KNOT THEORY
BANACH CENTER PUBLICATIONS, VOLUME 42
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1998

ESTIMATING THE STATES OF THE KAUFFMAN BRACKET SKEIN MODULE

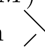
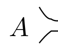
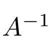
DOUG BULLOCK

Department of Mathematics
Boise State University
Boise, Idaho 83725, U.S.A.
E-mail: bullock@math.idbsu.edu

Abstract. The states of the title are a set of knot types which suffice to create a generating set for the Kauffman bracket skein module of a manifold. The minimum number of states is a topological invariant, but quite difficult to compute. In this paper we show that a set of states determines a generating set for the ring of $SL_2(\mathbf{C})$ characters of the fundamental group, which in turn provides estimates of the invariant.

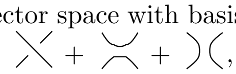
1. Introduction. Skein modules were introduced by Józef Przytycki in 1987 as a class of 3-manifold invariants somewhat analogous to homology groups. The basic idea is to divide the linear space of all links by a set of meaningful relations. The known skein relations for various polynomial invariants of links are obvious examples. We will consider the module corresponding to the Kauffman bracket polynomial.

Let M be a 3-manifold. Its Kauffman bracket skein module is an algebraic invariant, $K(M)$, built from the set of all framed links in M. By a framed link we mean an embedded collection of annuli considered up to isotopy in M. The set of framed links is denoted $\mathcal{L}_M$ and it includes the empty link $\emptyset$.

Let R denote the ring of Laurent polynomials $\mathbf{Z}[A, A^{-1}]$ and $R(\mathcal{L}_M)$ the free R-module with basis $\mathcal{L}_M$. Let $S(M)$ be the smallest submodule of $R(\mathcal{L}_M)$ containing all possible expressions of the form  $- A$  $- A^{-1}$ , or $\bigcirc + A^2 + A^{-2}$. The first relation, called a skein relation, involves three links embedded identically except as the diagrams indicate, with framing annuli assumed to lie flat in the page. The second relation, called a framing relation, tells how to remove a trivial component from a link. We define $K(M)$ to be the quotient $R(\mathcal{L}_M)/S(M)$.

In a previous work [5] the author developed a connection between $K(M)$ and the set of $SL_2(\mathbf{C})$ characters of $\pi_1(M)$, which was exploited at the level of linear spaces and

1991 *Mathematics Subject Classification*: Primary 57M99; Secondary 32G81.
The paper is in final form and no version of it will be published elsewhere.

linear functionals. This paper extends the idea to multiplicative structures. We will be working with a vector space $V(M)$ closely related to $K(M)$. Let $\mathcal{H}_M$ denote the set of (unframed) links in M, including $\emptyset$, but considered only up to homotopy. Let $\mathbf{C}\mathcal{H}_M$ be the complex vector space with basis $\mathcal{H}_M$. It contains a subspace $W(M)$ generated by all skein relations , and $\bigcirc + 2$. The vector space $V(M)$ is the quotient $\mathbf{C}\mathcal{H}_M/W(M)$.

If $K(M)$ is specialized at $A = -1$ the skein relations imply that crossings can be changed at will. Hence, the only difference between this specialization and $V(M)$ is the use of complex coefficients. There is a commutative multiplication on $\mathbf{C}\mathcal{H}_M$, given by $L_1 L_2 = L_1 \cup L_2$, for which $\emptyset$ is a unit. Since $W(M)$ is an ideal the multiplication descends to $V(M)$, which, as an algebra, is finitely generated [4, Theorem 1].

Our main result is that $V(M)$ maps onto the ring of $SL_2(\mathbf{C})$ characters. In Section 2 we review the necessary character theory and prove this theorem. As a corollary, we obtain a lower bound on the number of generators of $V(M)$ as an algebra. This in turn provides a lower bound on a previously intractable invariant of manifolds, the aforementioned minimum number of states. Section 3 contains a precise definition of states and estimates of the invariant for a number of examples.

