A TQFT FOR WORMHOLE COBORDISMS
OVER THE FIELD OF RATIONAL FUNCTIONS

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Abstract. We consider a cobordism category whose morphisms are punctured connected sums of $S^1 \times S^2$'s (wormhole spaces) with embedded admissibly colored banded trivalent graphs. We define a TQFT on this cobordism category over the field of rational functions in an indeterminant A. For r large, we recover, by specializing A to a primitive 4rth root of unity, the Witten-Reshetikhin-Turaev TQFT restricted to links in wormhole spaces. Thus, for r large, the rth Witten-Reshetikhin-Turaev invariant of a link in some wormhole space, properly normalized, is the value of a certain rational function at $e^{\frac{\pi i}{2r}}$. We relate our work to Hoste and Przytycki's calculation of the Kauffman bracket skein module of $S^1 \times S^2$.

Introduction. We wish to consider links and graphs in a connected sum of $S^1 \times S^2$'s. Borrowing a phrase from the physicists, we will use the term 'wormhole space' to describe this kind of oriented 3-manifold with a relatively simple topology. Let $\mathbb{Q}(A)$ denote the field with involution consisting of the rational functions in the indeterminant A, with involution given by sending A to A^{-1}.

In section one, we associate to a framed link L in a wormhole space M an invariant $< L >\in \mathbb{Q}(A)$. This generalizes the Kauffman Bracket [K] version of the Jones polynomial [J]. Let $G \subset M$ be a banded trivalent graph in the sense of [BHMV] (4.5) with an admissible coloring where the set of colors is taken to be all nonnegative integers. We will also define $< G >\in \mathbb{Q}(A)$. Here we make use of Roberts' [R] fusion technique for simplifying bracket calculations. The well-definedness of the invariant rests ultimately on the well-definedness of the Witten-Reshetikhin-Turaev invariant [W] [RT]. See also [L] and [KL]. It follows that for a link in a wormhole space, the Witten-Reshetikhin-Turaev invariant for r large is given by the evaluation of a rational function in A at $e^{\frac{\pi i}{2r}}$, which we denote A_r. In the introduction to [La], Lawrence remarks that while the Witten-

Reshetikhin-Turaev invariant of a link in S^3 (being essentially the Jones polynomial) can be defined for all values of A, this is not so far known to be true for links in any other 3-manifolds. We now know how to do this for links in wormhole space. The function is rational except at even roots of unity of small order where it may be discontinuous. These discontinuities are unavoidable, as is demonstrated in §5.

In section two, we define the related TQFT using the universal construction given in [BHMV]. We describe now the cobordism category $\mathcal{C}$ on which the TQFT is defined. A punctured wormhole space is a wormhole space with the interiors of some smooth closed 3-balls deleted. The boundary of a punctured wormhole space is a collection of 2-spheres. A punctured wormhole space with a properly embedded admissibly colored graph (M, G) has boundary a disjoint union of 2-spheres with colored banded points. The objects of $\mathcal{C}$ are disjoint unions of 2-spheres with colored banded points. We also include the empty surface. A morphism from object Σ_1 to object Σ_2 is a triple (M, G, f). Here f is a diffeomorphism of $\partial(M, G)$ to $-\Sigma_1 \coprod \Sigma_2$, and M is a punctured wormhole space or a disjoint union of punctured wormhole spaces or is the empty set. It is pleasant that no additional further structure such as a 2-framing or p_1 structure is needed. This is because our 3-manifolds are so simple.

Since $<>$ is a multiplicative involutive invariant, the universal construction provides a cobordism generated quantization functor from $\mathcal{C}$ to the category of vector spaces over $\mathbb{Q}(A)$. We show that the vector spaces are finite dimensional and that the tensor product axiom holds. Thus we have a TQFT [A] which assigns $V(\Sigma)$ to an object Σ and $Z_M : V(\Sigma_1) \to V(\Sigma_2)$ to a morphism M from Σ_1 to Σ_2. If Σ is the 2-sphere with $2k$ banded points all colored one, the $\dim(V(\Sigma))$ is the kth Catalan number. A version of this TQFT with many fewer objects and morphisms was introduced in [G, §4].

