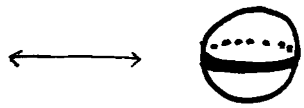


Fig. 10(g)

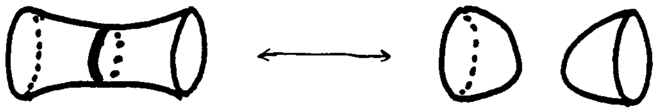


Fig. 10(h)

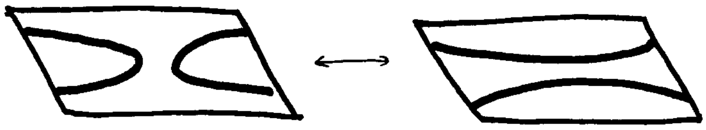


Fig. 10(i)

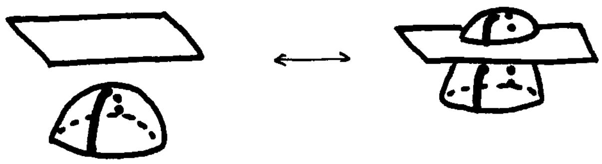


Fig. 10(j)

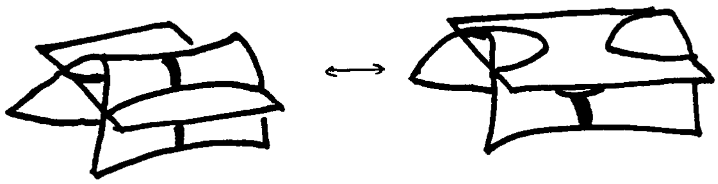


Fig. 10(k)

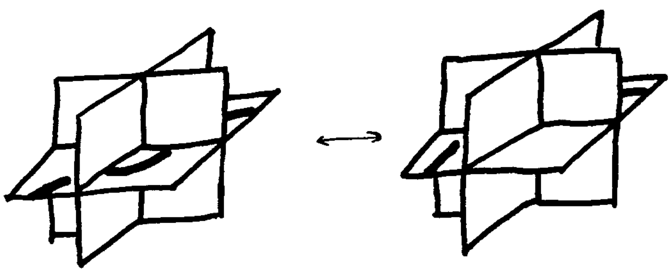


Fig. 10(l)

Similarly, in Figure 10, move (e) looks like two copies of Figure 1(e), and move (f) looks like five copies of Figure 1(g). A move of type (g) introduces (or eliminates) a sphere of the double point set which contains a circle of branch points (shown as a bold curve in Figure 10); this move is analogous to move (c) for 2-dimensional knots. In move (h) a circle of branch points is seen to disappear (locally), separating the double point set; this is analogous to a move of type (d) for 2-dimensional knots. For a move of type (i), we look at a plane of double points and we see two arcs of the branch set, drawn in Figure 10(i) as a hyperboloid. The branch set changes from one type hyperbola to the other. In a move of type (j), a paraboloid containing a branch point arc goes through a plane of double points; this is analogous to the 2-dimensional move of type (f). A move of type (k) shown in Figure 10 is similar to the previous move (j) except we have a hyperbola and plane. Finally, for a move of type (l) we see a triple point of double points where one plane contains an arc of branch points; this arc moves in its plane through the triple point.

5. Geometric calculations. A principal use of Theorem 1 is to show two given knottings are isotopic and we will now show examples. There are two approaches we may use. The most straightforward is to draw the projection of a given knotting, then apply a sequence of local knot moves as shown in Figure 1. However, in practice, the drawing of the projection of a surface knotting in $\mathbb{R}^3$ without computer assistance is rather difficult, thus this approach is limited to relatively simple isotopies of relatively simple knots. A potentially useful method is to look at a projection of a knotting, then at the projection link and try to accomplish the moves by making the appropriate changes on the projection link. This idea has particular appeal for the case of a knotting of a 2-sphere for then the projection link is drawn on a 2-sphere which then can be represented as a drawing of the projection of a link in a plane. Given such a drawing one of the more natural things to do is, if possible, to change the projection link using moves as indicated in Figure 2. There is, however, a serious but not necessarily insurmountable problem in doing this—a realization problem.

5.1. *Realizing changes for projection links.* The realization problem is whether or not changes in the projection link, as shown in Figure 2 and characterized in our discussions and Figures 6–9, can, in fact, correspond to knot moves. A basic problem here is that the projection link, labeled so as to identify branch points and points corresponding to the same k-fold points, describes the projection of a knotting *as a set* but not *as a subset of* $\mathbb{R}^3$.

We discuss the various seven moves.

Given a part of a projection link in which we find two circles bounding disks B_1 and B_2 as described for a move of type (a), this can always be realized as a knot move. This is because, in the projection in $\mathbb{R}^3$, B_1 and B_2 fit together to form a 2-sphere which must bound a 3-ball. Thus in $\mathbb{R}^3$ the planes must have fit together as shown in the right hand side of Figure 1(a), thus allowing us to realize this change as a knot move.

However, given a pair of paired arcs and disks B_1 and B_2 as described in Figure 4(a) it may not be possible to change to Figure 4(b) by a knot move of type (b). We can realize this as a knot move if and only if two arcs γ_1 and γ_2 as shown (where we choose them so that their endpoints are identified) form a circle which bounds a disk B_3 in $\mathbb{R}^4$ such that this disk only meets the projection of the knotting in the curve $\gamma_1 \cup \gamma_2$.

