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CHORD DIAGRAMS IN THE CLASSIFICATION OF MORSE–SMALE FLOWS ON 2-MANIFOLDS

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1. Introduction. Let $\mathfrak{X}(M)$ denote the space of smooth flows on a compact, connected 2-manifold M with C^∞ topology. The problems of (1) finding a suitable equivalence relation R in $\mathfrak{X}(M)$, and (2) classifying the equivalence classes of $\mathfrak{X}(M)$ mod R are among the most important in the generic theory of flows. Topological conjugacy can be regarded as a suitable equivalence relation in $\mathfrak{X}(M)$. There is a well known approach to studying the dynamical systems when the problem of topological classification of smooth flows is reduced to that of classification of the corresponding combinatorial schemes up to isomorphism. To put it in another way, one tries to assign a combinatorial scheme to each smooth flow on M in such a way that two flows will be topologically equivalent if and only if the corresponding schemes are isomorphic in natural sense. In particular, if the class of smooth flows under consideration consists of those which don't contain the nonclosed Poisson stable trajectories both in positive and negative directions, the combinatorial schemes give us a complete topological invariant of them [1]. For the Morse–Smale flows on closed oriented 2-manifolds their distinguished graph is an invariant of such kind [8]. The above mentioned approach to the problem has been developed in [5]. In particular, the problem of topological classification of minimal Morse–Smale flows without closed orbits on closed oriented 2-manifolds has been reduced to the identification of words up to some equivalence relation. In Section 1 of our paper we shall use a similar construction to find a complete topological invariant of Morse–Smale flows without closed orbits on closed 2-manifolds. This invariant is defined in terms of so-called chord diagrams. The chord diagrams appear to encode closed generic plane curves [9] or patterns of generic singular knots and links in sphere S^3 [3], but their real meaning differs from ours. The main result of the first Section is Theorem 1.2.

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In Section 2 we consider the isotopy equivalence relation in $\mathfrak{X}(M)$. This is a more subtle equivalence than the topological one. A necessary and sufficient condition for two minimal Morse–Smale flows without closed orbits on a closed 2-manifold to be isotopically equivalent is given in terms of generators of the fundamental group $\pi_1(M)$ (Theorem 2.4). One example of two topologically equivalent but isotopically nonequivalent minimal Morse–Smale flows without closed orbits on a torus is given.

To prove Theorem 2.4 we shall use some results of Epstein concerning the isotopies of closed curves on a 2-manifold [4]. The results obtained here seem to us to be new.

For the convenience of the reader we shall repeat the relevant material from [1, 6, 7] without proofs, thus making our exposition self-contained. We assume throughout that M is a smooth (C^∞), connected, closed 2-manifold (so that M is a PL-manifold) unless otherwise stated. Since M is compact we won't distinguish between the smooth vector field X (dynamical system) on M and the smooth flow φ^X generated by X. In the majority we follow the notation used in [6,7].

1. Encoding of Morse–Smale systems on two-manifolds by chord diagrams. We introduce the notion of Morse–Smale system following J. Palis and W. de Melo. For more detailed relevant information about Morse–Smale systems see also [7].

Let N be a smooth compact manifold.

Definition 1.1. A smooth vector field X on N will be called a *Morse–Smale system* provided

1) X has a finite number of singular elements (singular points and closed orbits), each of hyperbolic type;

2) If σ_1 and σ_2 are singular elements of X, then the stable manifold $W^s(\sigma_1)$ associated with σ_1 and unstable manifold $W^u(\sigma_2)$ associated with σ_2 have transverse intersection;

3) The set $\Omega(X)$ consisting of non-wandering points of X coincides with the union of singular elements of X.

Let $p \in N$. Denote by $\omega(p)$ the set $\{q \in N | X_{t_n}(p) \to q$ for some sequence $t_n \to \infty\}$, and by $\alpha(p)$ the set $\{q \in N | X_{t_n}(p) \to q$ for some sequence $t_n \to -\infty\}$. It is known [7] that for Morse–Smale systems on a compact smooth 2-dimensional manifold N the following properties are valid:

a) There exists no trajectory joining any two saddle points of X;

b) For each $p \in N$ there are singular elements α_i and α_j such that $\omega(p) = \alpha_i$ and $\alpha(p) = \alpha_j$.

For abbreviation, we write M-S system instead of Morse–Smale system. Analogously, we write M-S flow instead of Morse–Smale flow.

Let $b_i(X)$ ($\overline{b_i}(X)$) denote the number of singular points (closed orbits) of index i of an M-S system X on an n-dimensional smooth manifold N. The following notion was introduced by Sharko [11].

Definition 1.2. Let X be an M-S system on a compact smooth n-dimensional manifold N. Then X is called *minimal* provided there is no M-S system Y on N with the following property:

for each i, $1 \le i \le n$, $b_i(Y) \le b_i(X)$, $\overline{b_i}(Y) \le \overline{b_i}(X)$ and
there exists j, $1 \le j \le n$, such that $b_j(Y) < b_j(X)$, or $\overline{b_j}(Y) < \overline{b_j}(X)$.

Note that every M-S system on a compact manifold contains at least one source and one sink [7]. In the sequel an M-S system will always be assumed to be an M-S system without closed orbits.

Let E be a smooth function from the compact manifold N into R and let $\triangle$ denote the set of critical points of E. Denote by $XE(p)$ the derivation of the function E along the vector $X(p)$, where $p \in N$. It is well known that for every nondegenerate point c_i of E there exists a coordinate system (N_i, x_i) such that

$$E \circ x_i^{-1} = E(c_i) + Q(x),$$

where Q is a nonsingular quadratic form in x whose index is the same as the index of the Hessian of E at c_i.

Definition 1.3. Let X be a smooth vector field (flow) on N without closed orbits. Then a function E from N into R will be called an *energy function* for X provided

1) $XE(p) < 0$ for all $p \in M - \triangle$, i.e. E is decreasing along the trajectories of X or the trajectories of X are transversal to the level lines of E;

2) $\triangle$ consists of nondegerate critical points, i.e. $\triangle$ consists of critical points where the Hessian of E has nullity 0 (so $\triangle$ is a finite set $\{c_1, \dots, c_n\}$).

3) there exists a constant $k \ge 0$ such that on each N_i

$$-XE(p) \ge k \cdot d(p, c_i)^2 \quad \text{for } p \in N_i,\ i = \overline{1, n}.$$

For the definition of energy function in a general case see [6].

Meyer shows (Theorem 1, [6]) that if X is a M-S system then there exists an energy function E for the system.

Definition 1.4. Let X be an M-S system on a smooth compact manifold M. An energy function E for X will be called a *nice energy function* for X provided all the sources of X lie in $E^{-1}(1)$, all the saddle points of X lie in $E^{-1}(0)$, and all the sinks of X lie in $E^{-1}(-1)$.

The construction of Theorem 2 [6] could be made to yield a nice energy function for every M-S system on N.

Definition 1.5. Let X and Y be two flows on a compact smooth manifold N. Then X and Y will be called *topologically equivalent* if there exists a homeomorphism $h : N \to N$ which sends the trajectories of X into the trajectories of Y. Then we shall say that X and Y are topologically equivalent under the homeomorphism h.

Definition 1.6. Two functions E and E' from N to R are said to be *topologically equivalent* if there exist homeomorphisms f and g, $f : N \to N$ and $g : R \to R$ such that the following diagram commutes:

$$\begin{array}{ccc} N & \xrightarrow{E} & R \\ {\scriptstyle f}\downarrow\uparrow{\scriptstyle f^{-1}} & & {\scriptstyle g^{-1}}\uparrow\downarrow{\scriptstyle g} \\ N & \xrightarrow{E'} & R \end{array}$$

By Meyer's Proposition (see [6], p. 1039) the topological equivalence of flows X and Y on a compact 2-dimensional smooth manifold follows from the topological equivalence of corresponding nice energy functions E and E' for X and Y respectively.

Let X be an M-S flow on M. (Recall that M is assumed to be a smooth closed 2-manifold and X is assumed to be an M-S flow without closed orbits).

