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CALCULATION OF THE CASSON–WALKER–LESCOP INVARIANT FROM CHORD DIAGRAMS

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In [5, 6] T. Q. T. Le, J. Murakami, T. Ohtsuki and the author gave an invariant of three-manifolds by using the Kontsevich integral. It turns out that it contains the order of the first homology and the Casson–Walker–Lescop invariant (C. Lescop's generalization [8] to closed 3-manifolds of the Casson–Walker invariant which is K. Walker's generalization [9] to rational homology 3-spheres of the Casson invariant [1] for integral homology 3-spheres).

Since the definition in [5, 6] uses complicated integrals, it is convenient for topologists (including the author himself) to give a combinatorial approach to our invariant. Unfortunately, the author cannot give a combinatorial proof of its existence but can explain how to calculate it without using integrals. If you can draw circles and dotted lines, you can calculate our invariant. (Of course the author assumes you can calculate fractions.)

There is nothing new in this article; no precise definitions or proofs. If the reader is interested in our invariant, please try our paper [6].

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Key words and phrases: Casson–Walker–Lescop invariant, Kontsevich integral, Vassiliev invariant, chord diagram.

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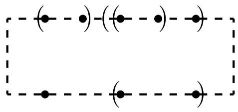
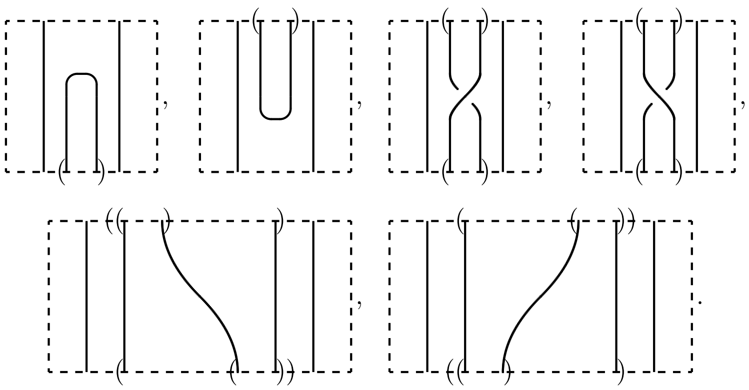
1. q-tangles. In this section we introduce the notion of q-tangles (or non-associative tangles). For more systematic treatment we refer the reader to [2]

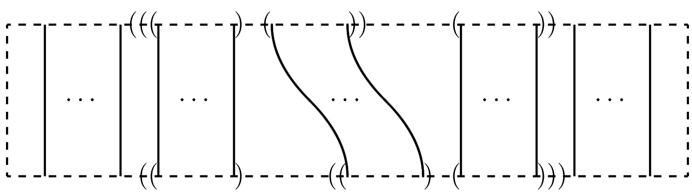
Let $\mathcal{B}$ be a family of triples (B, P_0, P_1), where $B = [0,1] \times [0,1]$, P_0 and P_1 are sets of n and m points on its bottom $[0,1] \times \{0\}$ and top $[0,1] \times \{1\}$ respectively together with pairs of parentheses describing which points are closer.



An elementary q-tangle is a set of strings S in $(B, P_0, P_1) \in \mathcal{B}$ with $\partial S \in [0,1] \times \{0,1\}$, $S \cap [0,1] \times \{0\} = P_0$, and $S \cap [0,1] \times \{1\} = P_1$ which is constructed from one of the following types of strings by repeatedly replacing a string with some parallel parenthesized strings.



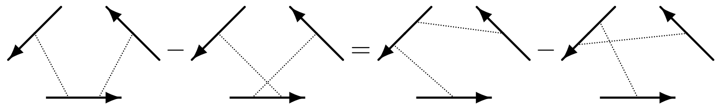
So for example the following is also an elementary q-tangle.



A q-tangle is a composition of several elementary q-tangles. Here two elementary q-tangles in $(B, P_0, P_1), (B', P'_0, P'_1) \in \mathcal{B}$ can be composed by identifying $[0,1] \times \{1\} \subset B$ and $[0,1] \times \{0\} \subset B'$ if $P_1 = P'_0$. For a given framed link (link diagram modulo isotopy and Reidemeister moves II, III), we can change it to a q-tangle. An example is given in Figure 1.

2. A framed link invariant $\check{Z}_f$. In this section, we define Kontsevich's universal Vassiliev invariant [4] via q-tangles.

