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# SPHERICAL DETECTORS OF GRAVITATIONAL WAVES

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**Abstract.** Resonant mass detectors of GWs of spherical shape constitute the fourth generation of such kind of antennae, and are scheduled to start operation in the near future. In this communication I present a general description of the fundamental principles underlying the physics of this kind of detector, as well as of the motion sensor set suitable to retrieve the information generated by the incidence of a GW on the antenna.

**1. Introduction.** GW detection research started in the early 1960's thanks to the pioneering work of J. Weber [1], and has been going on ever since. Disproval of original claims of event sights motivated further efforts in the direction of improving the sensitivity of the detectors, whereby a sophisticated technology of cryogenic cooling of bars began to develop in the late 1970's. By cooling the alluminum bars to liquid helium temperatures it is possible to damp *thermal noise* in it, which strongly tends to blur the rather weak GW signals expected, but it also enables the use of very sensitive *SQUID* amplifiers for a better performance of the detector system. Resonant detectors of this kind constituted what is often called *second generation* detectors, and they started operation towards the mid 1980's, and still are in function today [3]. *Third generation* detectors include an improved technology which enables them to reach thermodynamic temperatures in the range of a few milli-Kelvin, and have begun taking data recently [2, 4]. The attained sensitivity of second generation detectors is reported to be about $h = 10^{-19}$ for millisecond bursts of supernova radiation [3], and is expected to improve by nearly an order of magnitude with the third generation of ultracryogenic antennae.

In spite of this remarkable sensitivity, it turns out that it is just sufficient to see supernova explosions in our galaxy; the *event rate* for such explosions is however too low (only a few per *century*), which makes it highly necessary to enhance the detector capabilities in order to *stretch* its scope beyond the galaxy into the Virgo cluster and

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perhaps even further out. This would obviously increase the event rate and therefore the chances of seeing GWs.

A further *fourth generation* of resonant GW detectors is now being *projected* at several places in the world, with the aim of satisfying the requirement of an even better sensitivity. These detectors are planned to have *spherical* shape, rather than *cylindrical*, as had the previous ones. There are a number of reasons of different nature which support this choice. A rather fundamental one is that a solid sphere having the same (lowest) resonance frequency as a cylinder is about 20 times more massive than the latter, whence at least this factor is gained in *energy* sensitivity by use of such device, if operated in like laboratory circumstances. But there are others too, as I shall presently discuss.

In this paper I will concentrate on the *theoretical* aspects of the functioning of a spherical GW antenna, which means that some of the details of its practical implementation will be left aside. I will clearly differentiate the following two parts of the problem: in the first part I will consider the interaction between an incoming metric GW and a solid elastic sphere of uniform density; this analysis enables the discussion of its response and GW energy absorption *cross section*, i.e., of the sphere's sensitivity. In the second part, I will address the problem of how the sphere's excitations can be actually *sensed* by means of a suitable set of motion sensors attached to its surface.

Much of the material I will present here is the result of very recent original research, and has been presented to an international audience here in Warsaw for the first time. What follows is a summary of the most relevant ideas and conclusions; the reader will be opportunely referred to the appropriate bibliography where he/she will find further technical information.

**2. The general formulae.** It will be assumed that we have a homogeneous *solid* sphere of uniform density $\rho$, total mass $M$ and radius $R$. If it is hit by a *force density* $\mathbf{f}(\mathbf{x},t)$, its *vibration modes* will be excited; these are described by the field of displacements $\mathbf{u}(\mathbf{x},t)$, which satisfy the elastic equations

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mu\nabla^2\mathbf{u} - (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f}(\mathbf{x},t) \tag{1}$$

where $\lambda$ and $\mu$ are the sphere's Lamé coefficients [5]. This is a *non-relativistic* equation, but we feel firmly justified in using it, as we do not expect relativistic speeds in any GW excitations —they should come in relatively low frequency ranges.

The first thing we want to address is this: how do we express the GW force density acting on the solid?