DEFINITION 1.7. A trajectory L of X will be called *singular* provided a) L is a singular element of X (singular point) or b) L is a trajectory joining the singular point to a saddle point.

Denote by C the union all of singular trajectories of X.

Remark 1.1. Every connected component of $M \setminus C$ is a simply connected domain, which is homeomorphic to a simply connected domain of R^2 (actually an open 2-cell), and is filled with nonclosed trajectories which are Poisson unstable both in the positive and negative directions. Each such component looks as in Fig. 1. We shall call the components of $M \setminus C$, which look as in Fig. 1, the *cells of type A*.

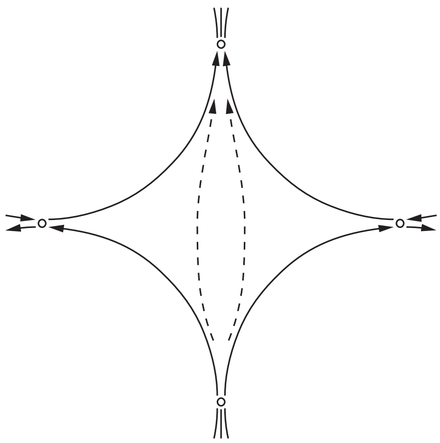

Fig. 1

For more detailed relevant information and definitions concerning the structure of connected components of $M \setminus C$ in the general case of flows on a closed, smooth 2-manifold see [1,2].

The set consisting of all singular trajectories and cells of type A determines the cellular decomposition of M.

Let u_s be an arbitrary sink of X. Let A_s be the family consisting of all cells B of type A such that $u_s \in \overline{B}$, and let C_s be the family of trajectories of X which join the sink u_s to the saddle points of X. Put $B_s = (\bigcup A_s) \cup (\bigcup C_s) \cup \{u_s\}$. In view of Remark 1.1 the following assertion holds.

PROPOSITION 1.1. *B_s is an open 2-cell. Moreover, if u_s and u_t are two distinct sinks then $B_s \cap B_t = \emptyset$.*

Definition 1.8. By an *n-component chord diagram* G we mean n distinct planar circles $S_1, \dots, S_n$ with several chords having distinct end points. The number of chords in the chord diagram will be called the *order* of the diagram. If the chords of G are additionally equipped with sign "+" or "−", such a chord diagram will be called *signed.* If all the circles of the chord diagram are oriented we will call it *oriented.* The ends of chords will be called the *distinguished points.* The chord which joins u to v is denoted by $\{u, v\}$.

Remark 1.2. Chords can join points of distinct circles as depicted in Fig. 2. The geometry of the chords is irrelevant.

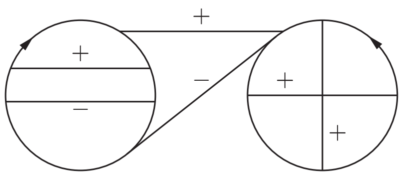

Fig. 2

Let G and G' be two n-component, oriented, signed chord diagrams consisting of circles $S_1, \dots, S_n$ and $S'_1, \dots, S'_n$ respectively.

Definition 1.9. Two chord diagrams G and G' will be called *isomorphic* if there exist homeomorphisms h_i, $h_i : S_i \to S'_{j_i}$, $i = \overline{1, n}$, where $j_i \neq j_k$ if $i \neq k$, which preserve the orientations of all the circles and preserve all the chords as well as their signs. The isomorphism class of the chord diagram G is denoted by (G).

We can think of two isomorphic chord diagrams as being the same.

Now we shall describe the correspondence between M-S flows on closed connected 2-manifolds and oriented signed chord diagrams.

Let X be an M-S flow on M with sources $v_1, \dots, v_n$, sinks $u_1, \dots, u_l$ and saddle points $w_1, \dots w_m$. If $m = 0$, then M is the sphere S^2. It is well known that the set of topological equivalence classes of M-S flows under consideration on S^2 consists of only one element [7]. We shall assign to every such M-S flow on S^2 the chord diagram consisting of one circle (with arbitrary orientation) and having no chord. So we assume that $m > 0$.

It follows from [6] that there exists a nice energy function E for X. By definition of a nice energy function we have $E(v_1) = \dots = E(v_n) = 1$, $E(w_1) = \dots = E(w_m) = 0$ and $E(u_1) = \dots = E(v_l) = -1$. The trajectories of X are transverse to the level lines of E. For small enough $\varepsilon > 0$, the level line $E^{-1}(1 - \varepsilon)$ consists of m disjoint simple closed curves $\check{S}_1, \dots, \check{S}_n$ on M. Each $\check{S}_i$ bounds a disk D_i in M. Fix an orientation on each D_i. Denote by $e^i_1, \dots, e^i_{p_i}$ all the points of the set $C \cap \check{S}_i$ (recall that C is the union of singular trajectories of X). For every point e^i_j, $1 \leq j \leq p_i$, there exists only one trajectory which joins v_i to some saddle point u_s. The choice of orientation on D_i determines the cyclic sequence of points $e^i_1, \dots, e^i_{p_i}$, which corresponds to the circuit of S_i in accordance with the induced orientation on it. We assign a circle S_i to every connected component $\check{S}_i$ of the level line $E^{-1}(1 - \varepsilon)$, by choosing a homeomorphism $g_i : S_i \to \check{S}_i$, $i = \overline{1, n}$.

It will cause no confusion if we use the same letter e^i_j to designate the point e^i_j and its preimage $g_i^{-1}(e^i_j)$. For every i choose the orientation of S_i which is carried from $\check{S}_i$ by the homeomorphism g_i^{-1}, $i = \overline{1,n}$. Then the points $e^i_1, \dots, e^i_{p_i}$ will be taken as distinguished on S_i, $i = \overline{1,n}$. Two distinguished points $e^i_t \in S_i$, $e^j_s \in S_j$, will be joined in a chord diagram G_X consisting of the circles $S_1, \dots, S_n$, if and only if there exist trajectories L_i, L_j and a saddle point v such that $\alpha(L_i) = v_i$, $\alpha(L_j) = v_j$, $L_i \cap \check{S}_i = e^i_t$, $L_j \cap \check{S}_j = e^j_s$, and $\omega(L_i) = \omega(L_j) = v$.

Next, the preimage $E^{-1}(1-\varepsilon, 1]$ consists of n open disks each of which contains the corresponding source. Let $\{e_1, e_2\}$ be an arbitrary chord in G_X, $e_i \in S_i$, $e_j \in S_j$. By definition of the chord in G_X there exist trajectories L_1 and L_2 of X such that $\alpha(L_1) = e_1$, $\alpha(L_2) = e_2$, and $\omega(L_1) = \omega(L_2)$. Denote by $L(e_i, e_j)$ the submanifold $E^{-1}[-1, 1-\varepsilon] \cap (L_1 \cup L_2 \cup \omega(L_1))$ of the manifold $E^{-1}[-1, 1-\varepsilon]$. Let $U(e_i, e_j)$ be a tubular neighborhood of $L(e_i, e_j)$ in $E^{-1}[-1, 1-\varepsilon]$. $U(e_i, e_j)$ can be regarded as a ribbon homeomorphic to the rectangle $[-1//2, 1//2] \times [-1, 1]$, which is glued along the sides a and b to the disks D_i and D_j respectively, where $a = \{-1//2\} \times [-1, 1]$ and $b = \{1//2\} \times [-1, 1]$. The choice of orientation on D_i induces an orientation on the side a and hence an orientation on the ribbon $U(e_i, e_j)$. Similarly the choice of orientation on D_j induces an orientation on $U(e_i, e_j)$. Then we shall equip the chord $\{e_i, e_j\}$ of G_X with the sign "+" if the two orientations of $U(e_i, e_j)$ coincide and with the sign "−" in the opposite case. Thus we have assigned a chord diagram G_X to every M-S flow on M. It is clear that the definition of chord diagram G_X doesn't depend on the choice of $\varepsilon > 0$ or on the choice of homeomorphisms g_i. However such correspondence isn't determined uniquely and depends on the choice of orientation on each disk D_i.