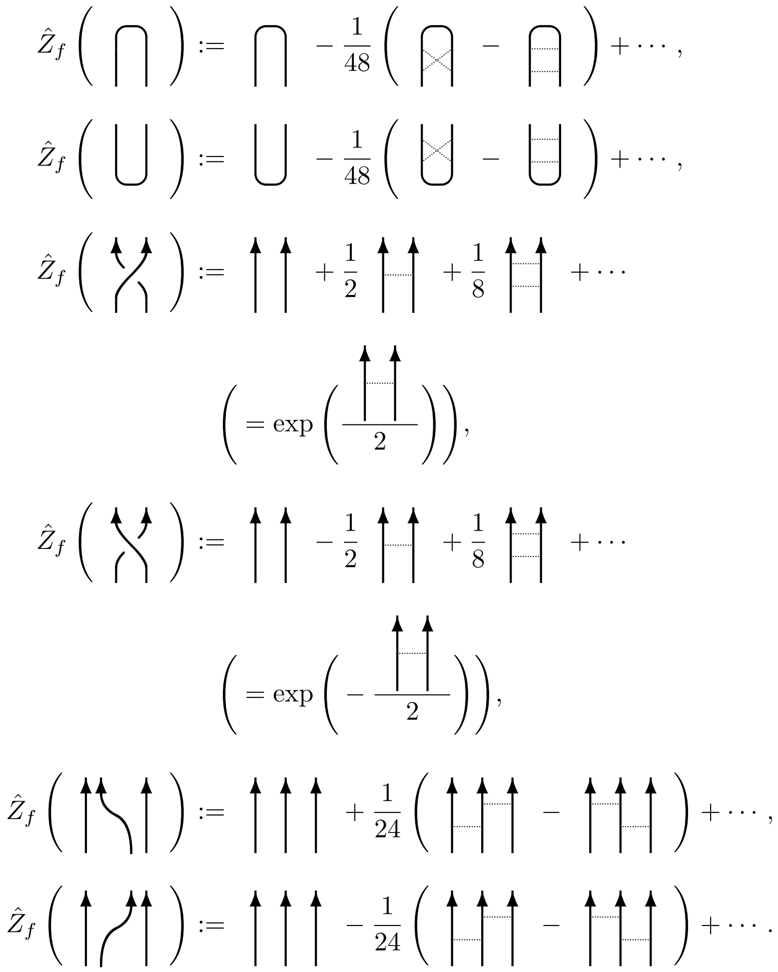
A chord diagram is a union of circles S^1 with chords which are homeomorphic to an interval I such that their boundaries are on circles. We will use solid lines for circles and dotted lines for chords. Let $\mathcal{A}^{(\ell)}$ be the linear span over $\mathbf{C}$ of chord diagrams with ℓ circles modulo the following 4-term relation.



$\mathcal{A}^{(\ell)}$ is graded by the number of chords.

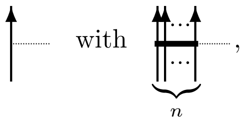
Now for a given framed link $\mathcal{L}$, lower degree terms of its Kontsevich invariant $\hat{Z}_f(\mathcal{L}) \in \mathcal{A}^{(\ell)}$ can be calculated as follows.

First we define $\hat{Z}_f$ for elementary q-tangles. (So in this case $\hat{Z}_f$ takes values in a linear combination of chord diagrams on arcs rather than on circles.) We define

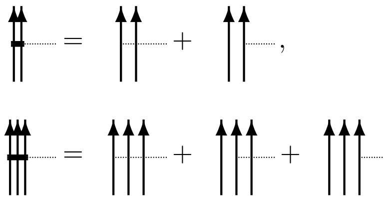


Here orientations of arcs are irrelevant in the first two. For precise values of the first, the second, the fifth, and the sixth, we refer the reader to [6].

If an arc is replaced with n parallel arcs we replace

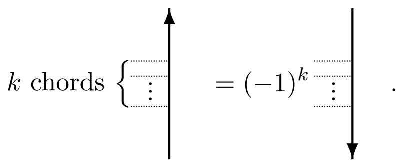


where



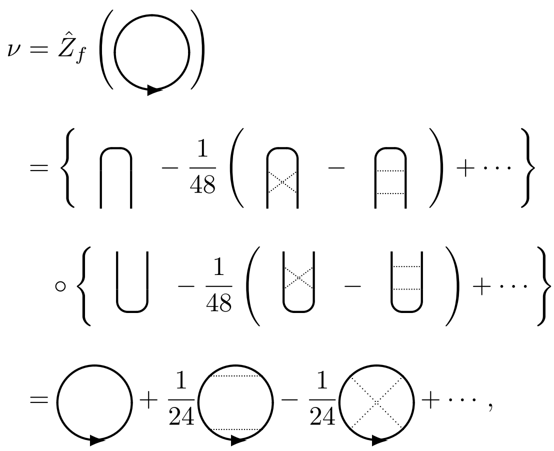
and so on.