*2.1. The GW tidal forces: their monopole-quadrupole structure.* We shall adopt a point of view whereby GW forces are regarded as *tidal* forces arising as a consequence of the non-vanishing of the GW Riemann tensor. If the *wavelength* of the incoming radiation is *long* compared to the sphere's dimensions (radius) then it can be shown that

$$f_i(\mathbf{x},t) = \rho c^2\, R_{0i0j}(t)\, x_j \tag{2}$$

where $R_{0i0j}(t)$ are the "electric" components of the GW Riemann tensor evaluated at the *centre* of the sphere, and $x_i$ is a Cartesian coordinate of a point in the solid relative to its centre. Eq. (2) has been obtained from the standard *geodesic deviation* equation. If the assumption is made that the GW amplitude is given by the *small* metric perturbations $h_{\mu\nu}$, then

$$R_{0i0j} = \frac{1}{2}\,(h_{ij,00} - h_{0i,0j} - h_{0j,0i} + h_{00,ij}) \tag{3}$$

if only first order terms in the $h$'s are retained. If General Relativity is assumed to be the theory which correctly describes GWs then it is possible to make a choice of *gauge* where only the term $h_{ij,00}$ survives in the right hand side of (3). However, *I shall not make this assumption here*, but will allow for alternative theories, too. As we shall shortly see, it is possible to calculate the sphere's response to *arbitrary metric* GWs, which gives the spherical antenna the possibility of *experimentally* setting bounds on the predictions of other hypothetical theories of the gravitational interaction. We thus keep (3) as it stands.

$R_{0i0j}(t)$ is a symmetric 3-tensor, and therefore the following decomposition can be established [6]:

$$\mathbf{f}(\mathbf{x},t) = \mathbf{f}^{(S)}(\mathbf{x})\,g^{(S)}(t) \;+\; \sum_{m=-2}^{2} \mathbf{f}^{(m)}(\mathbf{x})\,g^{(m)}(t) \tag{4}$$

with

$$f_i^{(S)}(\mathbf{x}) = \rho\,E_{ij}^{(S)}\,x_j\,, \qquad g^{(S)}(t) = \frac{4\pi}{3}\,E_{ij}^{*(S)}\,R_{0i0j}(t)\,c^2 \tag{5}$$

$$f_i^{(m)}(\mathbf{x}) = \rho\,E_{ij}^{(m)}\,x_j\,, \qquad g^{(m)}(t) = \frac{8\pi}{15}\,E_{ij}^{*(m)}\,R_{0i0j}(t)\,c^2 \tag{6}$$

and

$$E_{ij}^{(S)} = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

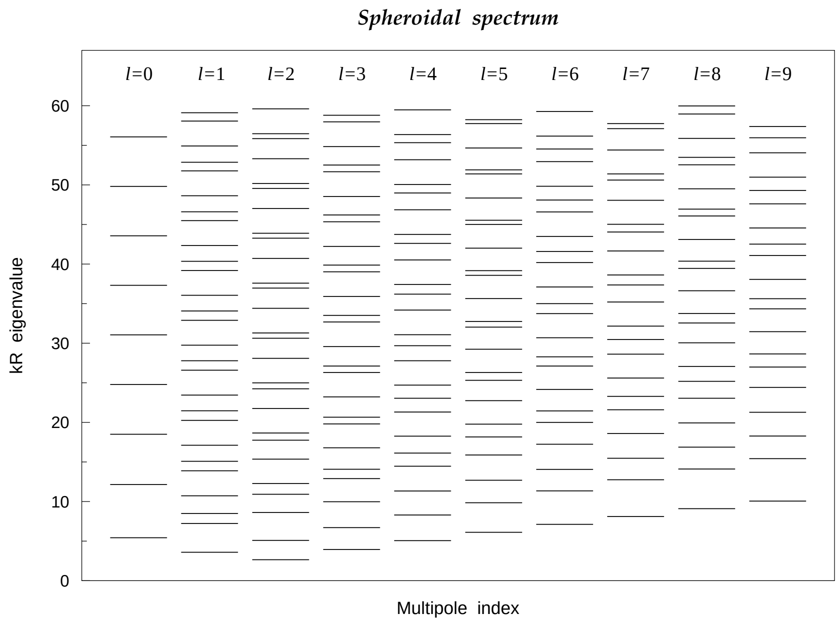
$$E_{ij}^{(0)} = \left(\frac{5}{16\pi}\right)^{\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\;,\quad E_{ij}^{(\pm 1)} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & \mp 1 \\ 0 & 0 & -i \\ \mp 1 & -i & 0 \end{pmatrix} \tag{8}$$

