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EDGE NUMBER RESULTS FOR PIECEWISE-LINEAR KNOTS

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Abstract. The minimal number of edges required to form a knot or link of type K is the edge number of K, and is denoted e(K). When knots are drawn with edges, they are appropriately called piecewise-linear or PL knots. This paper presents some edge number results for PL knots. Included are illustrations of and integer coordinates for the vertices of several prime PL knots.

1. Introduction. An elementary piecewise-linear knot invariant is the edge number e(K) of K, which is the minimal number of edges required to form the given knot or link K. It is already known [4, 6, 7, 9] that the edge numbers of prime knots[1] are given as follows:

$$(\dagger)\qquad \begin{gathered} e(unknot) = 3, \quad e(3_1) = 6, \quad e(4_1) = 7 \\ e(K) = 8 \text{ for } K = 5_1, 5_2, 6_1, 6_2, \text{ or } 6_3 \text{ and } e(K) \geq 8 \text{ for any other knot.} \end{gathered}$$

The minimal number of edges required to form some links and composite knots is also known [1, 4, 5]. For example, a composition of n trefoils has an edge number of $2n + 4$ [1]. Exact edge number results are difficult to obtain. In fact, edge numbers have not been determined in general for the most basic types of knots, the torus knots. Lower and upper bounds have been determined for torus knots, which together with certain additional hypotheses, yield exact results in these special cases [4, 5].

The purpose of this paper is to present edge number results which were obtained using a program called KED [3] written by K. Hunt as part of his thesis research in computer science at the University of Iowa. Each one of the several drawings of prime knots provided in Section 3 were produced using a similar type of program called MING [10], written by Professor Y-Q. Wu.

The work presented here was completed while at The University of Iowa.

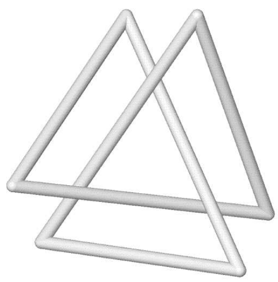
The paper is in final form and no version of it will be published elsewhere.

[1] Notation used in this paper is adopted from [8].

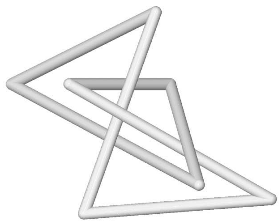
2. Edge number results. Since exact edge number results had previously been obtained for all prime knots with crossing number less than seven [4, 6, 7, 9], the prime knots with a crossing number of seven were logically the next ones to be considered. These knots were all constructed using only nine edges, therefore the result (†) stated in the introduction implies that each must have an edge number equal to either 8 or 9. A few knots with crossing number 8 were also realized, specifically the non-alternating ones, since these were suspected to have smaller edge numbers. 8_{19} and 8_{20} were drawn with eight edges[2] and 8_{21} was constructed from nine edges. Therefore, $e(8_{19})=8=e(8_{20})$ and $e(8_{21})=8$ or 9.

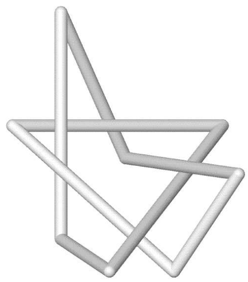
Each of the knots whose illustrations appear in Section 3 were obtained using KED [3], which gives the coordinates of the vertices of each of the knots produced as numbers with 16 decimal places. "Nice" integer coordinates were more desirable. In the table provided in Section 3, notice that each of the knots was, in fact, realized with integer coordinates (x, y, z) such that $-10 \leq x, y, z \leq 10$. These were obtained by examining the coordinates of each knot individually, manually inputting the "nice" integer coordinates nearby the original ones, and then inspecting the result to check that the knot type was preserved in the process. In a few cases, the coordinates had to be adjusted again in order to maintain the same knot type as the original.