If the orientation of an arc is reversed we put



For a framed link $\mathcal{L}$, we first isotope it to a q-tangle and decompose it into some elementary q-tangles. Then $\hat{Z}_f(\mathcal{L})$ is defined as the composition of $\hat{Z}_f$ of each elementary q-tangle. Here the composition of chord diagrams on arcs is defined by attaching arcs to one another.

Next we define a version of $\hat{Z}_f(\mathcal{L})$, $\check{Z}_f(\mathcal{L})$ which will be used to define a three-manifold invariant. Put



where $\circ$ denotes the composition of chord diagrams. We define $\check{Z}_f(\mathcal{L})$ as the connect-sum of $\hat{Z}_f(\mathcal{L})$ and ℓ ν's. Here we connect-sum ν one to each circle in $\hat{Z}_f(\mathcal{L})$. (To connect-sum two chord diagrams (on circles), we first cut two circles into two arcs and then connect them into a circle. The points where we cut circles do not matter thanks to the four-term relation.) Note that $\check{Z}_f(\mathcal{L})$ is also an invariant of framed links.

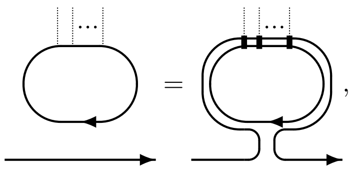
3. A three-manifold invariant Λ_1. In this section we describe relations on $\mathcal{A}^{(\ell)}$ to construct a three-manifold invariant.

We add the following two relations to make $\check{Z}_f$ invariant under the orientation reversing and the second Kirby move (handle sliding) respectively.

$$D' = (-1)^k \, D$$

(Orientation independence relation)

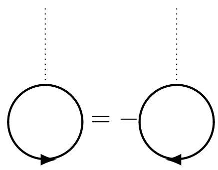
and



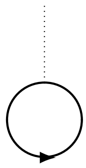
(KII relation)

where D' is the chord diagram obtained from D with the orientation of a circle reversed and k is the number of chords attached to the circle.

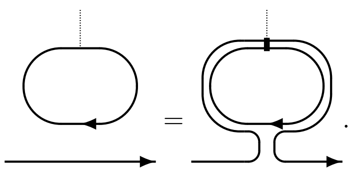
Using these relations we deduce the following useful relations. From the orientation independence relation with $k = 1$, we have



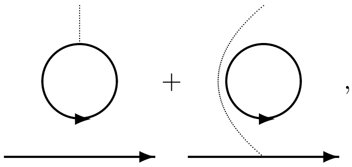
and so

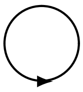
 $= 0.$

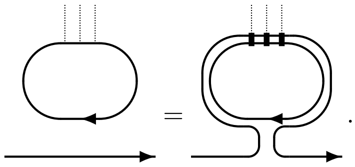
From the KII relation with one chord, we have



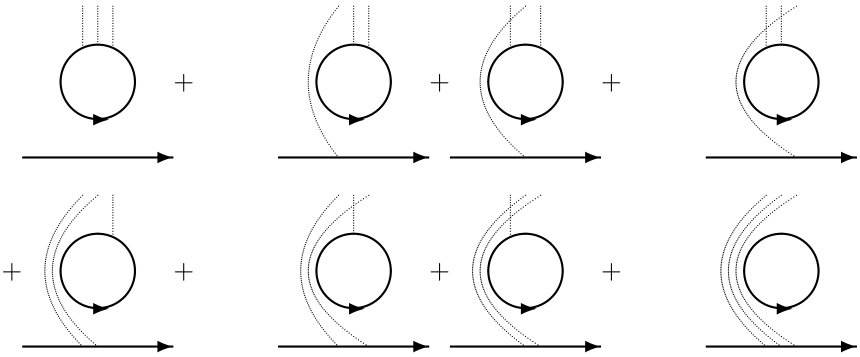
Since the right hand side equals

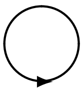
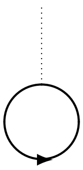


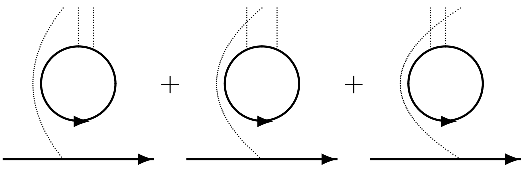
we have  = 0. From the KII relation with three chords, we have



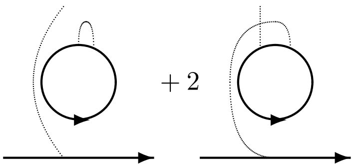
The right hand side equals



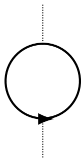
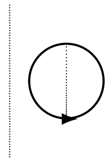
and since  =  = 0,we have

(3.1)  = 0.