Given a circle with disk B_1 as described in Figure 2(c), this will correspond to a knot move of type (c) since in $\mathbb{R}^3$ the disk B_1 will have a spherical image, this will bound a ball, allowing us to reconstruct the situation shown in Figure 1(c).

The move of type (d) is a very useful one as we will see, but has a subtlety that deserves special mention. Consider the cross cap Figure 11(a); this is the projection of an embedded projective plane in $\mathbb{R}^4$. The double point set of this projection is clearly a

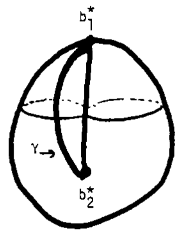


Fig. 11(a)

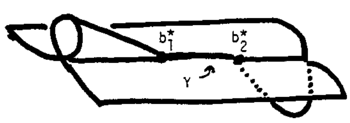


Fig.11(b)

circle containing two branch points, call them b_1 and b_2. This circle corresponds to the center-line of the Möbius band (if one thinks of the projective plane as the union of a Möbius band and a disk). It is tempting to think one could take an arc, γ, from b_1^* and b_2^* and use it as indicated to make a knot move of type (c). (Figure 11(a) shows γ drawn on the crosscap.) This would eliminate the branch points giving us an immersed projective plane. But this is impossible since no immersed projective plane in $\mathbb{R}^3$ is the projection of an embedded projective plane in $\mathbb{R}^4$. What went wrong? It is not a problem of the height relations. A careful look at a neighborhood of the arc γ in $\mathbb{R}^3$ reveals the problem: it looks like Figure 11(b) and not like the figure on the left side of Figure 1(d). In Figure 1(d) the "cones" are on the same side, in 11(b) they are on opposite sides.

To realize the moves, given data as indicated by Figures 7, 8 and 9 for moves of type (e), (f) and (g), respectively, is no problem. As before, the disks $\{B_i\}$ indicated fit together, in $\mathbb{R}^3$, to form spheres which bound balls in $\mathbb{R}^3$ which in turn will allow us to realize the corresponding models as in Figure 1.

5.2. *Some non-elementary knot moves.* Another idea which seems natural to consider when examining a given projection link is to try and use the classical knot moves Ω_1, Ω_2 and Ω_3. It is clear that one cannot simply use these moves to change a projection link—one immediate objection is that the projection link is not just a collection of immersed circles but a collection of *paired* immersed circles. Nonetheless, these moves frequently do give rise to some important simple local isotopies.

If we find a simple loop in a projection link, then clearly in a portion of $\mathbb{R}^3$ our projection of the knotting must look like Figure 12(a). There is an isotopy which corresponds

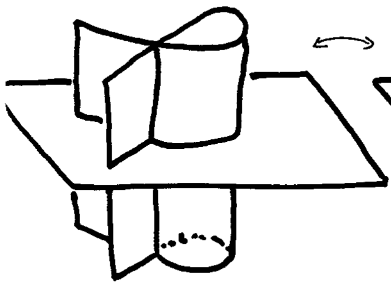


Fig. 12(a)

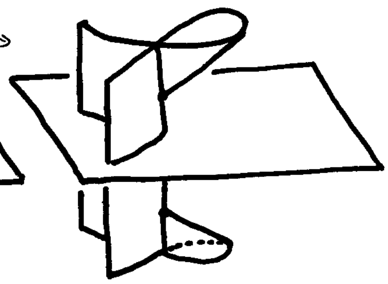


Fig.12(b)

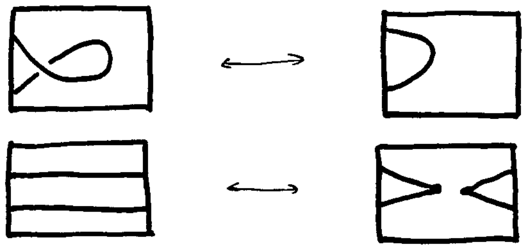


Fig. 12(c)

to Ω_1 whose projection will change Figure 12(a) to Figure 12(b). Figure 12(c) shows the local effect on the projection link. Clearly this move is obtained by a move of type (d) followed by one of type (f). If in our projection link, we attempt a move of type Ω_2, then in the projection, the situation must have looked as in Figure 13(a). In the projection the result will look like Figure 13(b); and Figure 13(c) is the local picture in the projection link if we consider the situation where A lies over B, and B lies over C.