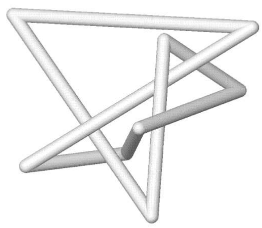
3. Prime knots with integer coordinates. This section contains illustrations of and integer coordinates for the prime knots with crossing number less than 8. Three non-alternating knots with crossing number eight are also included, as well as a symmetry presentation for the figure-eight knot 4_1. The drawings provided were produced by MING [10] using the cylinder-ball rendering and by choosing the knot color to be grey. In MING, there is also an option to obtain an image of the knot as it would be seen "viewed from above". In other words, the image has been projected down to the xy-plane, but in such a way that the crossings are still discernible. This option was selected, thereby producing illustrations which are easily interpreted. For example, the 3_1 knot shown below has leftmost vertex 5 and rightmost vertex 6.

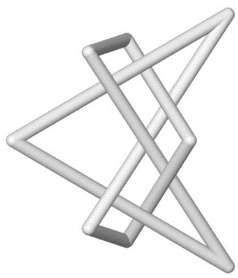
Knot	Integer coordinates of the vertices	MING illustration
3_1	1. (4, 9, 5) 2. (−7, −7, −5) 3. (7, −9, 5) 4. (−1, 9, −5) 5. (−9, −3, 5) 6. (9, −5, −5)	

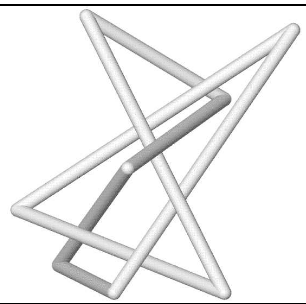
[2]K. Millett (personal communication) has realized 8_{20} with 8 edges of equal length.

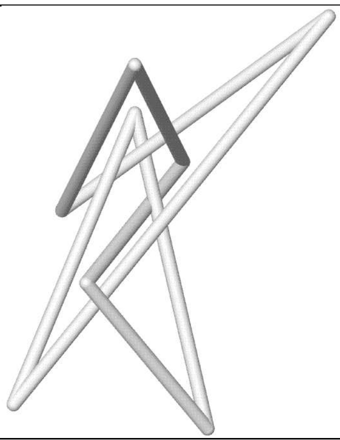
Knot	Integer coordinates of the vertices	MING Illustration
4_1	1. (9, −6, 3) 2. (−4, −7, 3) 3. (1, 7, 2) 4. (−9, 2, −10) 5. (4, −5, 10) 6. (2, 2, −2) 7. (−5, 2, 5)	

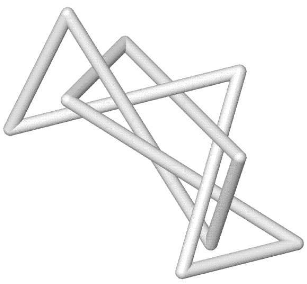
Knot	Integer coordinates of the vertices	MING Illustration
5_1	1. (−4, −6, 0) 2. (−1, −8, 9) 3. (6, 1, −8) 4. (−7, 1, 0) 5. (3, −8, 3) 6. (7, −2, 6) 7. (0, −1, −8) 8. (−4, 8, −1)	

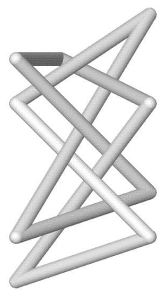
Knot	Integer coordinates of the vertices	MING Illustration
5_2	1. (3, 6, 2) 2. (9, 2, −4) 3. (1, −1, 6) 4. (0, −3, −5) 5. (−8, −4, 1) 6. (10, 7, 1) 7. (−9, 8, −6) 8. (2, −8, 3)	

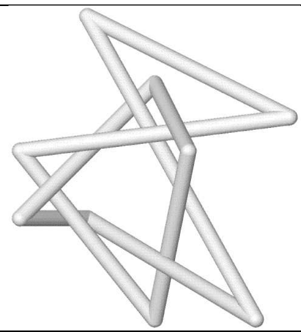
Knot	Integer coordinates of the vertices	MING Illustration
6_1	1. (0, −8, −5) 2. (−3, −7, 3) 3. (8, 10, −4) 4. (−9, 0, 5) 5. (8, −10, −5) 6. (−3, 7, 6) 7. (0, 8, −5) 8. (5, 0, 5)	