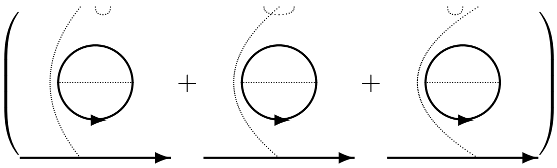
In particular, capping off the right two chords we have

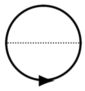
 = 0.

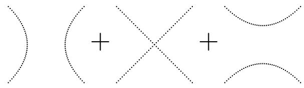
So we have

 $= -\dfrac{1}{2}$  .

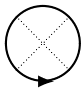
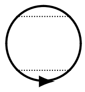
Using this relation we have from (3.1)

$-\dfrac{1}{2}$  $= 0.$

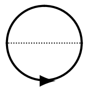
So we have (assuming that  $\neq 0$) the following 3-term relation.

 $= 0.$

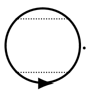
In particular, we have

 $= -2$  .

We denote $\mathcal{A}^{(\ell)}$ modulo the orientation independence relation and the KII relation by $\overline{\mathcal{A}}^{(\ell)}$. Let $\Lambda_1'(\mathcal{L})$ the image of $\check{Z}_f(\mathcal{L})$ in $\overline{\mathcal{A}}^{(\ell)}$. Then it is proved in [6] that $\Lambda_1'(\mathcal{L})$ is invariant under the second Kirby moves (handle slidings).

Now we must stabilize it by split union of $\infty_\pm$ with $\infty_\pm$ the trivial knot with ± 1 framing. To do this we introduce more equivalence relation in $\overline{\mathcal{A}}^{(\ell)}$. We call $D \in \overline{\mathcal{A}}^{(\ell)}$ and $D' \in \overline{\mathcal{A}}^{(\ell')}$ are stably equivalent if $D \sqcup \Theta_1 \sqcup \Theta_1 \sqcup \cdots \sqcup \Theta_1 = D' \sqcup \Theta_1 \sqcup \Theta_1 \sqcup \cdots \sqcup \Theta_1$ in $\overline{\mathcal{A}}^{(m)}$ for some m, where $\Theta_1 =$  and $\sqcup$ denotes the disjoint union of chord diagrams. Let $\overline{\mathcal{A}}_1^{(\infty)}$ denote $\bigcup_{\ell \geq 0} \overline{\mathcal{A}}^{(\ell)}$ modulo stable equivalence. If we regard $\sqcup$ as a multiplication in $\overline{\mathcal{A}}_1^{(\infty)}$, then the unit is $\emptyset (= \Theta_1 = \Theta_1 \sqcup \Theta_1 = \dots)$ and an element $D = a\emptyset + \dots$ with $a \neq 0$ is invertible with its inverse $D^{-\sqcup}$. Here $\overline{\mathcal{A}}_1^{(\infty)}$ is graded by the number of chords minus the number of circles. We define

$$\Lambda_1(\mathcal{L}) := 2^{\sigma_0(\mathcal{L})} \left(\Lambda_1'(\infty_+)^{-\sigma_+(\mathcal{L})\sqcup} \sqcup \Lambda_1'(\infty_-)^{-\sigma_-(\mathcal{L})\sqcup} \sqcup \Lambda_1'(\mathcal{L}) \right) \in \overline{\mathcal{A}}_1^{(\infty)},$$

where $D^{-n\sqcup}$ is the disjoint union of n copies of $D^{-\sqcup}$, $\sigma_\pm(\mathcal{L})$ is the number of $\pm$-eigenvalues of the linking matrix, and $\sigma_0(\mathcal{L})$ is the number of zeros of its eigenvalues. Now the main theorem of [5, 6] is as follows. We put $\Theta_2 =$  .