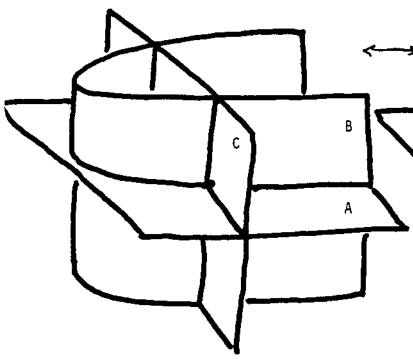


Fig. 13(a)

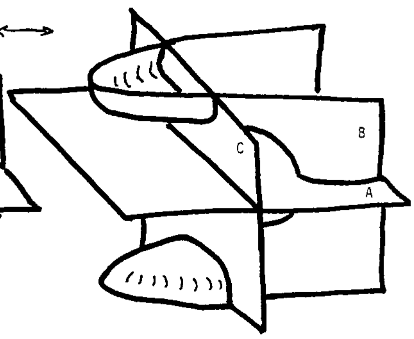


Fig.13(b)

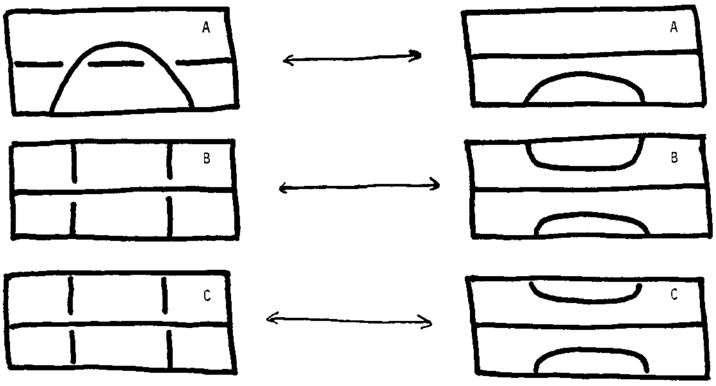
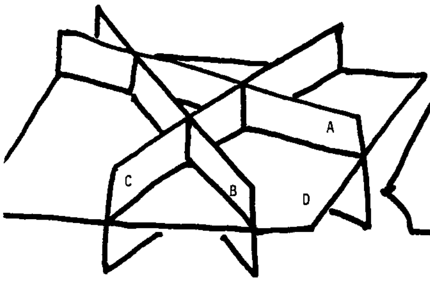


Fig. 13(c)

Finally for a knot move Ω_3 we will obtain figures as in Figure 14 where we illustrate the situation where A lies over B, B lies over C, and C lies over D.



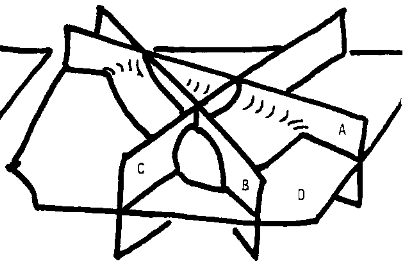


Fig. 14(a) Fig.14(b)

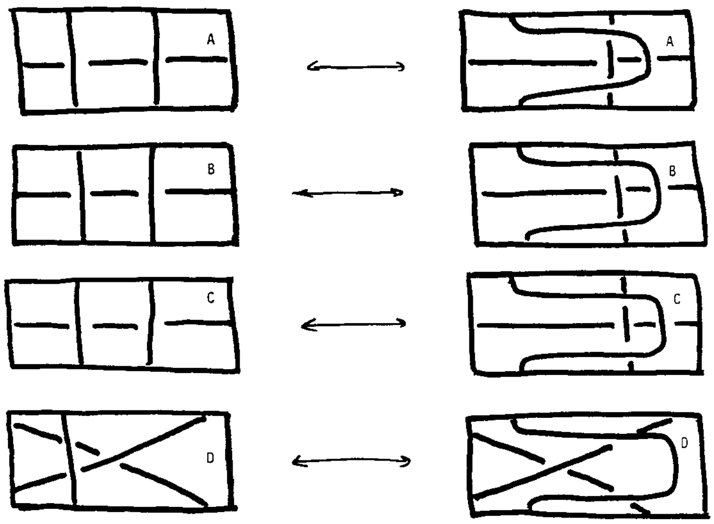


Fig. 14(c)

The local isotopies described above may be defined more precisely, using the trace of the isotopies Ω_1, Ω_2 and Ω_3. We will therefore use the notations $T\Omega_1$, $T\Omega_2$ and $T\Omega_3$, respectively, to describe these (non-elementary) knot moves. In trying to use these moves we may see the utility of the over-under convention we have adopted for the projection link. Although $T\Omega_1$, $T\Omega_2$ and $T\Omega_3$ are not elementary, they do have two advantages. First, there is no "realization problem"; if we see a possible move Ω_j then we can do a move $T\Omega_j$. Secondly, to recognize these moves one needs only look at one disk in the given 2-manifold, not two or three.

5.3. *Using knot moves on the Klein bottle.* We will illustrate the moves defined in the previous section for surfaces in $\mathbb{R}^4$.

A simple way to embed the Klein bottle in $\mathbb{R}^4$ is as follows: take a circle in 3-dimensional half-space, $\mathbb{R}^3_+$, which does not touch the bounding plane. If we spin this circle, we will obtain an embedded torus in $\mathbb{R}^4$, but if we give the circle a rotation of 180° about an axis in the plane of the circle as we spin it, we will obtain an embedding of a Klein bottle. By arranging our geometry appropriately we will obtain a projection of this embedding as shown in Figure 15(a)—this projection has two branch points, a and b,

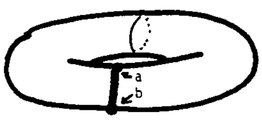


Fig. 15(a)

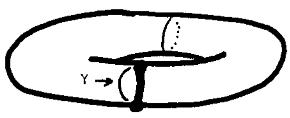


Fig.15(b)


Fig.15(c)

with an arc of crossing points connecting them. Let γ be an arc from a to b on the projection as shown in Figure 15(b); a careful examination of a neighborhood of γ shows that we may perform a knot move of type (d). If we do this knot move, we will obtain an immersion of the Klein bottle with a circle of self- intersection. Upon redrawing, one can see this is the familiar drawing as shown in Figure 15(c). A general version of this process of "eliminating branch points" will be discussed at the end of this section.