2. The ring of characters. For the rest of this article the term representation will refer to a homomorphism of groups $\rho : \pi_1(M) \to SL_2(\mathbf{C})$. The character of a representation ρ is the composition $\chi_\rho = \text{trace} \circ \rho$. The set of all characters will be denoted $X(M)$. For each $\gamma \in \pi_1(M)$ there is a function $t_\gamma : X(M) \to \mathbf{C}$ given by $\chi_\rho \mapsto \chi_\rho(\gamma)$. The following lemma appears to have been discovered independently by Vogt [13], Fricke [7] and Horowitz [9] as well as Culler and Shalen [6].

Lemma 1. *There exists a finite set* $\{\gamma_1, \ldots, \gamma_m\} \subset \pi_1(M)$ *such that every* t_γ *is an element of* $\mathbf{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$.

Theorem 1. (Culler-Shalen) *If every* t_γ *is in* $\mathbf{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$, *then* $X(M)$ *is a closed algebraic subset of* $\mathbf{C}^m$.

It follows from [6] that any two parameterizations of $X(M)$ are equivalent via polynomial maps, so their coordinate rings are isomorphic. We call this ring the *ring of characters of* $\pi_1(M)$ and denote it $\mathcal{R}(M)$.

Each knot K in $\mathcal{H}_M$ determines a unique t_γ as follows. Let $\vec{K}$ denote an unspecified orientation on K. Choose any $\gamma \in \pi_1(M)$ such that $\gamma \simeq \vec{K}$. Since trace is invariant under conjugation and inversion in $SL_2(\mathbf{C})$, it makes sense to write $\chi_\rho(K) = \chi_\rho(\gamma)$. Thus K determines the map t_γ. Conversely, any t_γ is determined by some K. This correspondence remains well defined at the level of $V(M)$.

Theorem 2. *The map* $\Phi : V(M) \to \mathcal{R}(M)$ *given by*

$$\Phi(K)(\chi_\rho) = -\chi_\rho(K)$$

is a well defined surjective map of algebras. If $V(M)$ *is generated by a set of knots* $K_1, \ldots, K_m$ *then* $-\Phi(K_1), \ldots, -\Phi(K_m)$ *are coordinates on* $X(M)$.

P r o o f. Let $\mathbf{C}^{X(M)}$ denote the algebra of functions from $X(M)$ to $\mathbf{C}$. If K is a knot in $\mathcal{H}_M$ let

$$\widetilde{\Phi}(K)(\chi_\rho) = -\chi_\rho(K)$$

and extend to $\mathbf{C}\mathcal{H}_M$, requiring that $\widetilde{\Phi}$ be a map of algebras.

Consider the image of $W(M)$ under $\widetilde{\Phi}$. For the framing relation $\bigcirc + 2\,\emptyset$ we have

$$\widetilde{\Phi}(\bigcirc + 2\,\emptyset)(\chi_\rho) = -\chi_\rho(\bigcirc) + 2 = -\mathrm{tr}(\mathrm{Id}) + 2 = 0.$$

Let $L + L_0 + L_\infty$ be a skein relation in which the first two terms are knots, and denote the components of the third term by K_1 and K_2. Choose a base point and two loops a and b in $\pi_1(M, *)$ so that, for some orientation, $ab \simeq \vec{L}$. With favorable orientations on the other knots we have $ab^{-1} \simeq \vec{L}_0$, $a \simeq \vec{K}_1$, and $b \simeq \vec{K}_2$. Choose any χ_ρ. Setting $A = \rho(a)$ and $B = \rho(b)$, we see that

$$\begin{aligned}\widetilde{\Phi}(L + L_0 + L_\infty)(\chi_\rho) &= -\chi_\rho(L) - \chi_\rho(L_0) + \chi_\rho(K_1)\chi_\rho(K_2) \\ &= -\mathrm{tr}(AB) - \mathrm{tr}(AB^{-1}) + \mathrm{tr}(A)\mathrm{tr}(B) = 0.\end{aligned}$$

These types of relations generate $W(M)$ as an ideal, so $\widetilde{\Phi}$ descends to a well defined map

$$\Phi : V(M) \to \mathbf{C}^{X(M)}$$

which is determined by its values on knots.