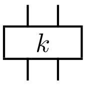
Theorem 3.1 *Let M be a 3-manifold and $\mathcal{L}$ a framed link presenting M. Then $\Lambda_1(\mathcal{L})$ defines an invariant of M and denoted by $\Lambda_1(M)$. Moreover let $\Lambda_{1,0}(M)$ and*

$\Lambda_{1,1}(M)$ *be the coefficients of* $\emptyset(=\Theta_1 \sqcup \Theta_1 \sqcup \cdots \sqcup \Theta_1)$ *and* $\Theta_2(=\Theta_2 \sqcup \Theta_1 \sqcup \Theta_1 \cdots \sqcup \Theta_1)$ *in* $\Lambda_1(M)$ *respectively. Then we have*

$$\Lambda_{1,0}(M) = |H_1(M;\mathbf{Z})| \quad \textit{and} \quad \Lambda_{1,1}(M) = -6\lambda(M),$$

where $|H_1(M;\mathbf{Z})|$ *is the order of the first homology* (*which is zero if it has infinite number of elements*) *and* $\lambda(M)$ *is the Casson–Walker invariant generalized by Lescop* (*which equals* $|H_1(M;\mathbf{Z})|\lambda_W(M)/2$ *if* M *is a rational homology* 3*-sphere and* $\lambda_C(M)$ *if it is an integral homology* 3*-sphere with* λ_W *and* λ_C *the Walker and the Casson invariant respectively* [1, 9]).

4. Example. In this section we calculate $\hat{Z}_f(\mathcal{K})$, $\check{Z}_f(\mathcal{K})$, and $\Lambda_1(\mathcal{K})$ up to degree two precisely, where $\mathcal{K}$ is the $(2,k)$-torus knot with n framing (k : odd).

We decompose $\mathcal{K}$ as in Figure 1. Here 

denotes k half-twists.

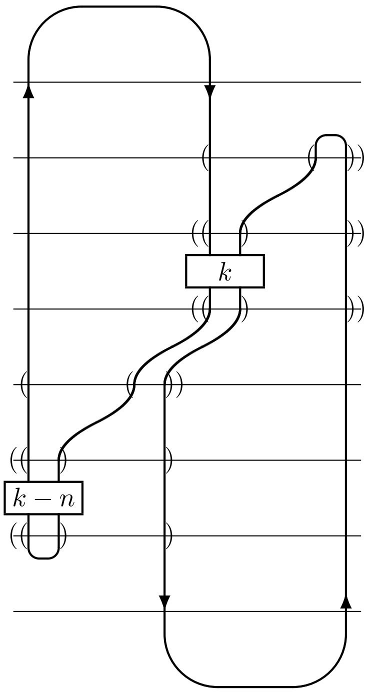
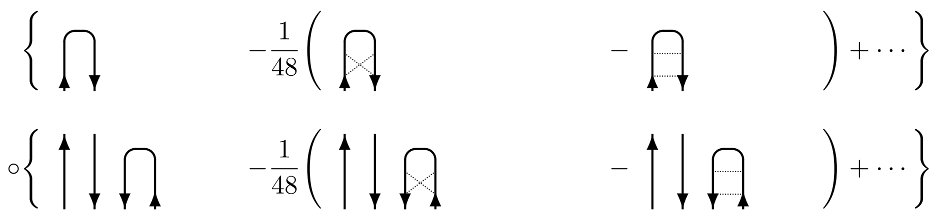
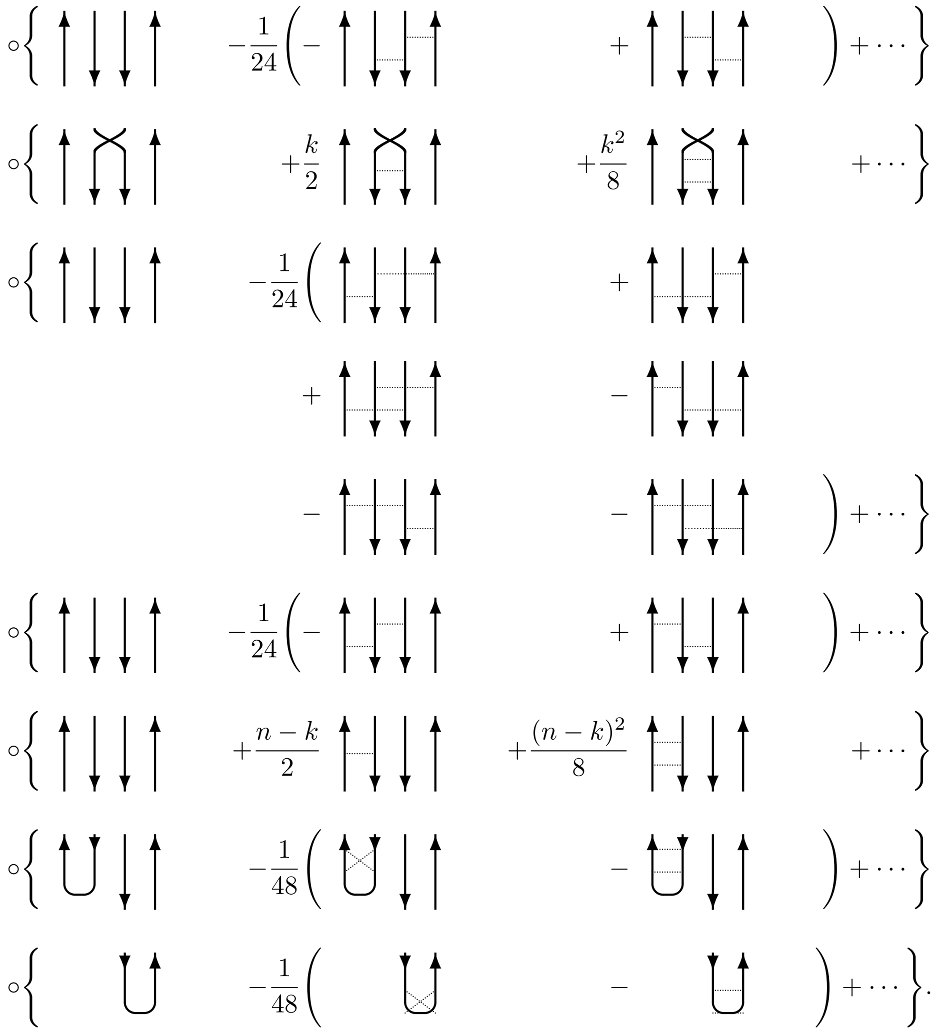