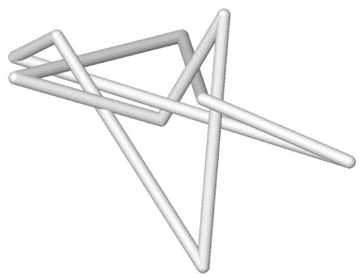
Knot	Integer coordinates of the vertices	MING Illustration
6_2	1. (−3, −10, 7) 2. (−7, −8, −10) 3. (−2, −1, 10) 4. (5, 4, −7) 5. (−5, 10, −2) 6. (5, −10, 10) 7. (−10, −4, −6) 8. (10, 9, −1)	

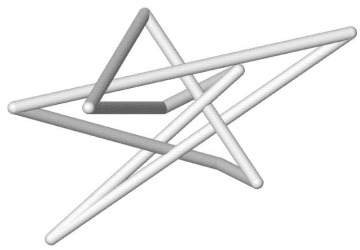
Knot	Integer coordinates of the vertices	MING Illustration
6_3	1. (2, −8, −6) 2. (−3, −2, −1) 3. (1, 3, −9) 4. (−1, 7, 10) 5. (−4, 1, −7) 6. (7, 9, −2) 7. (−6, −7, −4) 8. (−1, 5, −6)	

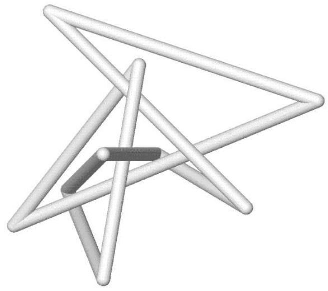
Knot	Integer coordinates of the vertices	MING Illustration
7_1	1. (−3, 5, 1) 2. (2, −3, −1) 3. (3, 0, 3) 4. (−1, 4, −3) 5. (−3, 2, 1) 6. (5, −3, −1) 7. (1, −4, 1) 8. (3, 3, 1) 9. (−5, 1, −2)	

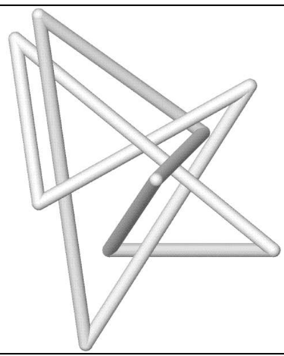
Knot	Integer coordinates of the vertices	MING Illustration
7_2	1. (−1, 1, 0) 2. (2, −2, 1) 3. (−1, −3, −2) 4. (2, 1, 2) 5. (0, 2, −3) 6. (−1, 2, 2) 7. (2, −1, −1) 8. (−1, −2, 3) 9. (2, 3, −3)	

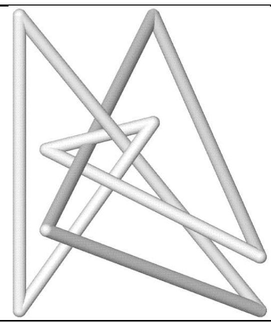
Knot	Integer coordinates of the vertices	MING Illustration
7_3	1. (0, 3, −2) 2. (−4, −1, 1) 3. (−2, −1, −2) 4. (3, −4, 4) 5. (−2, 5, −3) 6. (4, 2, 1) 7. (−4, 1, 0) 8. (0, −4, −1) 9. (1, 1, 3)	

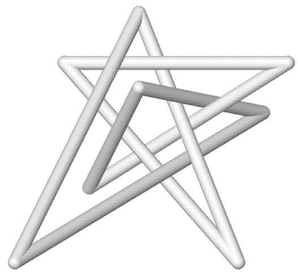
Knot	Integer coordinates of the vertices	MING Illustration
7_4	1. (−1, 1, 1) 2. (2, 6, 5) 3. (1, −6, 2) 4. (−6, 5, 2) 5. (−8, 4, −1) 6. (8, −1, 6) 7. (1, 2, 5) 8. (0, 5, −5) 9. (−8, 3, 4)	

