ESTIMATING THE STATES OF THE KAUFFMAN BRACKET SKEIN MODULE

DOUG BULLOCK
Department of Mathematics
Boise State University
Boise, Idaho 83725, U.S.A.
E-mail: bullock@math.idbsu.edu

Abstract. The states of the title are a set of knot types which suffice to create a generating set for the Kauffman bracket skein module of a manifold. The minimum number of states is a topological invariant, but quite difficult to compute. In this paper we show that a set of states determines a generating set for the ring of $SL_2(\mathbb{C})$ characters of the fundamental group, which in turn provides estimates of the invariant.

1. Introduction. Skein modules were introduced by Józef Przytycki in 1987 as a class of 3-manifold invariants somewhat analogous to homology groups. The basic idea is to divide the linear space of all links by a set of meaningful relations. The known skein relations for various polynomial invariants of links are obvious examples. We will consider the module corresponding to the Kauffman bracket polynomial.

Let $M$ be a 3-manifold. Its Kauffman bracket skein module is an algebraic invariant, $K(M)$, built from the set of all framed links in $M$. By a framed link we mean an embedded collection of annuli considered up to isotopy in $M$. The set of framed links is denoted $\mathcal{L}_M$ and it includes the empty link $\emptyset$.

Let $R$ denote the ring of Laurent polynomials $\mathbb{Z}[A, A^{-1}]$ and $R(\mathcal{L}_M)$ the free $R$-module with basis $\mathcal{L}_M$. Let $S(M)$ be the smallest submodule of $R(\mathcal{L}_M)$ containing all possible expressions of the form $-A \mathcal{L}_M - A^{-1} \mathcal{L}_M$, or $\mathcal{L}_M + A^2 + A^{-2}$. The first relation, called a skein relation, involves three links embedded identically except as the diagrams indicate, with framing annuli assumed to lie flat in the page. The second relation, called a framing relation, tells how to remove a trivial component from a link. We define $K(M)$ to be the quotient $R(\mathcal{L}_M)/S(M)$.

In a previous work [5] the author developed a connection between $K(M)$ and the set of $SL_2(\mathbb{C})$ characters of $\pi_1(M)$, which was exploited at the level of linear spaces and

\begin{flushleft}
1991 Mathematics Subject Classification: Primary 57M99; Secondary 32G81.
The paper is in final form and no version of it will be published elsewhere.
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linear functionals. This paper extends the idea to multiplicative structures. We will be working with a vector space $V(M)$ closely related to $K(M)$. Let $\mathcal{H}_M$ denote the set of (unframed) links in $M$, including $\emptyset$, but considered only up to homotopy. Let $\mathcal{C}H_M$ be the complex vector space with basis $\mathcal{H}_M$. It contains a subspace $W(M)$ generated by all skein relations $\frown + \frown + \frown$, and $\emptyset + 2$. The vector space $V(M)$ is the quotient $\mathcal{C}H_M/W(M)$.

If $K(M)$ is specialized at $A = -1$ the skein relations imply that crossings can be changed at will. Hence, the only difference between this specialization and $V(M)$ is the use of complex coefficients. There is a commutative multiplication on $\mathcal{C}H_M$, given by $L_1L_2 = L_1 \cup L_2$, for which $\emptyset$ is a unit. Since $W(M)$ is an ideal the multiplication descends to $V(M)$, which, as an algebra, is finitely generated [4, Theorem 1].

Our main result is that $V(M)$ maps onto the ring of $SL_2(\mathbb{C})$ characters. In Section 2 we review the necessary character theory and prove this theorem. As a corollary, we obtain a lower bound on the number of generators of $V(M)$ as an algebra. This in turn provides a lower bound on a previously intractable invariant of manifolds, the aforementioned minimum number of states. Section 3 contains a precise definition of states and estimates of the invariant for a number of examples.

2. The ring of characters. For the rest of this article the term representation will refer to a homomorphism of groups $\rho : \pi_1(M) \to SL_2(\mathbb{C})$. The character of a representation $\rho$ is the composition $\chi_\rho = \text{trace} \circ \rho$. The set of all characters will be denoted $X(M)$.

For each $\gamma \in \pi_1(M)$ there is a function $t_\gamma : X(M) \to \mathbb{C}$ given by $\chi_\rho \mapsto \chi_\rho(\gamma)$. The following lemma appears to have been discovered independently by Vogt [13], Fricke [7] and Horowitz [9] as well as Culler and Shalen [6].