Fig. 1. $(2,k)$-torus knot with n framing

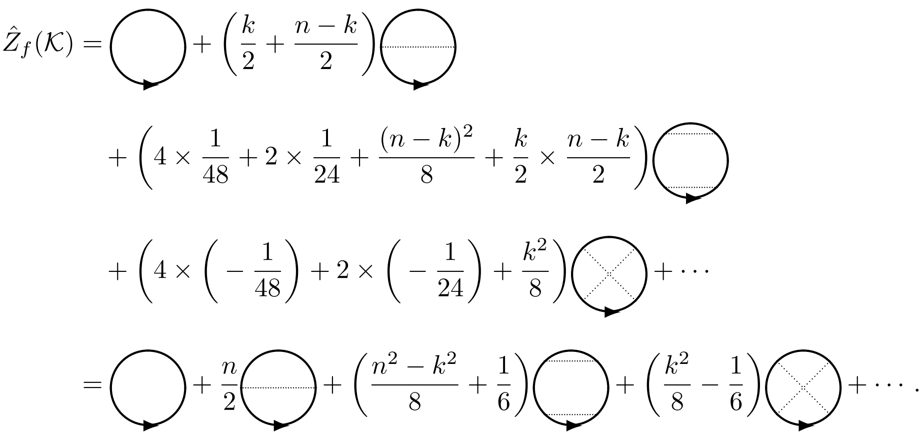
Then $\hat{Z}_f(\mathcal{K})$ is





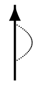
$-\frac{1}{24}$
$+\frac{k}{2}$
$+\frac{k^2}{8}$
$-\frac{1}{24}$
$-\frac{1}{24}$
$+\frac{n-k}{2}$
$+\frac{(n-k)^2}{8}$
$-\frac{1}{48}$
$-\frac{1}{48}$

So we have

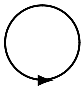
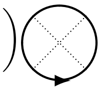


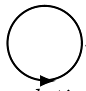
$\hat{Z}_f(\mathcal{K}) =$
$+ \left(\frac{k}{2} + \frac{n-k}{2}\right)$
$+ \left(4 \times \frac{1}{48} + 2 \times \frac{1}{24} + \frac{(n-k)^2}{8} + \frac{k}{2} \times \frac{n-k}{2}\right)$
$+ \left(4 \times \left(-\frac{1}{48}\right) + 2 \times \left(-\frac{1}{24}\right) + \frac{k^2}{8}\right)$
$+ \cdots$
$=$
$+ \frac{n}{2}$
$+ \left(\frac{n^2-k^2}{8} + \frac{1}{6}\right)$
$+ \left(\frac{k^2}{8} - \frac{1}{6}\right)$
$+ \cdots .$