Let $K_1, \ldots, K_m$ be generators of $V(M)$. Every element of $V(M)$ is a polynomial in these knots, so $\Phi(V(M)) \subset \mathbf{C}[-\Phi(K_1), \ldots, -\Phi(K_m)]$. Since each t_γ is some $-\Phi(K)$, Lemma 1 and Theorem 1 imply that the functions $-\Phi(K_i)$ are coordinates on $X(M)$. It follows that Φ maps onto $\mathcal{R}(M)$. ∎

3. Estimating the number of states. In this section we will apply Theorem 2 and other results to estimate the value of a 3-manifold invariant defined in [4]. The invariant is roughly the minimum number of knot types in a set of generators for $K(M)$, subject to the geometric condition that every resolving tree terminates in the set of generators.

The first step is to define what a resolving tree is. Let T be a finite, connected, contractible graph in which each vertex is labeled by a monomial in $R\mathcal{L}_M$, i.e. $\pm A^k L$. Assume there is a bivalent vertex labeled L. There is a well defined potential function on the vertices of T given by the number of edges in a path to L. Suppose also that T has the following properties.

1. Each vertex of T with non-zero potential is either univalent or trivalent.
2. Each non-univalent vertex is incident to two edges ending in higher potential vertices.
3. Let pL_1 be a non-univalent vertex. If qL_2 and rL_3 are the higher potential vertices specified by property 2, then $pL_1 - qL_2 - rL_3$ is a skein or framing relation.

The univalent vertices of a resolving tree are called *leaves*. The sum over all leaves is equal to L in $K(M)$, so T is called a *resolving tree for L*.

Given a set of module generators for $K(M)$, one may easily construct a resolving tree for any L such that the sum over all leaves is a linear combination of generators. However, leaves need not lie in the generating set, provided they cancel with other leaves. Hence the additional geometric condition alluded to above. Let $\mathcal{G}$ be a set of links that generate $K(M)$ as a module. If every link in M has a resolving tree with every leaf in $\mathcal{G}$ we say that $\mathcal{G}$ is *complete*.

Finally, there are manifolds for which $K(M)$ is an algebra, and we want to mimic that structure as closely as possible in a general setting. Let $\{K_1, ..., K_n\}$ be a collection of framed knots in M. For a positive integer t_i we form a link $K_i^{t_i}$ by taking t_i parallel copies of K_i, each one a push off along the framing. Let $\mathcal{G}$ be the set of all framed links of the form $K_1^{t_1} \cup K_2^{t_2} \cup \cdots \cup K_n^{t_n}$. (Note that this expression does not define a unique link; $\mathcal{G}$ must contain all possibilities.) If $\mathcal{G}$, together with $\emptyset$, is a complete generating set for $K(M)$, then we say $\{\emptyset, K_1, ..., K_n\}$ is a set of *states* for M.

In [4] it is shown that every compact orientable M admits a set of states, so we can define $s(M)$ to be the minimum number of states. Like any invariant defined by minimizing a geometric occurrence, $s(M)$ is frightfully difficult to compute. Estimates from above may be obtained by construction, but estimates from below require more subtle techniques. Fortunately we have the following inequalities.

PROPOSITION 1. *Let M be a compact, orientable 3-manifold.*

$$\begin{aligned} s(M) - 1 \quad &\geq \quad \textit{minimum number of generators for } V(M) & (1)\\ &\geq \quad \textit{minimum number of generators for } \mathcal{R}(M) & (2)\\ &= \quad \textit{smallest } m \textit{ such that } X(M) \subset \mathbf{C}^m & (3)\\ &\geq \quad \textit{dimension of } X(M). & (4) \end{aligned}$$

Furthermore, the last inequality is strict unless $X(M)$ is equivalent to affine space. In particular, it is strict if $X(M)$ is reducible or singular.