$$E_{ij}^{(\pm 2)} = \left(\frac{15}{32\pi}\right)^{\frac{1}{2}} \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

This decomposition fully displays the *monopole-quadrupole* structure of a general GW Riemann tensor: indeed its 6 independent components are seen to be expressible in terms of the one monopole amplitude $g^{(S)}(t)$ and the 5 quadrupole amplitudes $g^{(m)}(t)$ $(m=-2,...,2)$.

*2.2. The antenna response.* The next step is to solve the equations of motion (1). It is expedient to do so in terms of a *Green function* formalism. As shown in reference [6], the solution to the problem can be cast in the form of an orthogonal series:

$$\mathbf{u}(\mathbf{x},t) = \sum_N \omega_N^{-1}\mathbf{u}_N(\mathbf{x})\,\left[f_N^{(S)}\,g_N^{(S)}(t) \;+\; \sum_{m=-2}^{2} f_N^{(m)}\,g_N^{(m)}(t)\right] \tag{10}$$



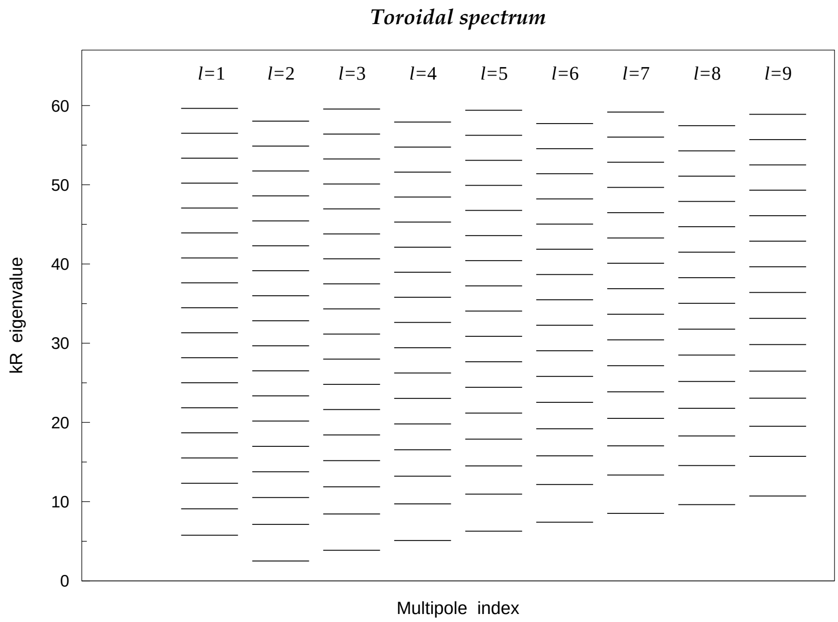




Figure 1. The homogeneous sphere's *spheroidal* (above) and *toroidal* eigenvalues for a few *multipole* families. Only the $l$=0 and $l$=2 *spheroidal* families couple to metric GWs, so the rest are given for completeness and *vetoing* purposes. The diagramme corresponds to a sphere with Poisson ratio $\sigma$=0.33. Plotted values correspond to *dimensionless* eigenvalues; actual frequencies can be obtained from these by a suitable conversion factor which depends on the solid's elastic properties and size —see [6] for further details.