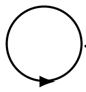
R e m a r k 4.1 *Let K be the underlying unframed knot of $\mathcal{K}$ and $\hat{Z}(K)$ be $\hat{Z}_f(\mathcal{K})$ modulo the following framing independence relation*

 $= 0.$

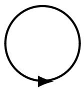
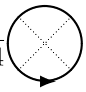
Since the second and the third terms vanish in $\hat{Z}_f(\mathcal{K})$, we have

$\hat{Z}(K) =$  $+ \left(\frac{k^2}{8} - \frac{1}{6}\right)$  $+ \cdots .$

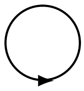
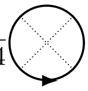
We will normalize it so that $\hat{Z}(\textit{trivial knot}) =$ *. Let $\hat{Z}(\textit{trivial knot})^{-\sharp}$ be the element in $\mathcal{A}^{(1)}$ modulo the framing independence relation above such that*

$\hat{Z}(\textit{trivial knot})^{-\sharp} \sharp \hat{Z}(\textit{trivial knot}) =$ .

Since

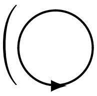
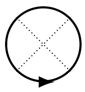
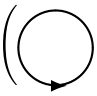
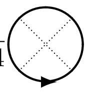
$\hat{Z}(\textit{trivial knot}) =$  $- \frac{1}{24}$  $+ \cdots$

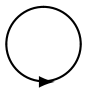
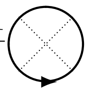
(put $k = 1$ in the formula of $\hat{Z}(K)$),

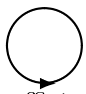
$\hat{Z}(\textit{trivial knot})^{-\sharp} =$  $+ \frac{1}{24}$  $+ \cdots .$

Therefore

(4.1) $\tilde{Z}(K) := \hat{Z}(K) \sharp \hat{Z}(\textit{trivial knot})^{-\sharp}$

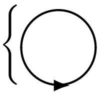
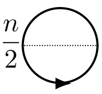
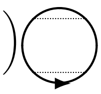
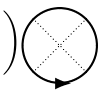
$= \Big($  $+ \left(\frac{k^2}{8} - \frac{1}{6}\right)$  $+ \cdots \Big) \sharp \Big($  $+ \frac{1}{24}$  $+ \cdots \Big)$

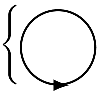
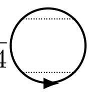
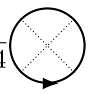
$=$  $+ \frac{k^2 - 1}{8}$  $+ \cdots$

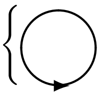
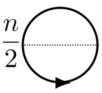
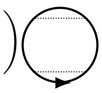
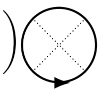
gives an invariant of an (unframed) knot K which is  *if K is trivial. (This coincides with $\tilde{Z}(K)$ in* [3]*.) It is well-known that its second coefficient coincides with the coefficient of z^2 in the Conway polynomial of K, which is equal to half the second derivative of the Jones polynomial at one. (Note that type two Vassiliev invariant for knots is unique up to constant.)*

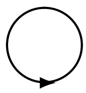
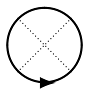
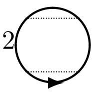
Now we return to the calculation of $\check{Z}_f(\mathcal{K})$. From the definition

$\check{Z}_f(\mathcal{K}) = \hat{Z}_f(\mathcal{K}) \sharp \nu$

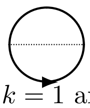
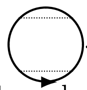
$= \Big\{$  $+ \frac{n}{2}$  $+ \left(\frac{n^2-k^2}{8} + \frac{1}{6}\right)$  $+ \left(\frac{k^2}{8} - \frac{1}{6}\right)$  $+ \cdots \Big\}$

$\sharp \Big\{$  $+ \frac{1}{24}$  $- \frac{1}{24}$  $+ \cdots \Big\}$

$= \Big\{$  $+ \frac{n}{2}$  $+ \left(\frac{n^2-k^2}{8} + \frac{5}{24}\right)$  $+ \left(\frac{k^2}{8} - \frac{5}{24}\right)$  $+ \cdots \Big\}.$

Since  $= 0$ and  $= -2$  in $\overline{\mathcal{A}}^{(\ell)}$, we have

$$\Lambda_1'(\mathcal{K}) = \frac{n}{2}\Theta_1 + \left(\frac{n^2-3k^2}{8} + \frac{5}{8}\right)\Theta_2 + \cdots,$$

where $\Theta_1 =$  and $\Theta_2 =$ .