$\mathcal{C}$ appears to be the largest category on which one may define a generic TQFT which will specialize to the Witten-Reshetikhin-Turaev SU(2) invariant for r large. Thus is how we were lead to it.

Hoste and Przytycki [HP] calculated $\mathcal{S}(S^1 \times S^2)$, the Kauffman bracket skein module of $S^1 \times S^2$. $\mathcal{S}(S^1 \times S^2)$ as a module over $\mathbb{Z}[A, A^{-1}]$ has a rank one free part, and some torsion. Thus a framed link in $S^1 \times S^2$ determines a Laurent polynomial in A. In section 3, we show that this polynomial agrees with our $< L >$. However our work provides an alternative method of calculating the Hoste-Przytycki polynomial. This polynomial is also (up to sign) the penultimate coefficient of $\Gamma(L)$, defined in [G]. The invariant $< L >$ is new for links in a connected sum of more than one $S^1 \times S^2$. The fourth section gives two worked examples. The fifth section, improves estimates on how large r must be to guarantee that $< G >$ evaluated at A at A_r is essentially the rth Witten-Reshetikhin-Turaev invariant.

We think that the TQFT introduced here is useful from a pedagogical point of view for the following reasons. On the one hand, the objects and morphisms of the cobordism category are simple topological spaces without any extra structure such as a 2-framing or p_1 structure. On the other hand, the theory is not completely trivial either.

1. Graphs in a wormhole space. Let r be an integer greater than two. By a graph, we will mean a trivalent banded colored graph as above. Let G be a graph in a worm-

hole space M. Suppose that k is greater than all the colors on G. Let $Z_r(M, G)$ denote the Witten-Reshetikhin-Turaev invariant. This is $\mathcal{I}_{A_r}(M, L)$ evaluated at the element of $\mathcal{S}(S^1 \times I)^{\otimes \#L}$ given by taking $S_i(\alpha)$ on a component colored i where L is the colored link obtained by expanding G [L]. Also $Z_r(M, G)$ is $< M, G >_{2r}$, where M has been given a p_1 structure whose σ invariant is zero [BHMV][MV]. We let

$$< G >_r = \frac{Z_r(M, G)}{Z_r(M)}$$

M is given by zero framed surgery on the unlink. The number of components of the unlink is the number of $S^1 \times S^2$ summands. We mark the components of the unlink with dots to indicate along which components 0-framed 1-surgery is to be performed, thinking perhaps of doing a 0-surgery along two three balls on ether side of the spanning disk [Ki, p. 5]. Thus (M, G) may be described by a trivalent colored graph drawn on blackboard with standard diagram of unlink whose components have been dotted. We may assume that G is transverse to the disks which bound the components of the unlink. Thus $Z_r(M, G)$ is the generalized bracket evaluation [KL] of this diagram after replacing each component of the unlink with the ω_{2r} of [BHMV]. $Z_r(M)$ is simply the generalized bracket evaluation of the unlink after replacing each dotted component with ω_{2r}. Thus if we were to change ω by a scale factor in the above computation, $< G >_r$ would remain unchanged. Thus we could as well take ω to be Lickorish's $\omega = \sum_i = \Delta_i S_i(\alpha)$ or [BHMV]'s Ω_{2r}.