To get around this difficulty in the case of an oriented 2-manifold M, we can choose, on each disk D_i, the orientation which is induced by a fixed orientation on M.

To get around this difficulty in the general case we must introduce a new equivalence relation on the set of chord diagrams.

Let G be a n-component, oriented, signed chord diagram of order m. Consider an arbitrary circle S_i of G. Change simultaneously the orientation of the circle S_i and the sign of each chord which has exactly one end on S_i. Denote by G' the new chord diagram. We shall call the chord diagrams G and G' *elementarily equivalent.*

DEFINITION 1.10. Two n-component chord diagrams G and H of order m will be called *equivalent* if and only if there exists a sequence of elementarily equivalent chord diagrams $G_1, \dots, G_l$ such that $G_1 = G$ and $G_l = H$. Denote by $[G]$ the class of equivalent diagrams which contains the chord diagram G.

Remark 1.3. The relation "to be isomorphic" is more subtle than the relation "to be equivalent" on diagrams.

Remark 1.4. If M is an oriented 2-manifold we may choose in every class $[G_X]$ a chord diagram with all chords having the sign "+". Thus in such a case we can restrict ourselves to the class of ordinary, non-signed, oriented chord diagrams.

It is easy to see that the correspondence $X \to [G_X]$, which assigns an equivalence class of chord diagrams to each Morse–Smale flow X without closed orbits on the closed

2-manifold doesn't depend on the choice of the orientation of disks D_i, and hence is well defined.

THEOREM 1.2. *Let X and Y be M-S flows without closed orbits on a connected, closed 2-manifold M. Then X and Y are topologically equivalent if and only if $[G_X] = [G_Y]$.*

Proof. Suppose that X and Y are topologically equivalent. In accordance with [6] there exist nice energy functions E_X and E_Y for X and Y respectively. Let $f : M \to M$ be the corresponding homeomorphism which sends the trajectories of X into the trajectories of Y, the sinks of X into the sinks of Y, the sources of X into the sources of Y and the saddle points of X into the saddle points of Y. Then f sends the trajectories joining the sources to the saddle points of X into similar ones of Y. By Theorem 2 [6], E_X and E_X are topologically equivalent under the homeomorphism f. The preimage $E_X^{-1}[1-\varepsilon, 1]$ is the union of disjoint disks $D_1, \dots, D_n$ with boundaries $\check{S}_1, \dots, \check{S}_n$, respectively. Then $E_X^{-1}(1)$ is the union of all the sources $v_1, \dots, v_n$ of X and $E_X^{-1}(1-\varepsilon) = \bigcup_{i=1}^n \check{S}_i$. Similarly $E_Y^{-1}[1-\varepsilon, 1]$ is the union of disjoint disks $D'_1, \dots, D'_n$ with boundaries $\check{S}'_1, \dots, \check{S}'_n$, respectively. Moreover, $E_Y^{-1}(1)$ is the union of all sources $v'_1, \dots, v'_n$ of Y and $E_Y^{-1}(1-\varepsilon) = \bigcup_{i=1}^n \check{S}'_i$. Since E_X and E_Y are topologically equivalent under the homeomorphism f, we have $f(D_i) = D'_{j_i}$, $f(\check{S}_i) = \check{S}'_{j_i}$. Without loss of generality we may assume that $j_i = i$ for every i, $1 \le i \le n$. Fix for every i, where $1 \le i \le n$, some orientation on D_i and choose the orientation of D'_i which comes from D_i by the homeomorphism $f|_{D_i} : D_i \to D'_i$. Then for each i the map $f|_{\check{S}_i} : \check{S}_i \to \check{S}'_i$ preserves the orientations of the circles under the fixed orientation of D_i, D'_i, $1 \le i \le n$. Thus $(g'_i)^{-1}(f|_{\check{S}_i})g_i : S_i \to S'_i$ also preserves the orientations, $1 \le i \le n$.

Next, each distinguished point e^i_r on S_i can be regarded as the intersection of the circle $\check{S}_i$ and some trajectory L of X, such that $\alpha(L) = v_i$ and $\omega(L)$ is a saddle point of X. Similarly each distinguished point e^i_q on S'_i can be regarded as the intersection of the circle $\check{S}'_i$ and some trajectory L' of Y, such that $\alpha(L') = v'_i$ and $\omega(L')$ is the saddle point of Y. For this reason f sends the distinguished points of S_i to those of S'_i, for each i, $1 \le i \le n$. Suppose that the points $e^i \in S_i$, $e^j \in S_j$ are joined by chord $\{e^i, e^j\}$ in G_X. It follows from the definition of G_X that there exist trajectories L_1, L_2 and a saddle point w of X such that $\alpha(L_1) = v_i$, $\alpha(L_2) = v_j$, $\omega(L_1) = \omega(L_2) = w$. Then the trajectory $f(L_1)$ joins the source $u'_i = f(u_i)$ to the saddle point $w' = f(w)$. Similarly the trajectory $f(L_2)$ joins the source $u'_j = f(u_j)$ to the saddle point w'. For this reason e'_i and e'_j are joined by the chord $\{e'_i, e'_j\}$ in G_Y.

Next, let e_i and e_j be two arbitrary distinguished points in G_X joined by the chord $\{e_i, e_j\}$. Suppose that the sign of $\{e_i, e_j\}$ is "+". Let us consider the ribbon $U(e_i, e_j)$ which is glued to the disks D_i and D_j along the sides a and b respectively. Then $U(e_i, e_j)$ is a tubular neighbourhood of $(L_1 \cup L_2 \cup \{w\}) \cap E_X^{-1}[-1, 1-\varepsilon]$ in the manifold $E_X^{-1}[-1, 1-\varepsilon])$. Since the sign of $\{e_i, e_j\}$ is "+", the orientation of $U(e_i, e_j)$ which is induced by the orientation of D_i is the same as the one induced by the orientation of D_j. Now let us consider the manifolds $T = D_i \cup U(e_i, e_j) \cup D_j$ and $T' = D'_i \cup U(e'_i, e'_j) \cup D'_j$, where $U(e'_i, e'_j) = f(U(e_i, e_j))$. Then $U(e'_i, e'_j)$ is a tubular neighbourhood of $(f(L_1) \cup f(L_2) \cup \{w'\}) \cap E_Y^{-1}[-1, 1-\varepsilon]$ in the manifold $E_Y^{-1}[-1, 1-\varepsilon]$, where $E_Y^{-1}[-1, 1-\varepsilon] = f(E_X^{-1}[-1, 1-\varepsilon])$ and $f|_T$ is the homeomorphism between T and T'.

Then the orientation of $U(e'_i, e'_j)$ induced by the one on D'_i is the same as that induced by the one on D'_j, so the sign of the chord $\{e'_i, e'_j\}$ is also "+". Reasoning similarly, we can show that the condition "the sign of $\{e_i, e_j\}$ is "−"" implies the condition "the sign of $\{e'_i, e'_j\}$ is "−"". Thus the signed oriented chord diagrams of G_X and G_Y are isomorphic and so $[G_X] = [G_Y]$.

Conversely, let X and Y be two M-S flows such that $[G_X] = [G_Y]$. Let G_X be an n-component chord diagram of order m. The number of sources of X is equal to n and coincides with the number of sources of Y. Denote the sources of X by $v_1, \dots, v_n$ and the sources of Y by $v'_1, \dots, v'_n$. It follows from the description of the correspondence $X \to [G_X]$ that the number of saddle points of X is equal to the number m of chords in G_X, so it coincides with the number of saddle points of Y. Denote by $w_1, \dots, w_m$ the saddle points of X and by $w'_1, \dots, w'_m$ the saddle points of Y. Next, let E_X and E_Y be nice energy functions for X and Y, respectively. Consider the level lines $E_X^{-1}(1-\varepsilon)$ and $E_Y^{-1}(1-\varepsilon)$. As has been stated above, $E_X^{-1}(1-\varepsilon)$ consists of n disjoint circles $\check{S}_1, \dots, \check{S}_n$ which bound disjoint disks $D_1, \dots, D_n$, respectively, and $E_X^{-1}[1-\varepsilon, 1] = \bigcup_{i=1}^n D_i$. Similarly $E_Y^{-1}(1-\varepsilon)$ consists of n disjoint circles $\check{S}'_1, \dots, \check{S}'_n$ which bound disks $D'_1, \dots, D'_n$ respectively and $\bigcup_{i=1}^n D'_i = E_Y^{-1}[1-\varepsilon, 1]$. It follows from the definition of correspondences $X \to [G_X]$ and $Y \to [G_Y]$ that for suitable choices of orientations on $S_1, \dots, S_n$, $S'_1, \dots, S'_n$ the chord diagrams G_X and G_Y will be isomorphic. Let $\varphi : G_X \to G_Y$ be the corresponding isomorphism between signed oriented diagrams G_X and G_Y.