where

$$f_N \equiv \frac{1}{M} \int_{\rm Solid} \mathbf{u}_N^*(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x})\, d^3x \tag{11}$$

and

$$g_N^{(\alpha)}(t) \equiv \int_0^t g^{(\alpha)}(t')\, \sin\omega_N(t-t')\, dt' \quad , \quad \alpha = S, m \tag{12}$$

and $\omega_N$ are the vibration *eigenfrequencies* of the antenna, whose (suitably normalised) wavefunctions are $\mathbf{u}_N(\mathbf{x})$ —see always [6] for details. Equation (10) is very general, and actually valid for *any* detector shape and boundary conditions. It shows that the *sphere* is the *optimum* GW detector shape; indeed, the *projection integrals* (11) are maximum for a maximum *overlap* between the driving forces' *form factors* $\mathbf{f}^{(\alpha)}(\mathbf{x})$ and the solid's wavefuntions $\mathbf{u}_N(\mathbf{x})$ —and this happens to be the case for a spherically shaped antenna, due to the canonical *multipole* structure of its specific wavefunctions [6].

It should also be stressed once more that (10) is the antenna's response to a *completely general metric* GW, i.e., it is valid no matter which is the *metric* theory which correctly describes the gravitational interaction.

A *spherical antenna* has *two families* of vibration eigenmodes: so called *toroidal*, or torsional, and *spheroidal* modes —see Figure 1 for a graphical representation of the eigenvalue spectrum. The former consist in purely torsional, or tangential, deformations, and are easily seen not to couple to GWs at all [6, 7]; the latter do couple to GWs and consist of a combination of tangential and radial deformations. We are therefore mainly interested in these[1]. If the spheroidal eigenfunctions are named $\mathbf{u}_{nlm}(\mathbf{x})$, it is not difficult to see that (10) reduces to

$$\mathbf{u}(\mathbf{x},t) = \sum_{n=1}^{\infty} \frac{a_n}{\omega_{n0}}\, \mathbf{u}_{n00}(\mathbf{x})\, g_{n0}^{(S)}(t) \;+\; \sum_{n=1}^{\infty} \frac{b_n}{\omega_{n2}} \left[\sum_{m=-2}^{2} \mathbf{u}_{n2m}(\mathbf{x})\, g_{n2}^{(m)}(t)\right] \tag{13}$$

where $a_n$ and $b_n$ are overlapping integrals. The reader should note that the subindex $N$ in equations (10)–(12) is a *multiple* index, actually $N=\{nlm\}$ in this case, yet the coefficients $a_n$ and $b_n$ are *fewer* than in the general case (they carry the *single* index $n$); this is due to the *good matching* between the sphere's eigenmode amplitudes and the GW's multipolar structure, as discussed earlier. What this practically means that *only* monopole ($l=0$) and quadrupole ($l=2$) *sphere's* modes can possibly be excited by an arbitrary GW. It should also be stressed that every $l$-pole mode is $(2l+1)$-fold degenerate, i.e., there are $(2l+1)$ eigenfunctions $\mathbf{u}_{nlm}(\mathbf{x})$ ($m=-l,...,l$) for *each* eigenfrequency $\omega_{nl}$.

*2.3. The sphere's GW energy absorption cross sections.* The antenna response (eq. (13)) can be *Fourier transformed*, and the detector's *vibration energy* for a given *frequency*, $E(\omega)$, calculated. If $\Phi(\omega)$ is the incoming GW's *flux density* then a *cross section* for GW energy absorption can be defined by

$$\sigma_{\rm abs}(\omega) = \frac{E(\omega)}{\Phi(\omega)} \tag{14}$$

[1] *Toroidal* modes can also be useful, in the sense that observation of excitations of them *cannot* be attributed to GWs, and can thus be used as *vetoes* on signals.