Suppose r is larger than the sum of the colors on the strands of G which pass through each disk. Then we may use recoupling theory to rewrite the evaluation as a linear combination of evaluations where at most a single strand passes through each disk. This is called fusion. One simply makes repeated use of the two strand fusion identity. See the identity of Figure 31 in [L] for example. Then we may discard all the terms in the linear combination where the single strand has a nonzero color using Lickorish's Lemma [L, Lemma 6]. This method of simplification is due to Roberts [R, Figure 7, Figure 16] [KL,§12.11]. Figure 1 shows this method applied to 3 strands. Recall a triple of nonnegative integers which are less than r are called r-admissible if $|i - j| \le k \le i + j$, $i + j + k \equiv 0 \pmod 2$, and $i + j + k \le 2r - 4$.
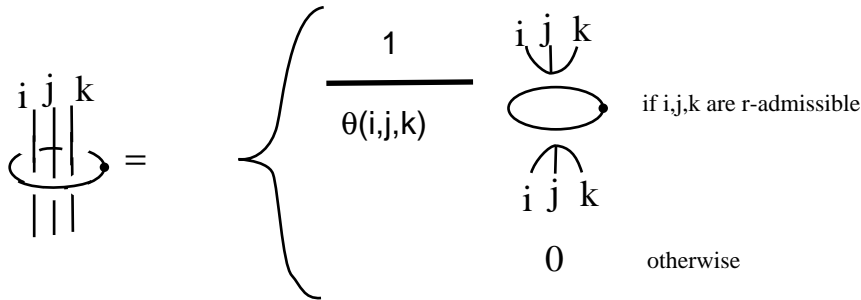

Figure 1   Fig. 1

In our situation, it is important to see that the above simplification can be made independent of r, as long as r is greater than the sum of colors passing through each disk. If

r is greater than the sum plus one, then all the two strand fusions may be done independently of r. If r is equal to the sum plus one, then all but the last fusion can be done with the same formulas as for larger r. In the last fusion, the expansion we get is obtained by taking the expansion for larger r and discarding all terms with a nonzero color on the single strand. However for larger r these same terms get discarded later using Lickorish's lemma.

One now has a linear combination of trivalent colored graphs completely unlinked from the unlink. Thus $< G >_r$ is given by the bracket evaluation of the graph after we delete the unlink. Let G' denote this new trivalent graph. $< G >_r$ is given by the bracket evaluation of G' after we set $A = A_r$.

Let $< G > \in \mathbb{Q}(A)$ be the bracket evaluation of G'. Then $< G >_r$ is simply $< G >$ evaluated at $A = A_r$. Since by, say, [BHMV] or [L], $< G >_r$ is a well defined isotopy invariant of G, we can conclude that $< G > \in \mathbb{Q}(A)$ is also. Here we use the elementary fact that if two rational functions agree at infinitely many distinct points then they must agree. This is an immediate consequence the fundamental theorem of algebra. It is important to realize that this argument shows that the result of calculating $< G >$ as above does not depend on the original surgery description of M or on any choices made in performing fusion.

We note that if L is a link diagram, then it describes a framed link in the wormhole space S^3. If we color this framed link one, we get a graph in S^3 which we also denote by L. In this situation $< L >$ is the ordinary Kauffman bracket [K] of the link diagram L, which in turn is version of the Jones polynomial [J]. Thus $< G > \in \mathbb{Q}(A)$ generalizes the Kauffman bracket or Jones polynomial of links in S^3 to graphs in a wormhole space.

LEMMA (1.1). *If $G \subset M$ meets an embedded 2-sphere $S \subset M$ in a single non-zero colored point, then $< G >$ is zero.*

P r o o f. Cut M along S and attach two 3-disks to obtain a space M'. If M' is connected, then M' is a wormhole space and M is obtained from M' by a 0-surgery. In this case Lickorish's lemma shows $< G >_r$ to be zero for r large. Thus $< G >$ must be zero.

If M' is disconnected, then M' is the disjoint union of two wormhole spaces and M is obtained by taking their connected sum. Now $< G >_r$ is zero for r large, making use of one basic properties of the Temperley-Lieb idempotents $f^{(n)} e_i = 0$ [L, Lemma 1]. Thus $< G >$ must be zero. ∎

This invariant is extended multiplicatively to disjoint union of wormhole spaces with graphs. This invariant is involutive: Let $-G$ denote G in M with the orientation on M reversed. Then $< -G > = \overline{< G >}$.