Proof. Since the non-empty states generate $V(M)$, we have (1). Theorem 2 implies (2). The definition of $\mathcal{R}(M)$ implies (3), and (4) is obvious. If equality holds in (4) then closure forces $X(M) = \mathbf{C}^m$, which is irreducible and smooth. ∎

The following is a list of the manifolds for which we have bounds on $s(M)$.

1. $F \times I$, F is a compact orientable surface with first Betti number $\beta(F)$.
2. $L(p, q)$, any lens space other than S^3.
3. M_q, the exterior of a $(2, q)$-torus knot.
4. $M_3(r)$, surgery on a right hand trefoil knot with integer framing r.

Deleting an open cell or capping a spherical boundary component has no effect on $K(M)$ or $X(M)$, so the list may be taken to include those modified manifolds as well.

PROPOSITION 2. $s(F \times I) \leq 2^{\beta(F)}$.

Proof. Follows from [4, Corollary 1]. ∎

PROPOSITION 3. $s(L(p, q)) \leq 2$.

Proof. Hoste and Przytycki [10, 11] have constructed generating sets that contain only 2 states. ∎

PROPOSITION 4. $s(M_q) \leq 3$.

Proof. Generating sets with 3 states are constructed in [2]. ∎

PROPOSITION 5. *If* $r \neq 1$ *or* 3 *then* $s(M_3(r)) \leq 3$. *If* $r = 1$ *or* 3 *then* $s(M_3(r)) \leq 2$.

Proof. Also by construction [3]. ∎

PROPOSITION 6. *If* $\partial F \neq \emptyset$ *then* $s(F \times I) \geq 3\beta(F) - 2$. *If* $\beta(F) = 3$ *this can be sharpened to* $s(F \times I) \geq 8$.

Proof. Since $\partial F \neq \emptyset$, $\pi_1(F \times I)$ is free of rank $\beta(F)$. It follows from [12, Theorem 2.2] that $X(F \times I)$ has dimension $3\beta(F) - 3$. If $\beta(F) = 3$ then $\pi_1(F \times I)$ is free on the set $\{a, b, c\}$. The assignments

$$x = t_a, \quad y = t_b, \quad z = t_c, \quad u = t_{ab},$$
$$v = t_{ac}, \quad w = t_{bc} \quad \text{and} \quad t = t_{abc}$$

realize $X(F \times I) \subset \mathbf{C}^7$. Horowitz [9] has shown that $X(M)$ is the zero set of

$$x^2 + y^2 + z^2 + u^2 + v^2 + w^2 + uvw - xyu - xzv - yzw - 4 + t^2 - txw - tyv - tzu + txyz.$$

The variety is singular at $(2, 2, 2, 2, 2, 2, 2)$, so Proposition 1 (4) applies. ∎

PROPOSITION 7. *If* F *is hyperbolic and* $\partial F = \emptyset$ *then* $s(F \times I) \geq 3\beta(F) - 5$.

Proof. It is implicitly shown in [8] that the dimension of $X(F \times I)$ is $6g - 6$. ∎

PROPOSITION 8. *If* F *is a torus then* $s(F \times I) \geq 4$.

Proof. Let $\pi_1(M) = \langle a, b \mid ab = ba \rangle$ with $x = t_a$, $y = t_b$ and $z = t_{ab}$. With these coordinates on $\mathbf{C}^3$, $X(M)$ is the zero set of

$$x^2 + y^2 + z^2 - xyz - 4,$$

which has dimension 2 and a singularity at $(2, 2, 2)$. ∎

PROPOSITION 9. $s(M_q) \geq 3$.