Fix in each $\check{S}_i$ ($\check{S}'_i$ respectively) the orientation which is carried by the homeomorphism g_i (g'_i respectively) from S_i (S'_i respectively), $i = \overline{1, n}$. Choose in each disk D_i (D'_i respectively) the orientation which agrees with the orientation of the circle $\check{S}_i$ ($\check{S}'_i$ respectively). Put $f_i = g'_i \cdot \varphi \cdot (g_i)^{-1}$ for each i, $1 \le i \le n$. Define the map $f : \bigcup_{i=1}^n \check{S}_i \to \bigcup_{i=1}^n \check{S}'_i$ by the formula:

$$f|_{\check{S}_i} = f_i \quad \text{for each } \check{S}_i,\ 1 \le i \le n.$$

Let us consider an arbitrary chord $\{e_i, e_j\}$ of G_X, where e_i is a distinguished point on the circle S_i and e_j is a distinguished point on the circle S_j. It follows from the definition of G_X that there exist trajectories $L(v_i, w)$, $L(v_j, w)$, and a saddle point w of X, such that $\alpha(L(v_i, w)) = v_i$, $\alpha(L(v_j, w)) = v_j$, $\omega(L(v_i, w)) = \omega(L(v_j, w)) = w$, and $e_i \in L(v_i, w)$, $e_j \in L(v_j, w)$. Denote by $L(v_i, w, v_j)$ the set $L(v_i, w) \cup L(v_j, w) \cup \{v_i, w, v_j\}$. Set $L(e_i, w, e_j) = L(v_i, w, v_j) \cap E^{-1}[-1, 1-\varepsilon]$. Since φ is an isomorphism of chord diagrams G_X and G_Y, there exists a chord $\{e'_i, e'_j\}$ of the chord diagram G_Y such that $f(e_i) = e'_i$, $f(e_j) = e'_j$, where e'_i and e'_j are some distinguished points on S'_i and S'_j respectively. This means that there are trajectories $L(v'_i, w')$, $L(v'_j, w')$, and a saddle point w' of Y, such that $\alpha(L(v'_i, w')) = v'_i$, $\alpha(L(v'_j, w')) = v'_j$, $\omega(L(v'_i, w')) = \omega(L(v'_j, w'))$, and $e'_i \in L(v'_i, w')$, $e'_j \in L(v'_j, w')$. Put $\tilde{f}(w) = w'$. Repeat this procedure with all the chords of G_X. Thus we have determined a map $\tilde{f} : \{\text{Saddle points of } X\} \to \{\text{Saddle points of } Y\}$. Since the order of G_X equals the order of G_Y, $\tilde{f}$ is a one-to-one correspondence between the set W of saddle points of X and the set W' of saddle points of Y.

Denote by $L(v'_i, w', v'_j)$ the set $L(v'_i, w') \cup L(v'_j, w') \cup \{v'_i, v'_j, w'\}$. Let further $B_X = \bigcup_{w \in W} L(v_i, w, v_j)$, $B_Y = \bigcup_{w' \in W'} L(v'_i, w', v'_j)$, $S = \bigcup_{i=1}^n S_i$, $S' = \bigcup_{i=1}^n S'_i$. It is obvious that the map $\tilde{f} : W \to W'$ can be extended to the homeomorphism $\tilde{f}_1 : B_X \to B_Y$, such that $\tilde{f}_1(v_i) = v'_i$ for each i, $1 \le i \le n$, and $\tilde{f}_1(e_j) = e'_j$ for each distinguished point e_j, $e_j \in \check{S}_j$. It follows from what has been said that the map $\tilde{f}_1$ can be extended to the map $\tilde{f}_2 : (S \cup B_X) \to (S' \cup B_Y)$ in such a way that $\tilde{f}_2|_S = f$ and $\tilde{f}_2|_{B_X} = \tilde{f}_1$. The next step of our construction consists in an extension of the map $\tilde{f}_2$ to a map $\tilde{f}_3 : D \cup B_X \to D' \cup B_Y$ satisfying the conditions

$$\tilde{f}_3(v_i) = v'_i, \qquad i = \overline{1,n},$$

where $D = \bigcup_{i=1}^n D_i$, $D' = \bigcup_{i=1}^n D'_i$.

Recall that the trajectories of X with $\alpha(L) = v_i$ intersect S_i transversally for each i, $i = \overline{1,n}$. Using a suitable isotopy $F : D' \times I \to D'$ which keeps the set S fixed, we can deform the homeomorphism $\tilde{f}_3$ to, say $\tilde{f}_4$, such that the following property is satisfied:

$\tilde{f}_4$ sends the pieces of trajectories $D_i \cap L$ with $\alpha(L) = v_i$ into the pieces of trajectories $D'_i \cap L'$ with $\alpha(L') = v'_i$ and $\tilde{f}_4(v_i) = v'_i$, $i = \overline{1,n}$.

It may be noted that the homeomorphism $\tilde{f}_4$ sends each trajectory L of X, which joins a source to a saddle point, into a similar trajetory of Y. Now let us consider, for each saddle point w_l of X, a tubular neighbourhood $T(w_l)$ of the submanifold $L(e_i, w_l, e_j)$ in $E_X^{-1}[-1, 1-\varepsilon]$. Recall that $T(w_l)$ is homeomorphic to a rectangle (ribbon) and the trajectories of X in the rectangle (ribbon) look as in Fig. 3, because w_l is an isolated singular point of hyperbolic type.

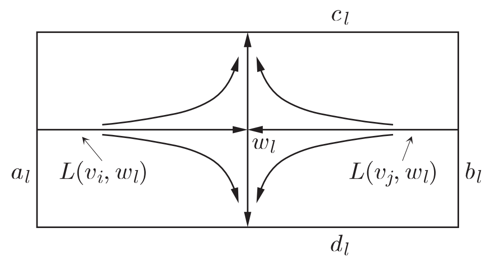

Fig. 3

It may be noted that each trajectory which crosses the sides a, b, c, d of the rectangle does so transversally. Let a_l and b_l be the sides of the ribbon $T(w_l)$ which are glued in M to the disks D_i and D_j respectively. Put $T = \bigcup_{w_l \in W} T(w_l)$, $a'_l = f(a_l)$, $b'_l = f(b_l)$, where $l = \overline{1,m}$. Then for each l, $1 \le l \le m$, there exists a tubular neighbourhood $T(w'_l)$ of the submanifold $L(e'_i, w'_l, e'_j)$ in $E_Y^{-1}[-1, 1-\varepsilon]$ such that $T(w'_l) \cap D'_i = a'_l$ and $T(w'_l) \cap D'_j = b'_l$. The trajectories of Y in $T(w'_l)$ also look as in Fig. 3. Taking into account that the sign of the chord $\{e_i, e_j\}$ in G_X coincides with the sign of the chord $\{e'_i, e'_j\}$ in G_Y, it may be proven that for each l, $1 \le l \le m$, there exists a suitable tubular neighbourhood $T(w'_l)$ of $L(e'_i, w'_l, e'_j)$ in $E_Y^{-1}[-1, 1-\varepsilon]$, and a homeomorphism $\psi_l : T(w_l) \to T(w'_l)$ such that $\psi_l|_{a_l} = \tilde{f}_4|_{a_l}$, $\psi_l|_{b_l} = \tilde{f}_4|_{b_l}$ and $\psi_l|_{T(w_l) \cap L(e_i, w_l, e_j)} = \tilde{f}_4|_{T(w_l) \cap L(e_i, w_l, e_j)}$. Moreover the homeomorphism ψ_l, $l = \overline{1,m}$, could be chosen in such a way that it sends the pieces of trajectories of X into those of Y. Put $T' = \bigcup_{l=1}^m T(w'_l)$. Define the map $\psi : T \cup D \to T' \cup D'$ by the formula

$$\psi|_D = \tilde{f}_4|_D, \qquad \psi|_{T(w_l)} = \psi_l|_{T(w_l)}, \quad \text{where } 1 \le l \le m.$$

It should be noted that each trajectory of X which crosses the boundary, $\partial(T \cup D)$, of the manifold $T \cup D$ does so transversally. Similarly each trajectory of Y which crosses the boundary, $\partial(T' \cup D')$, of the manifold $T' \cup D'$ does so transversally.