Note that $< G >$ is really an invariant of the pair (M, G). Thus if f is an orientation preserving diffeomorphism of M, then $< f(G) > = < G >$. Equivalently $< G >$ does not depend on a particular way of viewing M as the connected sum of some $S^1 \times S^2$'s.

2. A TQFT. Consider the cobordism category $\mathcal{C}$ described in the introduction. $<>$ is an involutive multiplicative invariant on the closed objects of $\mathcal{C}$. The universal construction of Blanchet, Habegger, Masbaum, and Vogel then provides a cobordism generated

quantization functor (Z, V) to the category of vector spaces over $\mathbb{Q}(A)$. To show that (Z, V) is a TQFT we must show that the tensor product axiom holds, and that the vector spaces associated to surfaces are finite dimensional . Recall elements of $V(\Sigma)$ are equivalence classes of linear combinations of graphs in a punctured wormhole space whose boundary is Σ. We note that $V(\Sigma)$ comes equipped with a nonsingular Hermitian form $< \, , \, >_\Sigma$.

LEMMA (2.1). *Let Σ be an object of $\mathcal{C}$. Let N be a punctured 3-sphere with boundary the underlying manifold of Σ. $V(\Sigma)$ is generated by graphs in N.*

P r o o f. Let (N, G) be a Wormhole space with graph whose boundary is Σ. We may represent G by a graph in a punctured 3-sphere N' together with a dotted unlink. Applying the method of calculation in §1, we obtain a linear combination of graphs G' in N'. Then we have the following equality of conjugate linear forms: $< (N, G), \, >_\Sigma = < (N', G'), \, >_\Sigma$. Thus (N, G) and (N', G') represent the same element in $V(\Sigma)$. ∎

LEMMA (2.2). *The natural map $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \coprod \Sigma_2)$ is surjective.*

P r o o f. Let (N, G) be a punctured 3-sphere with graph whose boundary is $\Sigma_1 \coprod \Sigma_2$. Let S be a 2-sphere in N such that Σ_1 and Σ_2 lie in different components of $N - S$. Let N' denote N surgered along S. We may use fusion to represent the same element of $V(\Sigma_1 \coprod \Sigma_2)$ as (N, G) by a linear combination of graphs in N which each meet S in at most a single point. By Lemma (1.1), we may discard those which meet S in a single point. Let G' be the resulting linear combination of graphs in N'. $< (N, G), \, >_\Sigma = < (N', G'), \, >_\Sigma$. Thus (N, G) and (N', G') represent the same element in $V(\Sigma)$. But such an element is in the image of $V(\Sigma_1) \otimes V(\Sigma_2)$. ∎

Let Σ be a 2-sphere with some banded colored points ℓ. S is the boundary of a 3-ball B. Let $\mathcal{G}$ be an embedded trivalent (noncolored) tree in B, such that $\mathcal{G} \cap S = \ell$. Admissible colorings of G give graphs in B and thus elements of $V(\Sigma, \ell)$. As in [BHMV] (see also [KL, Chapter 7]), one may show:

LEMMA (2.3). *The set of admissible colorings of a fixed tree $\mathcal{G}$ as above forms an orthogonal basis for $V(\Sigma)$ with respect to the Hermitian form $< \, , \, >_\Sigma$.*

THEOREM (2.4). *(Z, V) is a TQFT.*

P r o o f. By Lemmas (2.2) and Lemma (2.3), $V(\Sigma)$ is finite dimensional for any Σ. If we equip $V(\Sigma_1) \otimes V(\Sigma_2)$ with the tensor product of the forms on $V(\Sigma_1)$ and $V(\Sigma_2)$, the map $V(\Sigma_1) \otimes V(\Sigma_2) \to V(\Sigma_1 \coprod \Sigma_2)$ is an isometry , and thus injective. Thus the map is an isomorphism. ∎

PROPOSITION (2.5). *If Σ is a sphere with $2n$ points colored one, then $V(\Sigma)$ is given in the obvious way by the set of diagrams in the disk without crossings and with boundary $2n$ fixed points. Thus $\dim V(\Sigma)$ is the nth Catalan number $c(n) = \frac{1}{n+1} \binom{2n}{n}$.*