As has already become clear from the previous analysis, GW energy will be be transferred *exclusivey* to the monopole and/or quadrupole *spheroidal* modes of the antenna. In other words, cross sections will vanish for *all modes but these*. In reference [6] I have given a proof that

$$\sigma_{\rm abs}(\omega_{n0}) = K_S(\aleph)\,\frac{GMv_t^2}{c^3}\,(k_{n0}a_n)^2 \tag{15}$$

$$\sigma_{\rm abs}(\omega_{n2}) = K_Q(\aleph)\,\frac{GMv_t^2}{c^3}\,(k_{n2}b_n)^2 \tag{16}$$

where $\omega_{n0}$ is the $n$-th monopole harmonic, and $\omega_{n2}$ the $n$-th quadrupole harmonic; $K_S(\aleph)$ and $K_Q(\aleph)$ are values which are calculated based on a given assumption about which is the theory correctly describing GW physics, such theory being symbolically designated by $\aleph$. For example, if General Relativity is the correct theory then

$$\aleph = {\rm GR} \Rightarrow \begin{cases} K_S(\aleph) = 0 \\ K_Q(\aleph) = \frac{16\,\pi^2}{15} \end{cases} \tag{17}$$

whilst, if it is Brans-Dicke theory [8],

$$\aleph = {\rm BD} \Rightarrow \begin{cases} K_S(\aleph) = \frac{8\,\pi^2}{9}\,(3+2\Omega)^{-2}\,k\,\left[1+\frac{k\Omega}{(3+2\Omega)^2}\right]^{-1} \\ K_Q(\aleph) = \frac{16\,\pi^2}{15}\,\left[1+\frac{1}{6}\,(3+2\Omega)^{-2}\,k\right]\left[1+\frac{k\Omega}{(3+2\Omega)^2}\right]^{-1} \end{cases} \tag{18}$$

where $\Omega$ is the usual Brans-Dicke parameter $\omega$, renamed here to avoid confusion with frequencies, and $k$ is a dimensionless number of order 1 [9].

An interesting conclusion of equations (15) and (16) is that cross sections *scale* for higher harmonics *independently* of the underlying theory of gravity. In particular, it is seen that its value for the *second* quadrupole mode is only 2.61 times less than for the first, which means a spherical GW detector can be advantageously used for GW sensing at *two* quadrupole frequencies, in sharp contrast with the commonly used Weber bars, which can only sense one frequency. This remarkable fact was revealed for the first time, assuming General Relativity, in [10].

**3. The motion sensing problem.** So much, summarily, for the sphere's *potentialities*. But a major *practical* problem is this: how do we actually *sense* the eventual excitations (by a GW or other causes) of the sphere's vibrations, and convert them into valuable information?

Currently operating bars accomplish this by using a *resonant transducer* attached to one of its end faces. A resonant transducer is a device built in such a way that it has a fundamental vibraiton frequency *accurately tuned* to the frequency of the bar at which motions are to be sensed —always the first longitudinal mode of the cylinder. Such a resonator is *much less massive* than the bar, so resonant transfer of energy occurs between the two oscillating bodies (bar and sensor) which results in an *enhanced amplitude* motion of the attached resonator. This way one obtains a *mechanical amplification* system for the extremely tiny GW induced vibrations of the sphere. The transducer motions are thereafter converted into an electrical signal by means of suitable circuits [3], then further amplified and transferred to a readout device or disk archive.

The idea of using resonators for motion sensing in a spherical antenna has been considered a good one too by the experimentalists, who have recently set up a small scale prototype detector in the University of Louisiana [11, 12]. Taking full advantage of the sphere's capabilities means, however, that *several* resonators must be attached to its surface rather than just one, as in bars. This is due to the *degeneracy* of the sphere's vibration eigenmodes, which makes it possible to sense *all* the GW amplitudes of a given $l$ at a *unique* frequency $\omega_{nl}$[2]. More specifically, a minimum set of 5 are required to see the 5 quadrupole amplitudes $g^{(m)}(t)$, while only one is needed to see the monopole amplitude $g^{(S)}(t)$.