Lemma 1. There exists a finite set $\{\gamma_1, \ldots, \gamma_m\} \subset \pi_1(M)$ such that every $t_\gamma$ is an element of $\mathbb{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$.

Theorem 1. (Culler-Shalen) If every $t_\gamma$ is in $\mathbb{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$, then $X(M)$ is a closed algebraic subset of $\mathbb{C}^m$.

It follows from [6] that any two parameterizations of $X(M)$ are equivalent via polynomial maps, so their coordinate rings are isomorphic. We call this ring the ring of characters of $\pi_1(M)$ and denote it $\mathcal{R}(M)$.

Each knot $K$ in $\mathcal{H}_M$ determines a unique $t_\gamma$ as follows. Let $\vec{K}$ denote an unspecified orientation on $K$. Choose any $\gamma \in \pi_1(M)$ such that $\gamma \simeq \vec{K}$. Since trace is invariant under conjugation and inversion in $SL_2(\mathbb{C})$, it makes sense to write $\chi_\rho(K) = \chi_\rho(\gamma)$. Thus $K$ determines the map $t_\gamma$. Conversely, any $t_\gamma$ is determined by some $K$. This correspondence remains well defined at the level of $V(M)$.

Theorem 2. The map $\Phi : V(M) \to \mathcal{R}(M)$ given by

$$\Phi(K)(\chi_\rho) = -\chi_\rho(K)$$

is a well defined surjective map of algebras. If $V(M)$ is generated by a set of knots $K_1, \ldots, K_m$ then $-\Phi(K_1), \ldots, -\Phi(K_m)$ are coordinates on $X(M)$. 
Proof. Let $C^{X(M)}$ denote the algebra of functions from $X(M)$ to $C$. If $K$ is a knot in $\mathcal{H}_M$ let
\[ \tilde{\Phi}(K)(\chi_\rho) = -\chi_\rho(K) \]
and extend to $C\mathcal{H}_M$, requiring that $\tilde{\Phi}$ be a map of algebras.

Consider the image of $W(M)$ under $\tilde{\Phi}$. For the framing relation $\bigcirc + 2 \emptyset$ we have
\[ \tilde{\Phi}(\bigcirc + 2 \emptyset)(\chi_\rho) = -\chi_\rho(\bigcirc) + 2 = -\text{tr}(\text{Id}) + 2 = 0. \]

Let $L + L_0 + L_\infty$ be a skein relation in which the first two terms are knots, and denote the components of the third term by $K_1$ and $K_2$. Choose a base point and two loops $a$ and $b$ in $\pi_1(M,*)$ so that, for some orientation, $ab \simeq \tilde{L}$. With favorable orientations on the other knots we have $ab^{-1} \simeq \tilde{L}_0$, $a \simeq \tilde{K}_1$, and $b \simeq \tilde{K}_2$. Choose any $\chi_\rho$. Setting $A = \rho(a)$ and $B = \rho(b)$, we see that
\[ \tilde{\Phi}(L + L_0 + L_\infty)(\chi_\rho) = -\chi_\rho(L) - \chi_\rho(L_0) + \chi_\rho(K_1)\chi_\rho(K_2) = -\text{tr}(AB) - \text{tr}(AB^{-1}) + \text{tr}(A)\text{tr}(B) = 0. \]

These types of relations generate $W(M)$ as an ideal, so $\tilde{\Phi}$ descends to a well defined map $\Phi : V(M) \to C^{X(M)}$ which is determined by its values on knots.

Let $K_1, \ldots, K_m$ be generators of $V(M)$. Every element of $V(M)$ is a polynomial in these knots, so $\Phi(V(M)) \subset C[-\Phi(K_1), \ldots, -\Phi(K_m)]$. Since each $t_\gamma$ is some $-\Phi(K)$, Lemma 1 and Theorem 1 imply that the functions $-\Phi(K_i)$ are coordinates on $X(M)$. It follows that $\Phi$ maps onto $\mathcal{R}(M)$. ■

3. Estimating the number of states. In this section we will apply Theorem 2 and other results to estimate the value of a 3-manifold invariant defined in [4]. The invariant is roughly the minimum number of knot types in a set of generators for $K(M)$, subject to the geometric condition that every resolving tree terminates in the set of generators.

The first step is to define what a resolving tree is. Let $T$ be a finite, connected, contractible graph in which each vertex is labeled by a monomial in $R\mathcal{L}_M$, i.e. $\pm A^k L$. Assume there is a bivalent vertex labeled $L$. There is a well defined potential function on the vertices of $T$ given by the number of edges in a path to $L$. Suppose also that $T$ has the following properties.