P r o o f. Using the Kauffman relations, it is clear that this set generates $V(\Sigma)$. Consider the matrix we get when we pair this set of elements against itself under $< \, , \, >_\Sigma$. Its determinant is easily seen to be a polynomial of degree $n \, c(n)$ in $d = -A^2 - A^{-2}$. So this set of elements is linearly independent. ∎

3. Relation to Hoste and Przytycki's work. $\mathcal{S}(S^1 \times S^2)$ modulo its $\mathbb{Z}[A, A^{-1}]$-torsion submodule is isomorphic to $\mathbb{Z}[A, A^{-1}]$, [HP]. This quotient can be identified with $\mathbb{Z}[A, A^{-1}]$. The equivalence class of the free generator 1 given by the empty link is identified with one. Following Hoste and Przytycki, we let π denote the projection from $\mathcal{S}(S^1 \times S^2)$ to this quotient. Thus we have $\pi : \mathcal{S}(S^1 \times S^2) \to \mathbb{Z}[A, A^{-1}]$. The application of $< >$ to links L in $S^1 \times S^2$ colored one defines a $\mathbb{Z}[A, A^{-1}]$-module homomorphism to $\mathbb{Q}(A)$ which must vanish on the torsion submodule. Let $\emptyset$ denote the empty link in $S^1 \times S^2$, then $< \emptyset >= \pi(1) = 1 \in \mathbb{Z}[A, A^{-1}]$. This proves:

PROPOSITION (3.1). *If L is a framed link in $S^1 \times S^2$ colored one, then $< L >= \pi(L)$.*

The element of $\mathcal{S}(S^1 \times S^2)$ given by m standardly framed longitudes is denoted z^m. Let L_m denote this link viewed now as a closed morphism of $\mathcal{C}$. We have by [BHMV, (1.2)], $< L_m >= \mathrm{Trace}(Id_{S_m}) = \dim V(S_m)$. Using Proposition (2.5), we have a new proof of [HP, Corollary 5].

COROLLARY (3.2) (Hoste-Przytycki). *$\pi(z^{2n+1}) = 0$, and $\pi(z^{2n})$ is the nth Catalan number.*

We remark that it follows from Proposition (3.1) that, for a link L colored one in $S^1 \times S^2$, $< L >$ actually lies in $\mathbb{Z}[A, A^{-1}]$. This also follows from the fact shown in [G] that $\Gamma(\mathcal{T})$ has coefficients in $Z[A, A^{-1}]$. The first example in §4 shows this is not true for a knot colored one in the connected sum of two $S^1 \times S^2$s.

Although a relationship has been given between [HP] and [G, §4], the lower bounds given for the wrapping number of links in $S^1 \times S^2$ given by these two papers still seem different, but a detailed comparison has not been made.
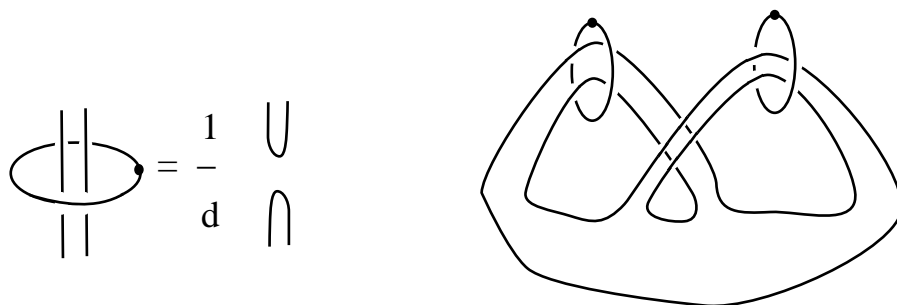
4. Two examples

Fig. 2

Using the fusion identity on the left of Figure 2, we compute $< K >= \frac{1}{d}$ for the knot in the connected sum of two $S^1 \times S^2$s pictured on the left.