We begin by setting up the general system of equations which has to be solved in relation to this problem. We assume that a set of $N$ *identical* resonators of mass

$$M_{\rm resonator} = \eta M \ , \qquad \eta \ll 1 \tag{19}$$

are attached to the sphere's surface at locations $\mathbf{x}_a$ ($|\mathbf{x}_a| = R$ for all $a = 1,...,N$); we shall further assume that these resonators have resonance frequency $\Omega$, and that they only couple to *radial* displacements of the sphere's surface. Under these circumstances the equations of motion of the complete system are

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2} - \mu\nabla^2\mathbf{u} - (\lambda+\mu)\,\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f}(\mathbf{x},t) \ + \\ + \ \eta M\Omega^2 \sum_{a=1}^{N} \delta^{(3)}(\mathbf{x}-\mathbf{x}_a)\ [\xi_a(t) - \mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)]\ \mathbf{n}_a \tag{20}$$

$$\ddot{\xi}_a(t) = -\Omega^2\ [\xi_a(t) - \mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)] \tag{21}$$

where $\mathbf{n}_a \equiv \mathbf{x}_a/R$ is the outward normal at point $\mathbf{x}_a$, and $\delta^{(3)}(\mathbf{x})$ is the usual Dirac density distribution. The meaning of the above is rather transparent: in the right hand side of (20) the force density caused by the attachement of resonators has been added to the GW force density $\mathbf{f}(\mathbf{x},t)$, while (21) is the equation of motion of each resonator. We have assumed that the latter behave like simple, non-damped harmonic oscillators, which is a fairly good approximation as we shall shortly see.

Eqs. (20) and (21) above constitute a relatively complicated system of coupled differential equations, whose solution, even if only *formal*, is not possible to write down. Thankfully, though, we are only interested in practice in the $N$ *measurable* quantities

$$q_a(t) \equiv \xi_a(t) - \mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t) \ \ , \qquad a = 1,\ldots,N \tag{22}$$

rather than in the *complete* solution. Even so, things are not easy. I shall attempt to give the reader a flavour of the main ideas and consequences which follow from the rigorous mathematical treatment of the problem.

*3.1. The frequency response of the coupled system.* Like before, a Green function formalism is the suitable tool to address the present situation. It should be noted, however, that the right hand side of (20) *contains the unknowns* $\mathbf{u}(\mathbf{x},t)$, and this results, combined with (21), into a system of integro-differential equations. As it turns out, it is possible to

[2]Actually $l = 0$ or $l = 2$, as discussed earlier.

reduce such a compliated system to a *linear* system of *algebraic* equations in the *Laplace transforms* of unknowns and sources. This is the system:

$$\hat{q}_a(s) = -\frac{s^2}{s^2+\Omega^2}\sum_\alpha\left\{\sum_{b=1}^N\left[\delta_{ab}+\eta\,\frac{s^2}{s^2+\Omega^2}\sum_\nu\frac{\Omega^2}{s^2+\omega_\nu^2}\,\chi_{ab}^{(\nu)}\right]^{-1}\times\right.$$
$$\left.\times\left[\sum_\mu\frac{B_{\mu,b}^{(\alpha)}}{s^2+\omega_\mu^2}\right]\right\}\hat{g}^{(\alpha)}(s) \qquad (23)$$

where $\mu,\nu$ are multiple indices $\{nlm\}$ each, and $B_{\mu,b}^{(\alpha)}$ is a term proportional to overlapping integrals of the tidal form factors —$a_n$ or $b_n$, see (13 above). $\chi_{ab}^{(\nu)}$ is a *diadic* product of sphere wavefunctions at locations $a$ and $b$, and for a *perfect sphere* is given by

$$\chi_{ab}^{(\nu)} \equiv \chi_{ab}^{(nl)} = \frac{2l+1}{4\pi}\,A_{nl}^2(R)\,P_l(\mathbf{n}_a\cdot\mathbf{n}_b) \qquad (24)$$

with $P_l$ a Legendre polynomial and $A_{nl}(R)$ a *radial function* coefficient —see [6]. Finally, $\hat{g}^{(\alpha)}(s)$ are the Laplace transforms of the corresponding GW Riemann tensor amplitudes. Equations (23) are somewhat complicated, but this is a result of their generality. We now examine their consequences in specific cases of interest.