It remains to prove that the homeomorphism ψ can be extended to the whole manifold M. Let $R_1, \ldots, R_s$ be the connected components of $\partial(T \cup D)$. Then $R'_1, \ldots, R'_s$ will be all the connected components of $\partial(T' \cup D')$, where $R'_i = \psi(R_i)$, $i = \overline{1,s}$. Each circle R_j, $j = \overline{1,s}$, is contained in some 2-cell B_j and bounds a disk C_j, so that a sink u_j is contained in each one. It follows from Remark 1.1 and Proposition 1.1 that the correspondence $R_j \to u_j$ between the connected components of $\partial(T \cup D)$ and the sinks of X is one-to-one. Therefore the number of the sinks of X is equal to the number of the sinks of Y. (The last assertion follows also from the Poincaré–Hopf theorem, if we take into account that the number of saddle points of X is the same as that of Y, and the number of sources of X is the same as that of Y). Without loss of generality we can assume that if a sink u_j of X corresponds to a connected component R_j, then the sink u'_j of Y corresponds to the connected component R'_j.

Put $\psi'_j = \psi|_{R_j}$, $j = \overline{1,s}$. Then ψ'_j is a homeomorphism of R_j onto R'_j. Extend ψ'_j to the whole disk C_j, $j = \overline{1,s}$. We may choose the extension h_j of ψ'_j in such a way that $h_j(u_j) = u'_j$ for each j, $j = \overline{1,s}$. Next we apply to each disk C'_j an isotopy, which keeps $\partial C'_j$ fixed, to obtain a homeomorphism $h'_j : C_j \to C'_j$ which sends the pieces of trajectories of X into the pieces of trajectories of Y. Finally we define the homeomorphism $h : M \to M$ by the formula

$$h|_{C_i} = h'_i|_{C_i}, \quad i = \overline{1,s},$$
$$h|_{T \cup D} = \psi|_{T \cup D}.$$

The homeomorphism h is well defined because $h|_{\partial C_i} = \psi|_{\partial C_i}$ by the construction of h.

Remark 1.5. In view of Theorem 1.2 and Remark 1.4, two M-S flows X and Y on an oriented closed 2-manifold M are topologically equivalent if and only if the oriented (nonsigned) chord diagrams G_X and G_Y corresponding to them are isomorphic.

Now we define the family $\mathcal{F}$ of contours of a chord diagram G_X (which is regarded as an oriented graph) in order to determine the corresponding circuits of the connected components R_j of $\partial(T \cup D)$, $j = \overline{1,s}$. Those contours are determined uniquely up to orientation by the following rules.

Rule 1. The preceding link to each chord in an arbitrary contour of $\mathcal{F}$ is an arc (of the circle) and conversely, the preceding link to each arc is a chord.

Rule 2. Each chord is contained twice in one contour or once in some pair of contours (possibly with different arrows).

Rule 3. Each arc is contained in a unique contour. Moreover the arc appears in such a contour only once.

Let $h = \{u, v\}$ be an arbitrary chord of the contour C, where $u \in S_i$, $v \in S_j$. Next, let a_0, a_1 be two arcs in S_i containing u, and let b_0, b_1 be two arcs in S_j containing v, so that the orientation of S_i induces the cyclic sequence $\ldots, a_0, u, a_1, \ldots$, and the orientation of S_j induces the cyclic sequence $\dots,\ b_0,\ v,\ b_1, \dots$ consisting of arcs and distinguished points. Suppose a_i is the arc preceding the chord h, where $i = 0$ or $i = 1$.

Rule 4. If the sign of h is "+", then the contour C contains the triple (a_i, h, b_i).

Rule 5. If the sign of h is "−", then the contour C contains the triple (a_i, h, b_{1-i}).

In other words, the contours of $\mathcal{F}$ correspond to circuits of the connected components of the boundary of the "thickened" chord diagram G.

Proposition 1.3. *There is a one-to-one correspondence between the family* $\mathcal{F}_G$ *of contours of the chord diagram* G_X *and the family of circuits of connected components* R_i, $i = \overline{1, s}$.

Proof. The assertion follows directly from the definition of the correspondence $X \to [G_X]$.

It is therefore reasonable to ask for which signed oriented chord diagrams G there exists a suitable M-S flow X on a closed 2-manifold such that $G_X = G$. The answer to this question is always affirmative: using the arguments similar to ones as in the proof of Theorem 1.2 one may construct, for a given signed oriented chord diagram G, an example of an M-S flow X on a closed 2-manifold such that $[G_X] = [G]$.

Denote by $\eta(M)$ the number of topological equivalence classes of minimal M-S flows on an oriented closed 2-manifold M.

Now we shall try to evaluate the upper and lower bounds for the number $\eta(M)$. First we need to establish some auxiliary assertions.

Let y be a singular point of an M-S field on M. Denote by $\operatorname{ind}(X, y)$ the index of the vector field X at y.

Proposition 1.4. *Let* X *be an M-S flow on a smooth, oriented, closed* 2*-manifold* M *of genus* s, *and let* G_X *be an* n*-component chord diagram of order* m *corresponding to* X (*up to equivalence*). *Let* r *be the number of contours of* $\mathcal{F}_G$. *Then*

$$n - m + r = 2 - 2s.$$

Proof. By definition of the chord diagram G_X the number of sources of X is equal to n, and the number of saddle points is equal to m. It follows from Proposition 1.3 and the proof of Theorem 1.2 that the number of sinks of X is equal to r. On the other hand we have $\operatorname{ind}(X, u) = 1$ if u is a sink or a source and $\operatorname{ind}(X, w) = -1$ if w is a saddle point. Now the assertion follows immediatly from the Poincaré–Hopf theorem.

Proposition 1.5. *Let* X *be an M-S flow on a smooth, nonoriented, closed* 2*-manifold* M *of genus* s, *and* G_X *a* n*-component chord diagram of order* m *corresponding to* X (*up to equivalence*). *Let* r *be the number of contours of* $\mathcal{F}_G$. *Then*

$$n - m + r = 2 - s.$$

Proof. The proof of Proposition 1.5. is completely analogous to that of Proposition 1.4.

PROPOSITION 1.6. *Let M be a closed oriented 2-manifold of genus s. Then*

$$\Big(\prod_{k=0}^{s-1} C^2_{4(s-k)-2}\Big)//4s \le \eta(M) \le (4s-3)\cdot(4s-3)!!$$

P r o o f. Our proof starts with the observation that each minimal M-S flow on a closed, oriented 2-manifold M of genus s has exactly one sink, one source and $2s$ saddle points. In view of Theorem 1.2, Remark 1.4, and Proposition 1.4, $\eta(M)$ is equal to the number $\mu(2s)$ of the equivalence classes of 1-component oriented chord diagrams G having exactly $2s$ chords and generating exactly one contour, i.e., with $|\mathcal{F}_G| = 1$. Denote by P the set $\{\exp(2\pi ik//4s),\ k = \overline{1,4s}\}$ and by E_{2s} the set consisting of 1-component chord diagrams on S of order $2s$ with distinguished points $\exp(2\pi ik//4s)$, $k = \overline{1,4s}$. The total number of ways to draw $2s$ chords in a circle is equal to $(4s-1)!! = (4s-1)\cdot(4s-3)\cdot\ldots\cdot 3\cdot 1$.