Consider the link $\mathcal{T}$ in $S^2 \times I$ pictured on the left of Figure 3. In [G], we calculated a matrix for $Z_{(S^2 \times I, \mathcal{T})}$ with respect to the basis given in Proposition (2.5). The trace of this matrix was $A^{-12} - A^{-8} - A^{-4} + 1 - 2A^4 + A^{12} - A^{16}$. This must be $< L >$ where L is the link in $S^1 \times S^2$ pictured on the right of Figure 3.
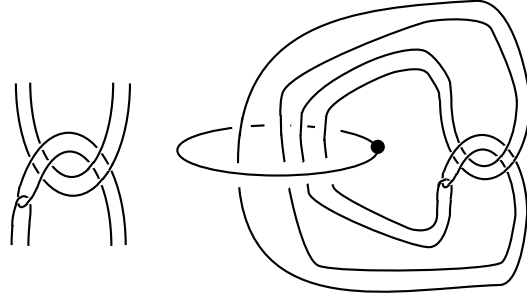
Fig. 3

We will now calculate $< L >$ directly by the method described in §1. We will make use of the fusion formula given in Figure 4. All unlabelled and undotted strands are colored one.
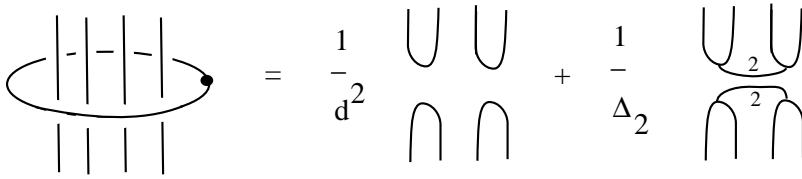

Fig. 4

Thus $< L >= \frac{1}{d^2} < G_1 > + \frac{1}{\Delta_2} < G_2 >$, where G_1 and G_2 are the graphs in S^3 pictured in Figure 5.

G_1 G_2
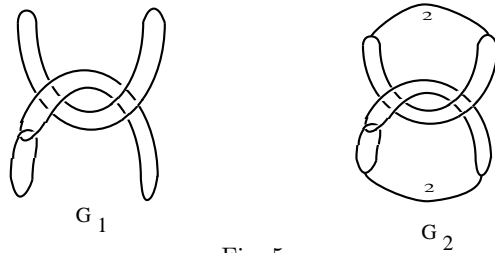
Fig. 5

We expand each one using Figure 6.
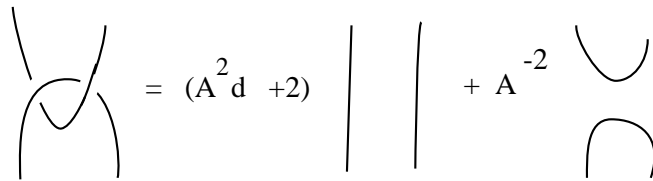

Fig. 6

$< G_1 >$ is easily seen to be $d^2(A^2 d + 2 + A^{-2} d)$. After we expand G_2 as in Figure 4, we may discard the term with coefficient A^{-2} using the vanishing of the idempotent

$f^{(2)}$ times a hook. We are left with two simply linked theta curves each with edges one, one and two. We may replace these with loops labelled two. See for instance [KL, p. 40]. Thus $< G_2 >$ is $A^2d + 2$ times the evaluation of a Hopf link colored two which is $A^{-16} \sum_{j= 0, 2, 4} \Delta_j A^{j(j+2)}$. This is a special case of a formula in [KL, p. 217] and is also easily found using [KL, 9.15]. Putting this together one calculates that $< L >$ is indeed the previously calculated trace.