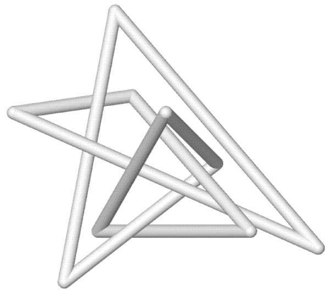
Knot	Integer coordinates of the vertices	MING Illustration
7_5	1. (−9, 1, 6) 2. (2, −2, −6) 3. (4, −3, 1) 4. (−2, 6, −7) 5. (−5, 1, 10) 6. (−1, 1, −9) 7. (3, 3, −2) 8. (−7, −6, −4) 9. (9, 5, −1)	

Knot	Integer coordinates of the vertices	MING Illustration
7_6	1. (−8, −4, −7) 2. (9, 3, 3) 3. (−6, 8, −1) 4. (5, −3, 1) 5. (0, 0, −7) 6. (−3, 0, 10) 7. (−5, −2, −10) 8. (−3, −7, 3) 9. (−1, 5, 0)	

Knot	Integer coordinates of the vertices	MING Illustration
7_7	1. (3, 2, −2) 2. (1, 0, 7) 3. (−1, −3, −6) 4. (6, −3, 2) 5. (−5, 6, 0) 6. (−4, −1, 0) 7. (5, 4, 0) 8. (−2, −7, −4) 9. (−4, 7, 3)	

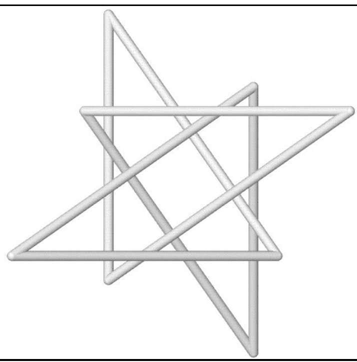
Knot	Integer coordinates of the vertices	MING Illustration
8_{19}	1. (−4, 6, 5) 2. (−4, −5, 3) 3. (1, 2, 3) 4. (−3, 1, 2) 5. (5, −3, 7) 6. (1, 6, −10) 7. (−3, −2, 10) 8. (5, −5, 0)	

Knot	Integer coordinates of the vertices	MING Illustration
8_{20}	1. (−3, 3, −3) 2. (4, −5, −1) 3. (0, 5, −3) 4. (−5, −5, 6) 5. (5, 1, −6) 6. (0, 2, 1) 7. (−2, −2, −5) 8. (6, 3, −1)	

Knot	Integer coordinates of the vertices	MING Illustration
8_{21}	1. (−3, 8, −1) 2. (−6, −8, −2) 3. (3, −1, −3) 4. (0, 2, 7) 5. (−4, −5, −7) 6. (5, −5, 5) 7. (−2, 3, −4) 8. (−9, 2, 0) 9. (9, −6, −1)	

It should be noted that coordinates for some of these knots have previously appeared elsewhere. Knot 4_1 appears in [9], knot 5_2 in [4], and each of the knots $3_1, 4_1, 5_2$, and 6_2 appear in [6].

In addition to the illustrations already provided in this paper, a more aesthetic result was also obtained. Based on the symmetry presentation of the figure-8 knot 4_1 which appeared in [2], a symmetry presentation was constructed using only 8 edges, as shown in the following table. It is called a symmetry presentation because a simple rotation of 90° counter-clockwise yields its mirror image, a knot with the same projection but with its crossings reversed. This result is especially interesting because it shows that this figure-8 knot and its mirror image are in the same path component of M_8 (hence too in M_n for $n \geq 8$), where M_n is the space of all knots with n edges [7].

Knot	Integer coordinates of the vertices	MING Illustration
4_1	1. (−3, −4, 0) 2. (7, 3, 7) 3. (−4, 3, 0) 4. (3, −7, −7) 5. (3, 4, 0) 6. (−7, −3, 7) 7. (4, −3, 0) 8. (−3, 7, −7)	

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