5.4. *Triviality of one-twist spun trefoil.* Next we consider projections of twist-spun knottings, for definitions see [ZM]. The most straightforward method of obtaining a projection of a twist-spun knotting is similar to that used to get the projection of the spun knotting. View the twisting as an isotopy h_t of a knotted arc, α, in half-space $\mathbb{R}^3_+$, where $0 \le t \le 2\pi$. Let α^*_t be the projection onto $\mathbb{R}^2_+$ of $h_t(\alpha)$. Then the projection of the twist spun knotting will be $\cup_{0\le t\le 2\pi} a^*_t$ where a^*_t is a path in $\mathbb{R}^2_+(t)$ corresponding to α^*_t, where using cylindrical coordinates (r, t, z), $\mathbb{R}^3_+(t)$ is the half-plane determined by fixed angle t. This is described in more detail in [RS2].

In Figure 16 we indicate how to obtain the projection link of the one-twist spun trefoil. The first step is to analyze the spinning isotopy, h_t, as a sequence of elementary knot moves. These are shown in Figure 16(a), for values t_i where $0 < t_1 < t_2 < \cdots < t_{11} < 2\pi$.

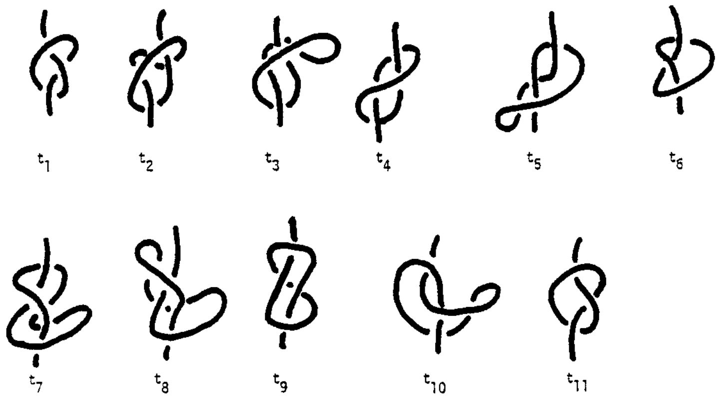


Fig. 16(a)

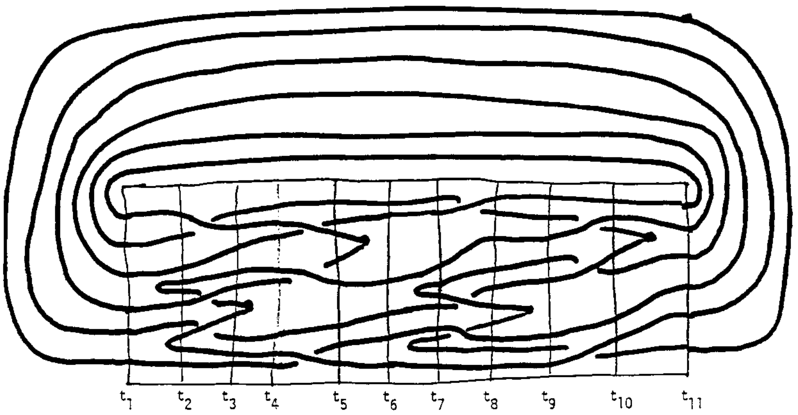


Fig. 16(b)

Think of this as an isotopy of a large proper subarc, α', of α, then consider the rectangle in the sphere corresponding to $[t_1, t_{11}] \times \alpha'$. This is the rectangle shown in Figure 16(b). The intersection of the projection link with the segment $\{t_i\} \times \alpha'$ corresponds to the double point set of the projection at $\{t_i\}$ as shown in Figure 16(a). In the region $[t_i, t_{i+1}] \times \alpha'$ we see the effect of a single knot move (heavy dots on the curves are to indicate branch points). To draw the completed projection link (as a subset of the plane) we connect the arcs within the rectangle with the arcs outside as shown. In Figure 17(a) we show three disks in the projection link, and note that the portion of the projection link within these is of the type shown in Figure 13 (with different relative positions of the heights of the three disks). Thus we can make a move of type $T\Omega_2$ and obtain a new projection link as shown in Figure 17(b). We may do another type $T\Omega_2$ move on the diagram shown in Figure 17(b). Recall that the projection link really lies on a sphere; one of the disks we need for our move will contain the point of the sphere not shown in our planar drawing in Figure 17(b). After our move we get Figure 17(c). The projection link now has been