Proof. Write $\pi_1(M_q)$ as $\langle a, b \mid (ab)^n a = b(ab)^n \rangle$, where $n = (q - 1)/2$. Let ω be the principle q-th root of -1 and let $p(y)$ be a polynomial whose roots are $\{\omega^i + \omega^{-i} \mid 1 \leq i \leq n\}$. If $x = t_a$ and $y = t_{ab}$ then, from [1, Propositions 9.1(i), A.4*.11(ii) and A.4*.13(i)], we know $X(M_q) \subset \mathbf{C}^2$ is the zero set of $p(y)(x^2 - y - 2)$. Its dimension is 1 and it has $n + 1$ components. ∎

PROPOSITION 10. *If* M *is* S^3 *or a punctured* S^3 *then* $s(M) = 1$. *Otherwise,* $s(M) \geq 2$.

Proof. Begin by capping all spherical boundary components with balls. This has no effect on either $K(M)$ or $s(M)$, so continue to denote the result by M. If M is closed, [4, Theorem 3] says that either $M = S^3$ or $s(M) \geq 2$. If $\partial M \neq \emptyset$ then $H_1(M, \mathbf{Z}/2\mathbf{Z})$ has positive rank. From [4, Lemma 6] we again have $s(M) \geq 2$. ∎

The estimates in Propositions 2–10 are summarized in Table 1. Those pertaining to handlebodies are obtained by considering the manifold to be a product of a planar surface and an interval.[1]

[1]When these results were presented at the Banach Center Mini Semester on Knot Theory, August 1995, they were the best known estimates. Recently, however, Przytycki and Sikora have announced that $s(F \times I) = 2^{\beta(F)}$, and that $s(M_3(6)) = 3$.

Manifold	Lower bound	Upper bound
B^3, $S^2 \times I$ and S^3	1	1
Solid torus	2	2
Genus 2 handlebody	4	4
Punctured torus $\times I$	4	4
Torus $\times I$	4	4
Genus 3 handlebody	8	8
Twice punctured torus $\times I$	8	8
$F \times I$; $\partial F \neq \emptyset$ and $\chi(F) < 0$	$3\beta(F) - 2$	$2^{\beta(F)}$
$F \times I$; $\partial F = \emptyset$ and $\chi(F) < 0$	$3\beta(F) - 5$	$2^{\beta(F)}$
$L(p, q)$	2	2
M_q	3	3
$M_3(1)$ and $M_3(3)$	2	2
$M_3(r)$; $r \neq 1$ or 3	2	3

Table 1. Summary of $s(M)$ estimates

References

[1] G. Brumfiel and H. M. Hilden, *$SL(2)$ representations of finitely presented groups*, Contemporary Mathematics **187** (1995).

[2] D. Bullock, *The $(2,\infty)$-skein module of the complement of a $(2, 2p+1)$ torus knot*, J. Knot Theory Ramifications **4** no. 4 (1995) 619–632.

[3] D. Bullock, *On the Kauffman bracket skein module of surgery on a trefoil*, Pacific J. Math., to appear.

[4] D. Bullock, *A finite set of generators for the Kauffman bracket skein algebra*, preprint.

[5] D. Bullock, *Estimating a skein module with $SL_2(\mathbf{C})$ characters,* Proc. Amer. Math. Soc., to appear.

[6] M. Culler and P. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. Math. **117** (1983) 109–146.

[7] R. Fricke and F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Vol. 1, B. G. Teubner, Leipzig 1897.

[8] W. Goldman, *The Symplectic Nature of Fundamental Groups of Surfaces*, Adv. Math. **54** no. 2 (1984) 200–225.

[9] R. Horowitz, *Characters of free groups represented in the two dimensional linear group,* Comm. Pure Appl. Math. **25** (1972) 635–649.

[10] J. Hoste and J. H. Przytycki, *The $(2,\infty)$-skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications **2** no. 3 (1993) 321–333.

[11] J. Hoste and J. H. Przytycki, *The Kauffman bracket skein module of $S^1 \times S^2$*, Math Z. **220** (1995) 65–73.

[12] W. Magnus, *Rings of Fricke characters and automorphism groups of free groups*, Math. Z. **170** (1980), 91-103.

[13] H. Vogt, *Sur les invariants fondamentaux des équations différentielles linéaires du second ordre*, Ann. Sci. École Norm. Supér. III. Sér. **6** (1889), 3–72.