Putting $k = 1$ and $n = \pm 1$, we have

$$\Lambda_1'(\infty_\pm) = \pm\frac{1}{2}\Theta_1 + \frac{3}{8}\Theta_2 + \cdots.$$

So

$$\Lambda_1'(\infty_\pm)^{-\sqcup} = \pm 2\Theta_1 - \frac{3}{2}\Theta_2 + \cdots.$$

Therefore we finally have

$$\begin{aligned}\Lambda_1(M_{\mathcal{K}}) &= \left\{2\,\mathrm{sign}(n)\Theta_1 + \frac{3}{2}\Theta_2 + \cdots\right\} \sqcup \left\{\frac{n}{2}\Theta_1 + \left(\frac{n^2-3k^2}{8} + \frac{5}{8}\right)\Theta_2 + \cdots\right\} \\ &= |n|(\Theta_1 \sqcup \Theta_1) + \mathrm{sign}(n)\left(\frac{n^2-3k^2}{4} + \frac{5}{4} - \frac{3|n|}{4}\right)(\Theta_2 \sqcup \Theta_1) + \cdots\end{aligned}$$

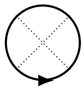
if $n \neq 0$ and

$$\Lambda_1(M_{\mathcal{K}}) = \left(\frac{-3k^2}{4} + \frac{5}{4}\right)\Theta_2 + \cdots = \left(\frac{-3k^2}{4} + \frac{5}{4}\right)(\Theta_2 \sqcup \Theta_1) + \cdots \in \overline{A}_1^{(\infty)}$$

if $n = 0$.

So the first coefficient (the coefficient of $\Theta_1 \sqcup \Theta_1$) is the order of the first homology of $M_{\mathcal{K}}$ (which is zero by definition if the homology has infinite number of elements). Moreover from C. Lescop's formula [8], the Casson–Walker–Lescop invariant of n-surgery of the $(2,k)$-torus knot is given as

$$\mathrm{sign}(n)\left\{v_2(K) - \frac{(|n|-1)(|n|-2)}{24}\right\},$$

where sign(0) = 1 and $v_2(K)$ is the coefficient of  in $\tilde{Z}(K)$. From (4.1), we know that $v_2(K) = \dfrac{k^2-1}{8}$. Therefore the second coefficient of our invariant coincides with $-6\lambda(M_{\mathcal{K}})$. This confirms our Theorem for the three-manifold obtained by Dehn surgery of the $(2,k)$-torus knot with n framing.

Our invariant described here was generalized to a series of invariants by T. Q. T. Le, J. Murakami, and T. Ohtsuki [7].

References

[1] S. Akbulut and J. D. McCarthy, *Casson's invariant for oriented homology* 3*-spheres - an exposition*, Mathematical Notes, vol. 36, Princeton University Press, 1990.

[2] D. Bar-Natan, *Non-associative tangles*, in: Geometric Topology (W. H. Kazez, ed.), AMS/IP Studies in Advanced Mathematics, vol. 2.1, American Mathematical Society and International Press, 1997, (1993 Georgia International Topology Conference, August 2–13, 1993, University of Georgia, Athens, Georgia), pp. 139–183.

[3] —, *On the Vassiliev knot invariant*, Topology **34** (1995), 423–472.

[4] M. Kontsevich, *Vassiliev's knot invariants*, Advances in Soviet Mathematics (1993), 137–150.

[5] T. Q. T. Le, H. Murakami, J. Murakami, and T. Ohtsuki, *A three-manifold invariant derived from the universal Vassiliev-Kontsevich invariant*, Proc. Japan Acad. Ser. A Math. Sci. **71** (1995), 125–127.

[6] —, *A three-manifold invariant via the Kontsevich integral*, preprint, Max-Planck-Institut für Mathematik, Bonn, MPI/95-62, 1995.

[7] T. Q. T. Le, J. Murakami, and T. Ohtsuki, *On a universal perturbative invariant of* 3*-manifolds*, to appear in Topology.

[8] C. Lescop, *Global surgery formula for the Casson-Walker invariant*, Annals of Mathematics Studies, vol. 140, Princeton University Press, 1996.

[9] K. Walker, *An extension of Casson's invariant*, Annals of Mathematics Studies, vol. 126, Princeton University Press, 1992.