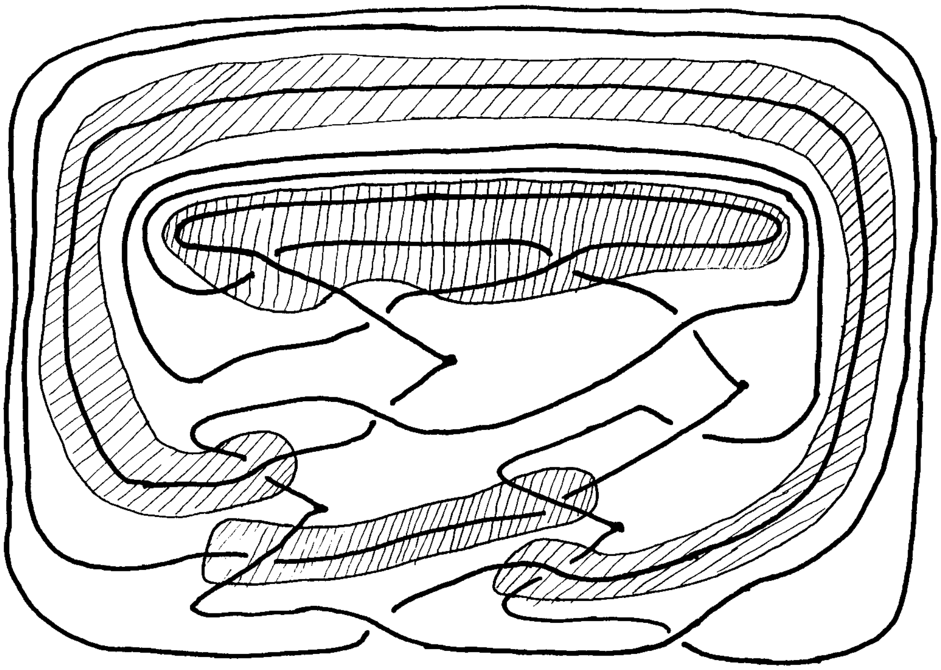


Fig. 17(a)

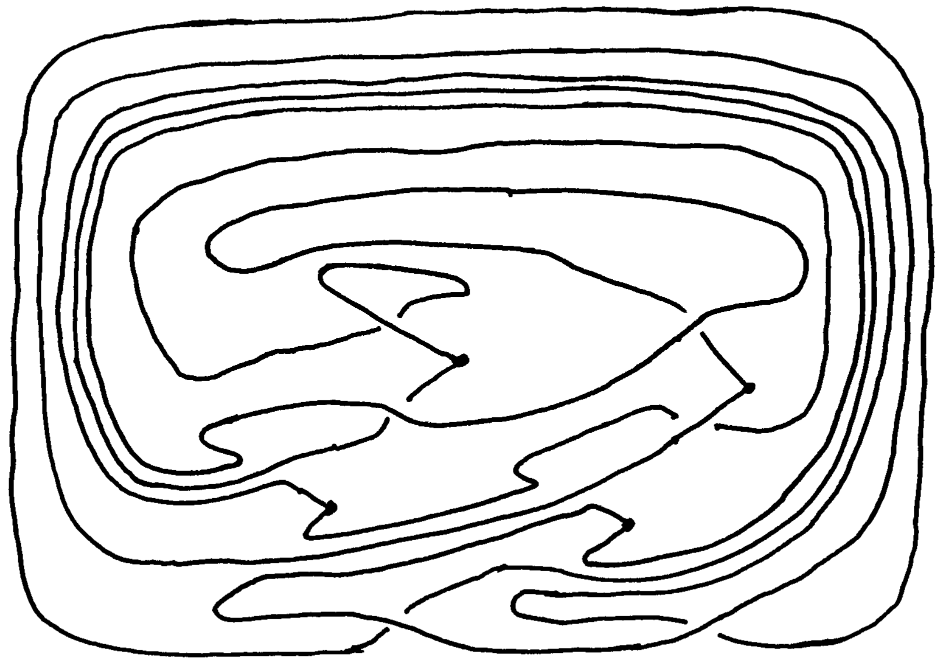


Fig. 17(b)

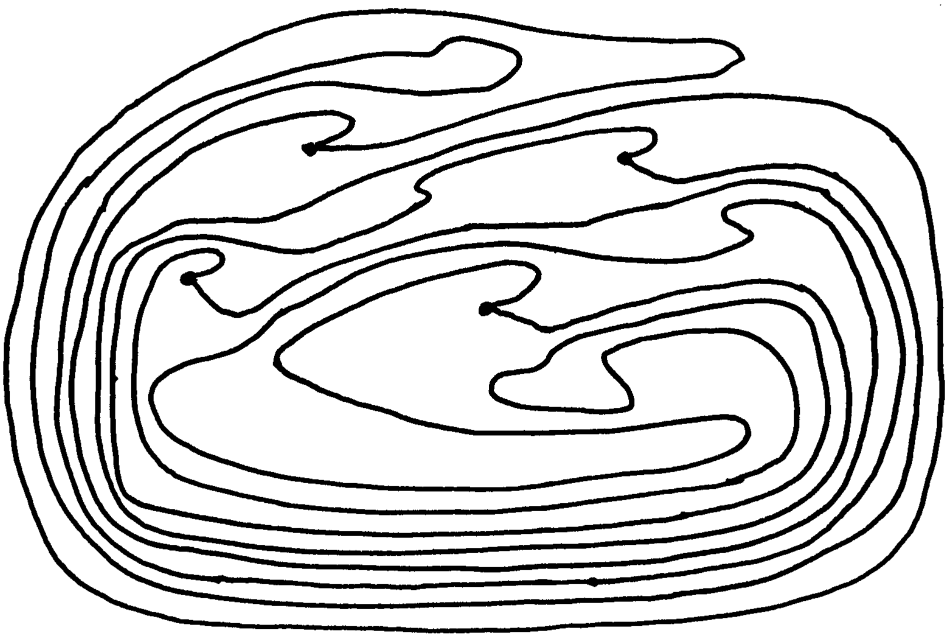


Fig. 17(c)

transformed to two disjoint circles on the sphere, each of which contains two branch points; after two moves of type (c) we will obtain the empty projection link. Thus we have explicitly shown an isotopy which "unknots" the one-twist spun trefoil. Of course the triviality of one-twist spun knotting, for any knotted arc, is well known [ZM], but our technique is quite different.