Thus the set E_{2s} consists of $2(4s-1)!!$ elements.

Set $T_{2s} = \{G \in E_{2s} | G$ consists of one contour$\}$. It is easily verified that $|T_{2s}| \le 2\cdot(4s-3)(4s-3)!!$. Taking into account that the equivalence class $[G]$ of G doesn't depend on the choice of orientation of S we obtain

$$\mu(2s) \le \frac{|T_{2s}|}{2} \le (4s-3)\cdot(4s-3)!!.$$

It remains to establish the second inequality. Fix two distinct points v_3, v_4 in $P - \{\exp 0, \exp(2\pi i//4s)\}$. Denote by $V(v_3,v_4)$ the set $\{\exp 0, \exp(2\pi i//4s), v_3, v_4\}$ and by E'_{2s-2} the set consisting of 1-component oriented chord diagrams of order $2s-2$ having their distinguished points in $P - V(v_3,v_4)$. Put

$$T'_{2s-2} = \{G \in E'_{2s-2} | \mathcal{F}_G \text{ consists of one contour}\},$$
$$T'_{2s}(v_3,v_4) = \{G \in T_{2s} | G \text{ has two chords having their ends in } V(v_3,v_4)\}.$$

It is easily seen that $|T'_{2s-2}| \le |T'_{2s}(v_3,v_4)|$. Next we note that if $\{v'_3, v'_4\}$ is another pair, $\{v'_3,v'_4\} \subset P - \{\exp 0, \exp(2\pi i//4s)\}$, then $T'_{2s}(v_3,v_4) \cap T'_{2s}(v'_3,v'_4) = \emptyset$. Thus we have $|T_{2s}| \ge C^2_{4s-2}\cdot|T'_{2s}(v_3,v_4)| \ge C^2_{4s-2}\cdot|T'_{2s-2}| \ge \ldots \ge C^2_{4s-2}\cdot C^2_{4s-6}\cdot\ldots\cdot C^2_6\cdot 1$.

In order to complete the proof of the second inequality we observe that there is a natural action of the dihedral group D_{2s} on T_{2s} (and on E_{2s} also). Each conjugacy class of such an action consists of no more than $4s$ elements. On the other hand two chord diagrams G_1, $G_2 \in T_{2s}$ are equivalent if and only if they lie in the same conjugacy class. Thus we obtain

$$\mu(2s) \ge \frac{|T_{2s}|}{4s} \ge \Big(\prod_{k=0}^{s-1} C^2_{4(s-k)-2})//4s\Big),$$

whence the assertion follows.

The upper bound is more exact than the one given in [5] for large s.

2. Isotopical classification of flows on two-manifolds. In the remainder of this paper we shall assume M to be a smooth, oriented, closed 2-manifold of nonpositive Euler characteristic. The aim of this section is to find a simple criterion of isotopical equivalence of minimal M-S systems (flows) on an oriented closed 2-manifold. We shall state the main assertion concerning this point and shall give only a sketch of the proof of the main result of Section 2 (Theorem 2.4). In the second section we use the notations of Section 1 and [4,10]. First we recall some needed notions.

Definition 2.1. Let g_1, $g_2 : A \to B$ be two embeddings of a topological space A into a topological space B. We call g_1 and g_2 *ambient isotopic* if there is a (not necessarily smooth) isotopy $G : A \times I \to A$ such that $g_2(x) = G(g_1(x), 1)$ for each $x \in A$. We shall say then that G is an ambient isotopy between g_1 and g_2. We shall say also that g is the *final homeomorphism* of the isotopy $G : A \times I \to A$ if $g(x) = G(x, 1)$ for each $x \in A$.

Let f_1, $f_2 : A \to B$ be two maps of a topological space A into a topological space B, and $A_1 \subseteq A$. The notation $f_1 \simeq f_2$ rel A_1 means that there exists a homotopy F between f_1 and f_2 which keeps A_1 pointwise fixed.

Definition 2.2. Let X and Y be two dynamical systems (flows) on the smooth compact manifold N. Then X and Y will be called *isotopically equivalent* if there is an isotopy $F : N \times I \to N$ such that X and Y are topologically equivalent under the final homeomorphism f of F.

Denote by R the real numbers. Let X be a dynamical system on the compact smooth manifold N. Denote by φ^X the flow which corresponds to X, $\varphi^X : N \times R \to N$.

Let $\varphi : N \times R \to N$ be a flow on N and $F : N \times I \to N$ be the isotopy of N with final homeomorphism f. Then $f \circ \varphi$ will, generally, be a topological flow on N.

Throughout Section 2, M denotes a connected, closed, oriented, smooth, 2-manifold.

Let X be a minimal M-S system on M. It follows from the results of Section 1 that X has one source, one sink and $2s$ saddle points, where s is the genus of M. Denote by v the single source of X and by $w_1, \ldots, w_{2s}$ the saddle points of X. For each w_i there are exactly two trajectories L_i and L'_i joining v to w_i. It is obvious that the set $C_i = L_i \cup L'_i \cup \{w_i\} \cup \{v\}$ is homeomorphic to the circle for each i, $i = \overline{1, 2s}$, and can be regarded as a simple closed curve in M.

Set $C_X = \bigcup_{i=1}^{2s} C_i$. C_X will be called the *graph* of the flow X. The choice of orientation on each C_i determines an element of the fundamental group $\pi_1(M, v)$ of M.

Denote by $\vec{C_i}$ the curve C_i equipped with a fixed orientation. The corresponding element of the fundamental group $\pi_1(M, v)$ will be denoted by $[\vec{C_i}]$. It follows from arguments similar to those used in Section 1 that $M - C_X$ is an open 2-cell and that the oriented curves $\vec{C_i}$ determine the generators $[\vec{C_1}], \ldots, [\vec{C}_{2s}]$ of $\pi_1(M, v)$. Then $M - C_X$ contains the single sink u of X.

To establish the main result of Section 2 we need to prove several auxiliary assertions.

Theorem 2.1. *Let X and Y be two minimal M-S dynamical systems on M with graphs C_X and C_Y respectively. Then X and Y are isotopically equivalent if and only if there exists an ambient isotopy F of M with the final homeomorphism f sending C_X into C_Y.*

Proof. Denote by u (u' respectively) the sink of X (of Y respectively), by v (v' respectively) the source of X (of Y respectively), and by $w_1, \ldots, w_{2s}$ ($w'_1, \ldots, w'_{2s}$ respectively) the saddle points of X (of Y respectively).

Let F be the isotopy of M under the condition of the theorem. Then F may be deformed to an isotopy F_1 of M with final homeomorphism f_1 which satisfies the above mentioned condition and sends the saddle points of X to the saddle points of Y. Take an isotopy F_2 of M with final homeomorphism f_2 which keep C_Y fixed and sends $f_1(u)$ to u'. Next we can construct an isotopy F_3 of M with final homeomorphism f_3 which keeps $C_Y \cup \{u'\}$ fixed and sends the trajectories $f_2 \cdot f_1(L)$ of the flow $f_2 \cdot f_1 \cdot \varphi^X$ with $\alpha(L) = w_i$ and $\omega(L) = u$ to the trajectories L' of φ^Y with $\alpha(L') = w'_i$ and $\omega(L') = u'$, $i = \overline{1, 2s}$. This is possible because the trajectories $f_2 \cdot f_1(L)$ and L' under consideration lie in the open 2-cell $U = M - C_X$ and join the same points. Thus the final homeomorphism $f_3 \cdot f_2 \cdot f_1$ of the isotopy $F_3 \cdot F_2 \cdot F_1$ sends all the singular trajectories of X into the singular trajectories of Y. Set $g = f_3 \cdot f_2 \cdot f_1$. To complete the proof it is sufficient to note that all the trajectories of both φ^X and $g \cdot \varphi^X$ in each connected component B of $M \setminus C$ look as in Fig. 1, where C is the union of all singular trajectories.