5. How large must r be. Suppose M is a wormhole space which has been identified as zero framed surgery on an unlink and that disjoint spanning disks $\mathfrak{D}_i$ have been chosen for the components of the unlink. Let G be a graph in M which is transverse to the $\mathfrak{D}_i$. $< G >_r$ may be calculated by the above fusion process, but if r is small, $< G >_r$ may not be $< G >$ evaluated at $A = A_r$. However if the sum of the colors on the strands passing through any of these disks is odd, then $< G >_r = 0$ for all r, and of course $< G >= 0$. Suppose now each of these color sums is even, let $\mathfrak{C}_i$ denote the list of colors on the strands passing though the ith disk. Here a list allows repetitions but is unordered.

Given a finite list $\mathfrak{C}$ of n colors, we may consider a 2-sphere $S(\mathfrak{C})$ with n framed colored points whose colors are given by this list. Fix a trivalent tree $T(\mathfrak{C})$ in the 3-ball with boundary these points, and assume r is greater than the maximum color plus two. Thus every color is less than $r - 1$. Let $V_r(S(\mathfrak{C}))$ denote the vector space denoted $V_{2r}(S(\mathfrak{C})) \otimes \mathbb{C}$ in [BHMV]. It is given by admissible colorings of $T(\mathfrak{C})$ by nonnegative integers less than $r - 1$ which extends the colorings of of the boundary with the extra condition that the sum of the colors on any three edges which meet at a point is less than or equal to $2r - 4$. If r is large enough , this extra condition does not exclude any admissible colorings. In this case we have $\dim V_r(S(\mathfrak{C})) = \dim V(S(\mathfrak{C}))$. Let $r(\mathfrak{C})$ be the least such r. It is easy to see that if $\mathfrak{C}$ is a list of $2m$ ones, then $r(\mathfrak{C}) = m + 2$, [G, 4.6].

A slight variant of the fusion process for calculating $Z_r(M, G)$ is given as follows. We consider a 3-ball neighborhood of $\mathfrak{D}_i$. The strands that go through each disk D_i represent an element of $V_r(S(\mathfrak{C}_i \amalg \mathfrak{C}_i))$. Here $\mathfrak{C}_i \amalg \mathfrak{C}_i$ denotes the list $\mathfrak{C}_i$ repeated twice. Consider the tree T_i in the 3-ball given as a connected sum of two 3-balls with one copy of $T(\mathfrak{C}_i)$ in each joined by an extra straight edge. An orthogonal basis for $V_r(S(\mathfrak{C}_i \amalg \mathfrak{C}_i))$ is given by colorings of T_i. We may expand the element given by the original strands in terms of this orthogonal basis. If $r \geq r(\mathfrak{C}_i)$, it is easy to see that the coefficients of a basis element which is given by a coloring for which the extra edge is colored zero is given by a rational function in A evaluated at $A = A_r$. The numerator of this rational function is simply the bracket evaluation of the expansion of the closure of the colored tree by straight strands. The denominator of this rational function is simply the bracket evaluation of the expansion of the double of the colored tree. On the other hand, the terms coming from colorings where the extra strand has a nonzero coloring contribute zero to the total evaluation of $Z_r(M, G)$ by Lickorish's Lemma. Thus we have:

PROPOSITION (5.1). *In the situation above, if r is greater than or equal to the maximum of $r(\mathfrak{C}_i)$, then $< G >_{|A=A_r}=< G >_r$.*

Now consider the example of a link L in $S^1 \times S^2$ considered in the previous section. It intersects a spanning disk in four points colored one. Thus if $r \geq 4$, we have $< L >_r=$

$< L >_{|A=A_r}$. On the other hand if $r = 3$, then the second term on the right hand side of the equation pictured in Figure 4 should be deleted. Thus $< L >_3 = \frac{1}{d^2} < G_1 >_{|A=A_3} = 1$. Whereas $< L >_{|A=A_3} = 4 - 3A_3$.

The above proposition may be used to give 'lower bounds' on $\{\mathfrak{C}_i\}$. In particular, we have the following result on the wrapping number of links in $S^1 \times S^2$.

PROPOSITION (5.2). *Let L be a link in $S^1 \times S^2$ with even wrapping number $w(L)$. If $< G >_{|A=A_r} \neq < G >_r$, then $w(L) \geq 2r - 2$.*

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