5.5. *Projections and projection links for twist spun knots.* The principle problem with the projection of the twist-spun knotting mentioned above is that it is necessary to express the isotopy which gives the twist as a sequence of elementary knot moves, a tedious, ad hoc process. The next description avoids these problems. For these examples we are concerned with the case of an arbitrary number of twists.

We begin our analysis by viewing the arc which we are twisting as being an arc with "the knotted part" contained in a thick disk in $\mathbb{R}^3_+$. The twisting isotopy is then thought of as a rotation of this disk with the "the knotted part" rigidly embedded inside, see Figure 18(a). (Note: In Figure 18 we will not draw the knot inside our thickened disk.) Next we deform this isotopy by tilting the plane of the disk slightly as shown in Figure 18(b).

Fig. 18(a)

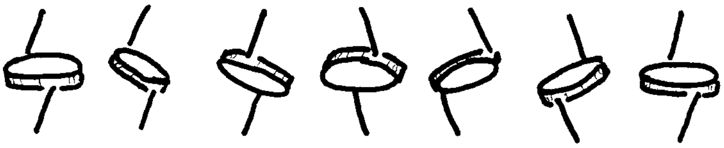


Fig. 18(b)

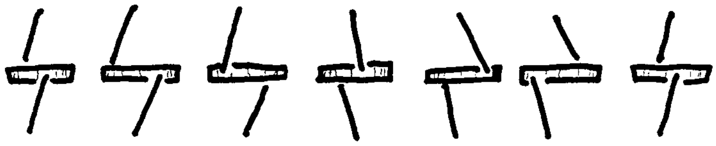


Fig. 18(c)

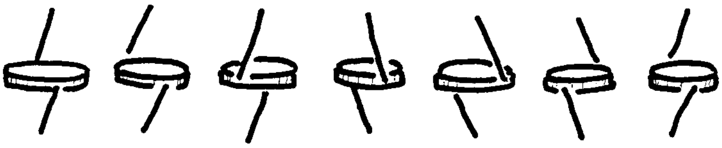


Fig. 18(d)

In Figure 18(c) we show this isotopy deformed so that the plane of the disk has tipped by 90°, here we see this disk rotating in a plane. We can then deform this isotopy so as to "look down at the disk as it spins" and we will get Figure 18(d). If the knot in the disk would be trivial then we could draw our projection link as shown in Figures 19(a) and 19(b). After a knot move of type (f) we obtain Figure 19(c) (which we can reduce to the empty projection link by a move of type (c).

Fig. 19(a)

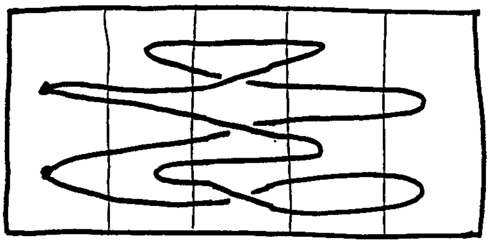


Fig. 19(b)

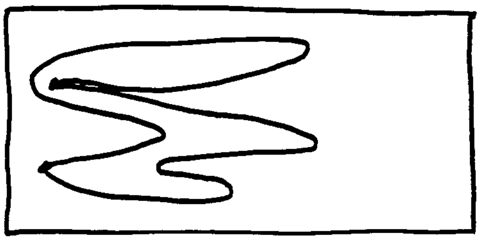


Fig. 19(c)

In Figure 20(a) we do the same construction with the one-twist spun trefoil. Figure 20(b) shows the portion of the projection link within a rectangle as has been done for

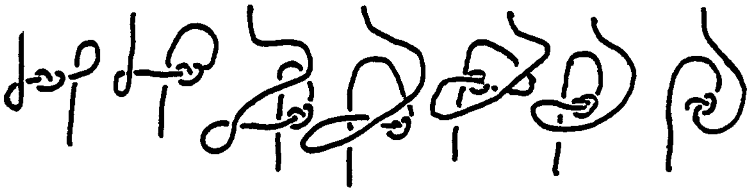


Fig. 20(a)

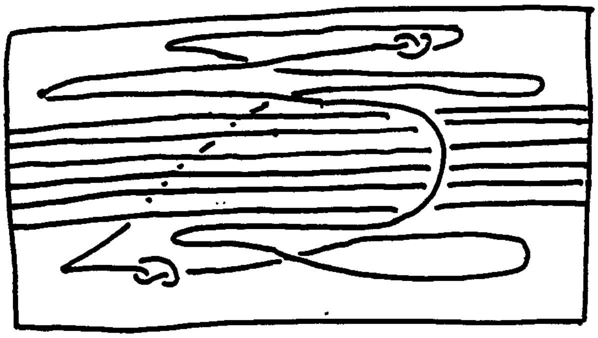


Fig. 20(b)