1. Each vertex of $T$ with non-zero potential is either univalent or trivalent.
2. Each non-univalent vertex is incident to two edges ending in higher potential vertices.
3. Let $pL_1$ be a non-univalent vertex. If $qL_2$ and $rL_3$ are the higher potential vertices specified by property 2, then $pL_1 - qL_2 - rL_3$ is a skein or framing relation.

The univalent vertices of a resolving tree are called leaves. The sum over all leaves is equal to $L$ in $K(M)$, so $T$ is called a resolving tree for $L$.

Given a set of module generators for $K(M)$, one may easily construct a resolving tree for any $L$ such that the sum over all leaves is a linear combination of generators. However, leaves need not lie in the generating set, provided they cancel with other leaves. Hence
the additional geometric condition alluded to above. Let $\mathcal{G}$ be a set of links that generate $K(M)$ as a module. If every link in $M$ has a resolving tree with every leaf in $\mathcal{G}$ we say that $\mathcal{G}$ is complete.

Finally, there are manifolds for which $K(M)$ is an algebra, and we want to mimic that structure as closely as possible in a general setting. Let $\{K_1, \ldots, K_n\}$ be a collection of framed knots in $M$. For a positive integer $t_i$ we form a link $K_{t_i}$ by taking $t_i$ parallel copies of $K_i$, each one a push off along the framing. Let $\mathcal{G}$ be the set of all framed links of the form $K_{t_1} \cup K_{t_2} \cup \cdots \cup K_{t_n}$. (Note that this expression does not define a unique link; $\mathcal{G}$ must contain all possibilities.) If $\mathcal{G}$, together with $\emptyset$, is a complete generating set for $K(M)$, then we say $\{\emptyset, K_1, \ldots, K_n\}$ is a set of states for $M$.

In [4] it is shown that every compact orientable $M$ admits a set of states, so we can define $s(M)$ to be the minimum number of states. Like any invariant defined by minimizing a geometric occurrence, $s(M)$ is frightfully difficult to compute. Estimates from above may be obtained by construction, but estimates from below require more subtle techniques. Fortunately we have the following inequalities.

**Proposition 1.** Let $M$ be a compact, orientable 3-manifold.

\[
s(M) - 1 \geq \text{minimum number of generators for } V(M) \geq \text{minimum number of generators for } R(M) = \text{smallest } m \text{ such that } X(M) \subset \mathbb{C}^m \geq \text{dimension of } X(M).
\]

Furthermore, the last inequality is strict unless $X(M)$ is equivalent to affine space. In particular, it is strict if $X(M)$ is reducible or singular.

**Proof.** Since the non-empty states generate $V(M)$, we have (1). Theorem 2 implies (2). The definition of $R(M)$ implies (3), and (4) is obvious. If equality holds in (4) then closure forces $X(M) = \mathbb{C}^m$, which is irreducible and smooth. □

The following is a list of the manifolds for which we have bounds on $s(M)$.

1. $F \times I$, $F$ is a compact orientable surface with first Betti number $\beta(F)$.
2. $L(p, q)$, any lens space other than $S^3$.
3. $M_q$, the exterior of a $(2, q)$-torus knot.
4. $M_3(r)$, surgery on a right hand trefoil knot with integer framing $r$.

Deleting an open cell or capping a spherical boundary component has no effect on $K(M)$ or $X(M)$, so the list may be taken to include those modified manifolds as well.

**Proposition 2.** $s(F \times I) \leq 2^{\beta(F)}$.

**Proof.** Follows from [4, Corollary 1]. □

**Proposition 3.** $s(L(p, q)) \leq 2$.

**Proof.** Hoste and Przytycki [10, 11] have constructed generating sets that contain only 2 states. □

**Proposition 4.** $s(M_q) \leq 3$. 

From [4, Lemma 6] we again have $s(M_\alpha(r)) \leq 3$. If $r = 1$ or $3$ then $s(M_\beta(r)) \leq 2$.

**Proof.** Also by construction [3].

**Proposition 6.** If $\partial F \neq \emptyset$ then $s(F \times I) \geq 3\beta(F) - 2$. If $\beta(F) = 3$ this can be sharpened to $s(F \times I) \geq 8$.