By the standard arguments of general position the next assertion seems to be evident.

Proposition 2.2. *Let X and Y be two minimal M-S systems* (*flows*) *on M with the same source v, and let C_X, C_Y be the graphs of X and Y respectively. Then there is an isotopy F of M with final homeomorphism f which keeps v fixed and satisfies the following condition*: *$f_1(C_Y) - \{v\}$ intersects $C_X - \{v\}$ in a finite number of points with each intersection transverse.*

From what has been outlined above the following conclusion may be drawn.

Proposition 2.3. *The manifold M admits a PL-structure $\mathcal{P}$ such that the graph C_X is a subpolyhedron of M with respect to it.*

Now we may state the main result of Section 2.

Theorem 2.4. *Let X and Y be two minimal M-S systems on a closed*, *oriented 2-manifold M of genus n with the same source v. Let $C_X = \bigcup_{i=1}^{2n} C_i$, $C_Y = \bigcup_{i=1}^{2n} C'_i$ be the graphs of X and Y respectively. Then X and Y are isotopically equivalent rel v if and only if for each i the closed curves C_i and C'_i are homotopic rel v*; *i.e. C_i and C'_i determine the same element of the fundamental group $\pi_1(M, v)$ under a suitable choice of orientations*, *$i = \overline{1, 2n}$.*

Fix a point z on the circle S.

In view of Propositions 2.2 and 2.3 and Theorem 2.1, to prove Theorem 2.4 it suffices to establish the following proposition.

Proposition 2.5. *Let M be a closed*, *oriented 2-manifold of genus n and let $C = \bigcup_{i=1}^{2n} C_i$, $C' = \bigcup_{i=1}^{2n} C'_i$ be two bouquets of simple closed curves $C_i = f_i(S)$, $C'_i = f'_i(S)$ respectively*, *for the same point v, $v = f_1(z) = f'_1(z)$, which satisfy the following conditions*:

a) *$C_i \cap C_j = C'_i \cap C'_j = \{v\}$ for all pairs (i, j) with $i \neq j$.*

b) *$\{f_i\}_{i=1}^{2n}$ generates the fundamental group $\pi_1(M, v)$ and $f_i \simeq f'_i$ rel z for each i, $1 \le i \le 2n$.*

c) *C is a subpolyhedron of M.*

d) *The manifold $C - \{v\}$ intersects $C' - \{v\}$ in a finite numbers of points with all such intersections transverse.*

Then there is an isotopy $G : M \times I \to M$ with final homeomorphism g, which keeps v fixed, such that $g(S_i) = S'_i$ for every i, $i = \overline{1, 2n}$.

Let us consider the bouquets C and C' on the manifold M.

Remark 2.1. Following the method used by D. B. A. Epstein to prove Theorem A1 [4] we can find an isotopy F_1 of M with final homeomorphism f_1 which keeps the point v fixed such that the following conditions will be satisfied:

1) both $f_1(C')$ and C are subpolyhedra of M;

2) the manifold $C - \{v\}$ intersects $f_1(C') - \{v\}$ in a finite number of points with all those intersections transverse.

Taking into account Remark 2.1 we may assume without loss of generality that condition c) of Proposition 2.5 is replaced by the following one:

c′) Both C and C' are subpolyhedra of M.

From this time on we shall work in PL-category.

Remark 2.2. Let g, $f : G \to M$ be two embeddings of a finite graph G (a one-dimensional connected polyhedron) into a closed, PL-2-manifold M. Then f and g are isotopic if and only if they are ambient isotopic. This follows from the criterion of ambient PL-isotopy (see Corollary 4.25 [10]).

To give a sketch of the proof of Proposition 2.5 we need to introduce several new notions.

Let $F : S \times I \to M$ be an isotopy of the circle S in a closed PL-2-manifold M with initial imbedding f and final imbedding g i.e. $f(x) = F(x, 0)$, $g(x) = F(x, 1)$ for all x, $x \in S$. Set $\sum_0 = f(S)$, $\sum_1 = g(S)$. Unless otherwise stated we assume that $\sum_0$ doesn't bound a disk in M (so $\sum_0$ isn't homotopic to the trivial loop [4]). Let K and L be the triangulations of $S \times I$ and M respectively such that $F : K \to L$ is a simplicial map. Determine the equivalence R_F on the set V of vertices of the complex K by the following formula:

$$(uR_F v) \Leftrightarrow (u, v \in V) \wedge (\exists s \in K : (\dim s = 1) \wedge (\partial s = \{u, v\}) \wedge (F(u) = F(v)))\,.$$

Let $p_1 : V \to V|R_F$ be the canonical map and let K_1 be the abstract simplicial complex on V corresponding to the geometric complex K. Then p_1 induces on the set $W = V|R_F$ the structure of a simplicial complex which will be denoted by (E, W). The map $p_1 : (K, V) \to (E, W)$ is simplicial by the construction of (E, W). Let N be the geometric realization of (E, W). Then p_1 can be extended uniquely to some simplicial map $q : N \to L$ such that the following diagram commutes:

$$\begin{tikzcd} & L \\ K \arrow[ur, "F"] \arrow[dr, "p_1"'] & \\ & N \arrow[uu, "q"'] \end{tikzcd}$$

Note that $q : N \to L$ is a nondegenerate simplicial map. Suppose that there exist two different simplexes $s = (x_1, y_1, z_1)$ and $t = (x_2, y_1, z_1)$ of N such that $q(s) = q(t)$. Glue in N the simplexes s and t with respect to q. Then we obtain a new simplicial complex N_1 and a simplicial map $r_1 : N \to N'$ such that $r_1(x_1) = r_1(x_2)$. There is only one simplicial map $q'_1 : N'_1 \to L$ for which the following diagram commutes:

$$\begin{tikzcd} & L \\ N \arrow[ur, "q"] \arrow[dr, "r_1"'] & \\ & N'_1 \arrow[uu, "q'_1"'] \end{tikzcd}$$

Next we apply to the polyhedron $|N'_1|$ a simplicial collapsing $\rho_1 : N'_1 \searrow N_2$ along the simplex $r_1(s)$ from the edge $r_1(y_1, z_1)$. There exists only one simplicial map $q_2 : N_2 \to L$ for which the following diagram commutes:

$$\begin{tikzcd} & L \\ N'_1 \arrow[ur, "q'_1"] \arrow[dr, "\rho_1"'] & \\ & N_2 \arrow[uu, "q_2"'] \end{tikzcd}$$

We continue the procedures of gluing and elementary collapsing as long as possible. Finally we obtain the sequence r_1, ρ_1, r_2, ρ_2,…,r_k, ρ_k of gluing and collapsing and the unique simplicial map q_{k+1} such the following diagram commutes:

$$\begin{tikzcd} L & & & & & & & & \arrow[llllllll, dashed] \\ K \arrow[u, "F"] \arrow[r, "p_1"'] & N \arrow[r, "r_1"'] & N'_1 \arrow[r, "\rho_1"'] & N_2 \arrow[r, "r_2"'] & N'_2 \arrow[r] & \cdots \arrow[r] & N'_k \arrow[r, "\rho_k"'] & N_{k+1} \arrow[u, "q_{k+1}"'] \end{tikzcd}$$

Definition 2.3. The polyhedron $q_{k+1}(|N_{k+1}|)$ will be called the *covering region* of the isotopy F and will be denoted by $Q(F)$.

Definition 2.4. Connected components of $Q(F) - (\sum_0 \cup \sum_1)$ are called the *components of the covering region* $Q(F)$. The set of components of the covering region $Q(F)$ is denoted by $S(F)$. If $Q(F) - (\sum_0 \cup \sum_1) = \emptyset$ we set $S(F) = \emptyset$.

Note that $q_{k+1} : N_{k+1} \to L$ is a nondegenerate simplicial map.

Let $D_i \in S(F)$. Put $U_i = q_{k+1}^{-1}(D_i)$. Let A_i be the set of connected components of U_i, $i = \overline{1, r}$, $A_i = \{A_i^1, \dots, A_i^{n_i}\}$.