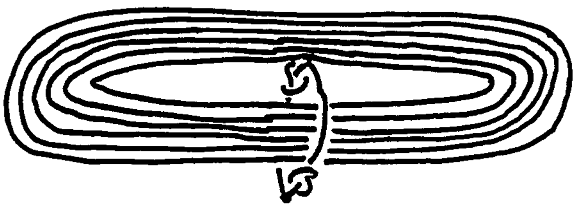


Fig. 20(c)

our previous examples. Performing a move of type (f) we can draw the projection link as shown in Figure 20(c). With careful graphical bookkeeping, it can be seen that the projection itself looks as shown in Figure 21(a). To understand Figure 21(a) we can divide it into three pieces; these are displayed in Figure 21(b). A similar analysis will show the

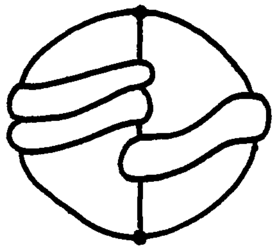


Fig. 21(a)

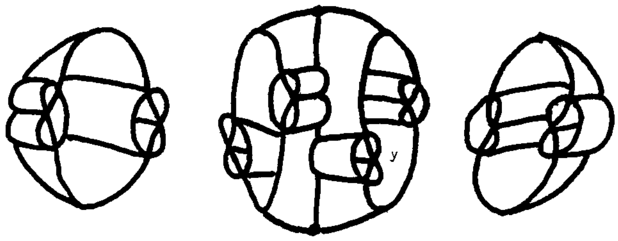


Fig. 21(b)

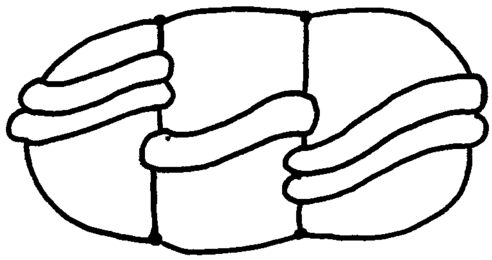


Fig. 21(c)

(non-trivial) two-twist spun trefoil to have a projection as indicated in Figure 21(c). By viewing Figure 21(a) as being a subset of S^3 by taking a different point (such as indicated by the point y in Figure 21(b)) for the point at infinity, one can see that the one-twist spun trefoil has a projection as shown in 22(a). We thus obtain the following description

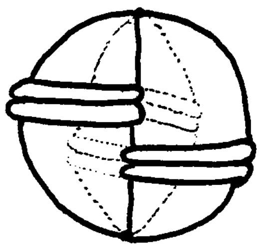


Fig. 22(a)

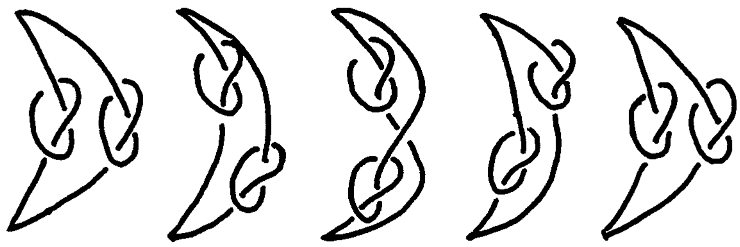


Fig. 22(b)

Fig. 22(c)

of the one-twist spun trefoil: Take the two knotted arcs joined at their endpoints a and b which lie in $\mathbb{R}^2 \subseteq \mathbb{R}^3_+$; spin this set while doing the interchange isotopy as shown in Figure 22(b). The subset of $\mathbb{R}^4$ we obtain is the one-twist spun trefoil. A similar analysis for the two-twist spun trefoil would show that it can be described as follows: Take the

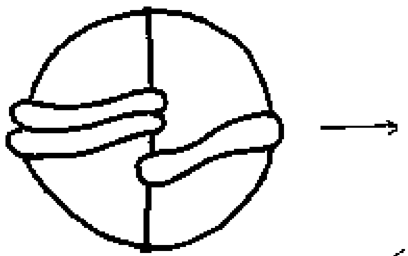


Fig. 23(a)

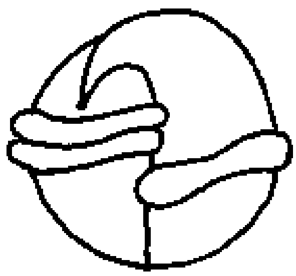


Fig. 23(b)

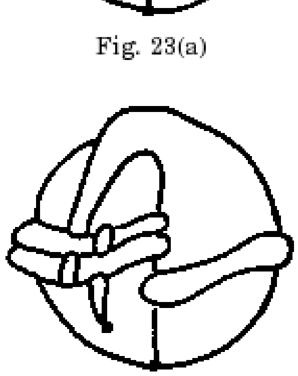


Fig. 23(c)

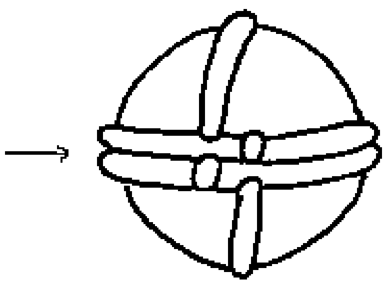


Fig. 23(d)

three joined arcs as shown in Figure 22(c) and spin, keeping the endpoints fixed while performing the isotopy which takes the arc on the left and passes it over the other two.