First of all we note the following general fact: the presence of an *inverse* matrix in equation (23) indicates that its *poles*, relative to the Laplace variable $s$, will give us the *resonance frequencies* of the coupled system {sphere + resonators}, while the *residues* at those poles will give us the corresponding *amplitudes*. In a practical situation the resonators will be tuned to one of the sphere's spheroidal eigenfrequencies, i.e.,

$$\Omega = \omega_{nl}\ , \qquad \omega_{nl}\ \text{fixed} \qquad (25)$$

and, more precisely, this will be either a quadrupole or a monopole frequency.

The *exact* solution to equation (23) cannot, once again, be found. We shall thus resort to approximate methods, based on the assumption that the dimensionless parameter $\eta$ is a small number —cf. (19). Although it is not strictly necessary, the further assumption that there are no other sphere's resonant frequencies in the vicinity of the chosen $\Omega=\omega_{nl}$ does simplify things, and will also be made here[3].

The system resonances are found to be at

$$\omega_{a\pm}^2 = \omega_{nl}^2\left(1\pm\sqrt{\frac{2l+1}{4\pi}}\,A_{nl}(R)\,\zeta_a\ \eta^{1/2}\right)+O(\eta) \qquad (26)$$

where $O(\eta)$ stands for terms of order $\eta$ or higher, and $\zeta_a^2$ is one of the eigenvalues of the $N{\times}N$ matrix $P_l(\mathbf{n}_a\cdot\mathbf{n}_b)$. Equation (26) thus says that the system frequencies, after resonators have been added, consist of $N$ *symmetric doublets* around the original $\omega_{nl}$. It so happens, however, that $P_l(\mathbf{n}_a\cdot\mathbf{n}_b)$ has $N{-}2l{-}1$ *identically null* eigenvalues if $N>2l+1$, which means that an *unshifted* frequency will survive whenever the latter inequality holds, but the amplitude of the modes at this frequency is *smaller*, by a

[3]This is definitely the case for the fundamental quadrupole mode of a sphere, and also for the first monopole mode; things are a little different for the second quadrupole mode, but I do not wish to enter into so much technical detail in this paper.

factor of order $\eta^{1/2}$, than the others'. More specifically, we are going to have a *single doublet* if we tune the resonators to a monopole sphere's resonance, or 5 *doublets* if a quadrupole frequency is selected. In the latter case, however, certain doublets may fall on top of one another, thereby reducing their actual number to fewer than 5; as we shall soon see, this happens when particular symmetries in the resonator distribution occur.

The *amplitudes* of the modes associated to the above frequency doublets are evaluated, as already mentioned, by the calculus of residues. The following is found:

$$\hat{q}_a(s) = \eta^{-1/2} \sum_\alpha \Lambda_a^{(\alpha)}(s;n,l)\ \hat{g}^{(\alpha)}(s) + O(\eta^0) \tag{27}$$

Two major features are displayed by this equation: first, the amplification factor $\eta^{-1/2}$ shows that the resonator amplitudes are enhanced relative to the sphere's due to *resonant energy transfer* between the large mass of the sphere and the small masses of the resonators; and second, there is a *pattern matrix* $\Lambda_a^{(\alpha)}(s;n,l)$ relating the system's response to the GW excitations $\hat{g}^{(\alpha)}(s)$. This matrix depends both on the selected frequency chosen for tuning the resonators and on the *geometry* of the layout. We come now to a more detailed discussion of these matters.

*3.2. Resonator layouts.* The following property of $\Lambda_a^{(\alpha)}(s;n,l)$ also holds: it vanishes unless $l$ is equal to the corresponding $l$ in the tuning frequency $\omega_{nl}$; in other words, if we tune our resonators to e.g. a *quadrupole* frequency ($l=2$) then their motion only couples weakly to the $l\neq 2$ sphere modes, actually a factor at least $\eta^{1/2}$ less intensely than to the quadrupole modes. This means that the resonators enable the observation of GWs in a relatively narrow bandwidth around the tuning frequency.

Thus, if we want to see (possible) *monopole* gravitational radiation then we need *one* resonator tuned to a sphere's monopole frequency —more than one is also OK, only it will in principle provide redundant information. Since monopole oscillations are spherically symmetric, it is irrelevant where we locate our resonator (or resonators); furthermore, the frequency doublets, see eq. (26) above, reduce to a single one and, if there are more than one sensors, a weakly coupled, unshifted frequency at $\omega_{n0}$. Sensing monopole GWs with a system like ours is thus straightforward, and we skip going into more technical detail.