**Proof.** Since $\partial F \neq \emptyset$, $\pi_1(F \times I)$ is free of rank $\beta(F)$. It follows from [12, Theorem 2.2] that $X(F \times I)$ has dimension $3\beta(F) - 3$. If $\beta(F) = 3$ then $\pi_1(F \times I)$ is free on the set $\{a, b, c\}$. The assignments

$$
x = t_a, \quad y = t_b, \quad z = t_c, \quad u = t_{ab},
$$

$v = t_{ac}, \quad w = t_{bc}$ and $t = t_{abc}$

realize $X(F \times I) \subset \mathbb{C}^7$. Horowitz [9] has shown that $X(M)$ is the zero set of

$$x^2 + y^2 + z^2 + u^2 + v^2 + w^2 + uvw - xuy - xzv - yzw - 4 + t^2 - txw - tyv - tzu + txz.$$

The variety is singular at $(2, 2, 2, 2, 2, 2, 2)$, so Proposition 1 (4) applies.

**Proposition 7.** If $F$ is hyperbolic and $\partial F = \emptyset$ then $s(F \times I) \geq 3\beta(F) - 5$.

**Proof.** It is implicitly shown in [8] that the dimension of $X(F \times I)$ is $6g - 6$.

**Proposition 8.** If $F$ is a torus then $s(F \times I) \geq 4$.

**Proof.** Let $\pi_1(M) = \langle a, b \mid ab = ba \rangle$ with $x = t_a$, $y = t_b$ and $z = t_{ab}$. With these coordinates on $\mathbb{C}^3$, $X(M)$ is the zero set of

$$x^2 + y^2 + z^2 - xyz - 4,$$

which has dimension 2 and a singularity at $(2, 2, 2)$.

**Proposition 9.** $s(M_q) \geq 3$.

**Proof.** Write $\pi_1(M_q)$ as $\langle a, b \mid (ab)^n a = b(ab)^n \rangle$, where $n = (q - 1)/2$. Let $\omega$ be the principal $q$-th root of $-1$ and let $p(y)$ be a polynomial whose roots are $\{\omega^i + \omega^{-i} \mid 1 \leq i \leq n\}$. If $x = t_a$ and $y = t_{ab}$ then, from [1, Propositions 9.1(i), A.4*.11(ii) and A.4*.13(i)], we know

$$X(M_q) \subset \mathbb{C}^2$$

is the zero set of $p(y)(x^2 - y - 2)$. Its dimension is 1 and it has $n + 1$ components.

**Proposition 10.** If $M$ is $S^3$ or a punctured $S^3$ then $s(M) = 1$. Otherwise, $s(M) \geq 2$.

**Proof.** Begin by capping all spherical boundary components with balls. This has no effect on either $K(M)$ or $s(M)$, so continue to denote the result by $M$. If $M$ is closed, [4, Theorem 3] says that either $M = S^3$ or $s(M) \geq 2$. If $\partial M \neq \emptyset$ then $H_1(M, \mathbb{Z}/2\mathbb{Z})$ has positive rank. From [4, Lemma 6] we again have $s(M) \geq 2$.

The estimates in Propositions 2–10 are summarized in Table 1. Those pertaining to handlebodies are obtained by considering the manifold to be a product of a planar surface and an interval.\(^1\)

\(^1\)When these results were presented at the Banach Center Mini Semester on Knot Theory, August 1995, they were the best known estimates. Recently, however, Przytycki and Sikora have announced that $s(F \times I) = 2^{d(F)}$, and that $s(M_2(6)) = 3$. 
Table 1. Summary of $s(M)$ estimates

<table>
<thead>
<tr>
<th>Manifold</th>
<th>Lower bound</th>
<th>Upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B^3$, $S^2 \times I$ and $S^3$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Solid torus</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Genus 2 handlebody</td>
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<td>4</td>
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<tr>
<td>Punctured torus $\times I$</td>
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<td>4</td>
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<tr>
<td>Torus $\times I$</td>
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<td>4</td>
</tr>
<tr>
<td>Genus 3 handlebody</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Twice punctured torus $\times I$</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>$F \times I$; $\partial F \neq \emptyset$ and $\chi(F) &lt; 0$</td>
<td>$3\beta(F) - 2$</td>
<td>$2\beta(F)$</td>
</tr>
<tr>
<td>$F \times I$; $\partial F = \emptyset$ and $\chi(F) &lt; 0$</td>
<td>$3\beta(F) - 5$</td>
<td>$2\beta(F)$</td>
</tr>
<tr>
<td>$M_4(p, q)$</td>
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<td>2</td>
</tr>
<tr>
<td>$M_4$</td>
<td>3</td>
<td>3</td>
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<tr>
<td>$M_4(1)$ and $M_4(3)$</td>
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<td>2</td>
</tr>
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<td>$M_4(r)$; $r \neq 1$ or $3$</td>
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References