Returning to the projection shown in Figure 21(a), we might wish to “simplify” it by eliminating the branch points. One way to do this is as shown in Figure 23. The idea here is to bring the two branch points close together, then use a knot move of type (d) to eliminate the branch points. (For more details, compare with Figure 24 which is later discussed.) Figure 24(a) shows another view of this knotting. To get a better look at details of this projection, we redraw the portions of the projection enclosed in the 3-balls, shown in Figure 24(b) These portions are shown in Figures 24(c) and (d). In the

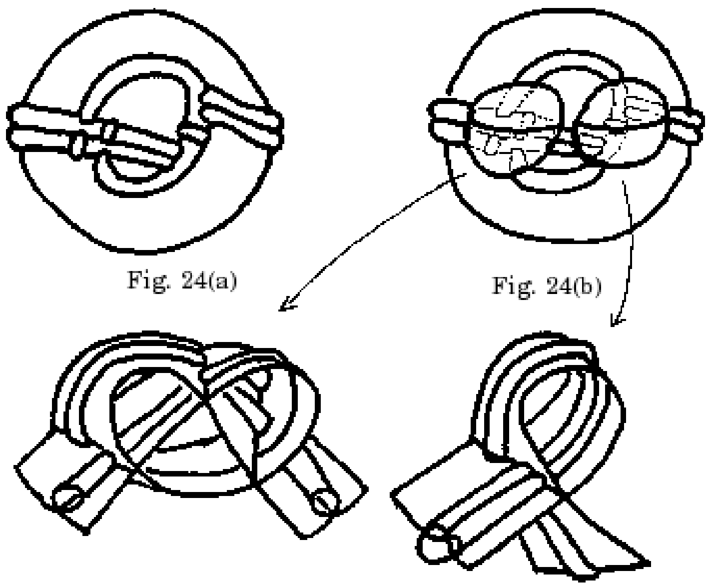


Fig. 24(a) Fig. 24(b)

Fig. 24(c) Fig. 24(d)

language introduced in [RS2] we can now describe this knotting as: The knot is obtained by taking a non-trivial link, L, of two components with a projection with two crossings, spinning the trefoil (with its usual projection) about one component, and spinning the trivial knot (with a one-crossing projection) about the other. In Figure 24(a), we think of one component of L as being equatorial, the other a small circle. However such a link can be drawn on the sphere so that both circles are nearly equatorial, thus obtaining a

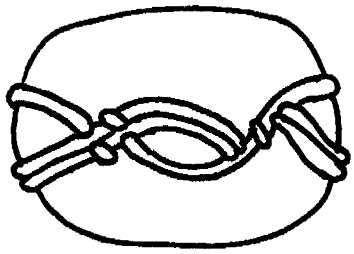


Fig. 25(a)

Fig. 25(b)

projection of our knotting as shown in Figure 25(a). Now the line in $\mathbb{R}^3$ through the north pole and south pole in this projection is the image under our projection of a plane in $\mathbb{R}^4$ and with this plane as our axis of rotation, we can see that the given knotting is the one obtained by projecting the spinning of the isotopy as shown in Figure 25(b) (similarly one can see that if we spin the isotopy which repeats that shown in Figure 25(b) twice, we obtain the two-twist spun trefoil).

6. History of this paper and update. At the request of the editors of these Proceedings, I append a note on the history of this paper. The contents of this paper originally constituted approximately 1/3 of a monograph "Projections and Moves of Higher Dimensional Knots". The submitted monograph was returned, unread, after several years, whereupon it was subdivided into three journal-sized papers, [RS3, RS4] and this paper. This paper was then submitted manuscript, returned, unread, after two years; resubmitted elsewhere and finally accepted for publication.

At this point I had gone on to other things (computational aspects related to these results) and lost enthusiasm to publish until encouraged by the editors of these Proceedings. During this time period, some of this material has appeared elsewhere:

• The graphics associated with the knot moves for surfaces in $\mathbb{R}^4$ have been reproduced in [RS6].

• Carter and Saito, following the author's papers in preprint form, as they mention, defined a refinement of the knot moves for surfaces by a further slicing, [C-S].

• Based on the preprint and other personal communications, the demonstration of the unknottedness of the 1-twist spun trefoil, Figures 17(a)–(c) has appeared in very nice computer graphics form in [H1], together with three-dimensional graphics showing the projections of the knot moves. Software [H2] allows visualization of projection links as shown in [H1].

• Computational work of the author has progress in two areas. A program to realize knot moves for knotted surfaces: a preliminary program to do similar calculations for classical knottings has been written [RS5, RS6]; knot moves for surfaces, however, involves problems in computational geometry that have made progress very slow. Videos [RS7, RS8, RS9] provide visualization of aspects of knotted surfaces and in each one can find many examples of isotopies of knotted surfaces and the knot moves discussed in this paper.

Bibliography update

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