Definition 2.5. The number n_i will be called the *index* of the component D_i.

It follows from Definitions 2.3 and 2.4 that the map $q_{k+1}|_{A_i^j} : A_i^j \to D_i$ is a homeomorphism for each i and j, $1 \le i \le |S(F)|$, $1 \le j \le n_i$. Then every nonempty component

D_i of $Q(F)$ is an open 2-cell or is homeomorphic to $S \times (-1,1)$. Moreover if F keeps some point in $\sum_0$ fixed, then $S(F)$ consists of 2-cells only.

PROPOSITION 2.6. *Let f_0 and f_1 be two embeddings of the circle S in M, $z \in S$, $f_0(z) = v$, $f_0 \simeq f_1$ rel z, and let F be the corresponding isotopy between f_0 and f_1. If $f_0(S)$ bounds no disk in M, then there is an isotopy F' between f_0 and f_1 such that $F'(S \times I) \subseteq Q(F)$. Moreover if L is a polyhedron in M such that $L \cap Q(F) \subset \{v\}$, then there is an ambient isotopy G between f_0 and f_1 which keeps L fixed.*

We omit the proof of this assertion. Now we are able to prove Proposition 2.5.

Proof. The proof is by induction on the number of generators of $\pi_1(M, v)$.

By assumption, $f_1 \simeq f_0$ rel z. Then, by Theorem 4.1 [4], there exists an ambient isotopy G between f_1 and f_0 which keeps v fixed. If g is the final homeomorphism of G then we have $g(C_1) = C'_1$.

Suppose that there exists an ambient isotopy $\tilde{G} : M \times I \to M$ with final homeomorphism $\tilde{g}$ which keeps v fixed and such that $\tilde{g}(C_i) = C'_i$ for all $i \le k$. We wish to prove that there exists an ambient isotopy $\tilde{G}_1 : M \times I \to M$ whose final homeomorphism $\tilde{g}_1$ keeps v fixed and satisfies the condition

$$\tilde{g}_1(C_i) = C'_i \quad \text{for each } i,\ 1 \le i \le k+1.$$

By assumption $f'_{k+1} \simeq f_{k+1}$ rel z. It follows from this that $\tilde{g} \circ f_{k+1} \simeq \tilde{g} f'_{k+1} \simeq f'_{k+1}$ rel z. By Theorem 4.1 [4], there is an ambient isotopy H between $\tilde{g} \circ f_{k+1}$ and f'_{k+1} which keeps the point v fixed. Consider the set $Q(H)$. From what has been said above it follows that all the components of $Q(H)$ are open 2-cells. Suppose that $S(H) = \emptyset$. Then $\tilde{g} f_{k+1}(S) = f'_{k+1}(S) = C'_{k+1}$. By Proposition 2.6 there is an ambient isotopy $\tilde{H} : M \times I \to M$ between $\tilde{g} \circ f_{k+1}$ and f'_{k+1} which keeps the set $\bigcup_{i=1}^{k} C'_i$ fixed.

It is easily seen that the isotopy $\tilde{H} \circ \tilde{G}$ satisfies the required property, i.e. $\tilde{H} \circ \tilde{G}$ keeps the point v fixed and the final homeomorphism $\tilde{h} \circ \tilde{g}$ sends C_i to C'_i for each i, $1 \le i \le k+1$.

Now suppose that $S(H) \ne 0$. Note that $\tilde{g} C_{k+1} \cap C'_i = C'_{k+1} \cap C'_i = \{v\}$ for each i, where $1 \le i \le k$. It follows from the definition of $Q(H)$ that if $Q(H) \cap C'_i$ consists of more than one point v for some i, where $1 \le i \le k$, then $S(H)$ consists of only one component A (which is a 2-cell) and $C'_i - \{v\} \subset A$. Then we have $\tilde{g} \circ f_{k+1} \simeq f'_i$ rel z or $f'_i \simeq f'_{k+1}$ rel z, contrary to the assumption. Thus $C'_i \cap Q(H) = v$ for each i such that $i \le k$. So by Proposition 2.6 there is an ambient isotopy $\tilde{H}$ between $\tilde{g} f_{k+1}$ and f'_{k+1} with final homeomorphism h_1 which keeps the polyhedron $\bigcup_{i=1}^{k} C'_i$ fixed. Then the isotopy $\tilde{H} \circ \tilde{G}$ satisfies the required property, i.e. $\tilde{H} \circ \tilde{G}$ keeps the point v fixed and the final homeomorphism $\tilde{h}_1 \circ \tilde{g}$ sends C_i to C'_i for each i, $1 \le i \le k+1$.

Theorem 2.4 admits an obvious generalization to the case where the sources of X and Y are different points of M. On the other hand, one can attempt to generalize this theorem to the case of nonoriented 2-manifolds or to the case of arbitrary M-S flows on oriented 2-manifolds, but we shall not develop this point here.

Finally, we give an example of two topologically equivalent but isotopically nonequivalent minimal M-S flows on a torus. It is easily seen that each minimal M-S flow on a torus has one sink, one source and two saddle points. By Theorem 1.2 and Proposition 1.3 a M-S system of that kind corresponds to a 1-component (nonsigned) chord diagram of order 2 with one contour. There are two non-isomorphic 1-component (non-signed) chord diagrams of order 2 which are depicted in Fig. 4.

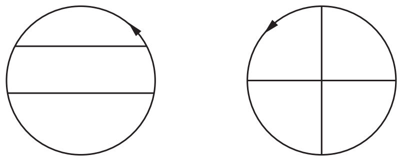

Fig. 4

It is easy to check that the first chord diagram generates 3 contours, so there is only one minimal M-S flow on a torus (see also [5]). To give the example of isotopically nonequivalent minimal M-S flows X and Y on the torus we shall use the representation of one as the rectangle with opposite sides glued together. We depict in Fig. 5 only the singular trajectories of flows X and Y, where the source of one flow is denoted by v, the sink of one is denoted by u and the saddle points of one are denoted by w_1, w_2.

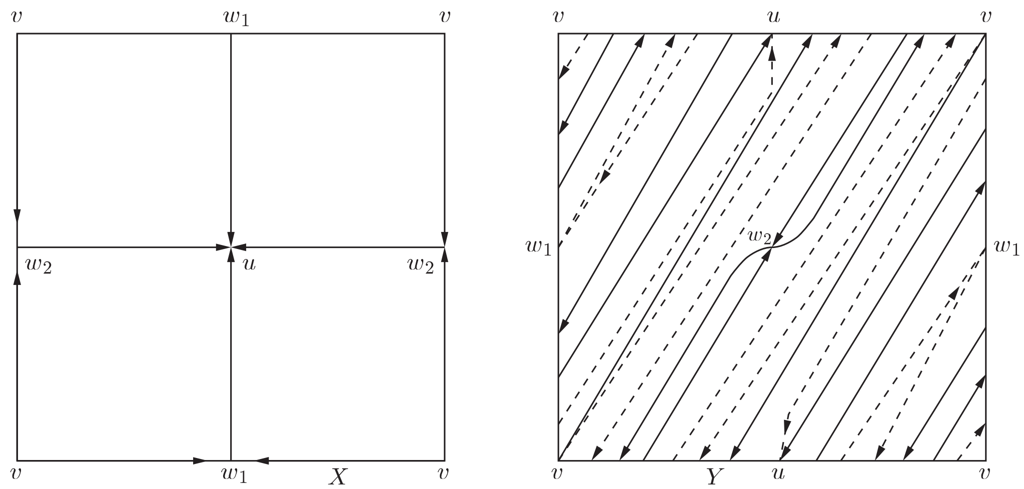

Fig. 5

In view of Theorem 2.4 the minimal M-S flows on a torus admit a simple isotopical classification but we shall not develop this point here.

R e m a r k 2.3. The author has observed later on that there is another proof of Theorem 2.4 based on Theorem 2.1 of this paper, the Dean–Nielsen Theorem (see [12]) and Theorem 6.3 of [4].

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