Our truly interesting concern is *quadrupole* radiation sensing. Here we find that the pattern matrix $\Lambda_a^{(\alpha)}(s;n,2)$ can be explicitely calculated to give (after rather laborious algebra)

$$\Lambda_a^{(m)}(s;n,2) = (-1)^N \sqrt{\frac{4\pi}{5}}\, b_n \ \times \\ \times \sum_{b=1}^{N} \left\{ \sum_{\zeta_c\neq 0} \frac{1}{2} \left[ \left(s^2+\omega_{c+}^2\right)^{-1} - \left(s^2+\omega_{c-}^2\right)^{-1} \right] \frac{v_a^{(c)} v_b^{(c)*}}{\zeta_c} \right\} Y_{2m}(\theta_b,\varphi_b) \tag{28}$$

where $b_n$ is an overlapping integral ($b_1/R=0.328$, $b_2/R=0.106$ for the lowest modes —cf. [9]), and $v_a^{(c)}$ is the eigenvector of the matrix $P_l(\mathbf{n}_a{\cdot}\mathbf{n}_b)$ corresponding to the eigenvalue $\zeta_c^2$.

The above equations are useful for *any* resonator distribution on the sphere surface. The question naturally arises as to whether there are preferred ones. Let us have a look at this in a bit more detail.

A recent proposal by the *LSU* people, called *TIGA* (for *T*runcated *I*cosahedral *G*ravitational *A*ntenna) [11, 12], consists in a set of 6 resonators attached to the centres of the *pentagonal* faces of a *truncated icosahedron.* This highly symmetric layout can be seen to result in a *single, quintuply degenerate doublet* of frequencies, plus a (weakly coupled) non-shifted frequency[4]. Degeneracy in this layout is an indication of its *isotropic* sensitivity: no matter where the GW comes from, *all* the energy absorbed by the detector will be deposited into oscillations at the single frequency doublet. As also shown by Johnson and Merkowitz, it is possible to make suitable linear combinations of the *six* resonator responses $\hat{q}_a(s)$, which they call *mode channels*, which are direct readouts of the *five* quadrupole GW amplitudes $\hat{g}^{(m)}(s)$ at the single frequency of the split doublet, whereby deconvolution of signal and incidence direction [6, 16, 15] can be readily accomplished.

Further investigation of the consequences of equation (28) has shown us that it is possible to think of alternatives to the *TIGA* layout which may be even advantageous in certain respects. We have for example proved [17] that *any* 5 resonator distribution with an axis of *pentagonal* symmetry results in a frequency multiplet consisting in *one* non-degenerate and *two* doubly degenerate frequency modes. Remarkably, GW amplitudes *selectively* couple to these modes, so that different *wave* modes are seen at different *detector* modes. More precisely, the system response is given by

$$\begin{aligned}\hat{q}_a(s) = -\eta^{-1/2}\sqrt{\frac{4\pi}{5}}\,b_n\Bigg\{\; & \frac{1}{2\zeta_0}\left[\left(s^2+\omega_{0+}^2\right)^{-1}-\left(s^2+\omega_{0-}^2\right)^{-1}\right]Y_{20}(\theta_a,\varphi_a)\,\hat{g}^{(0)}(s)\\ & +\frac{1}{2\zeta_1}\left[\left(s^2+\omega_{1+}^2\right)^{-1}-\left(s^2+\omega_{1-}^2\right)^{-1}\right]\\ & \qquad\times\left[Y_{21}(\theta_a,\varphi_a)\,\hat{g}^{(1)}(s)+Y_{2-1}(\theta_a,\varphi_a)\,\hat{g}^{(-1)}(s)\right]\\ & +\frac{1}{2\zeta_2}\left[\left(s^2+\omega_{2+}^2\right)^{-1}-\left(s^2+\omega_{2-}^2\right)^{-1}\right]\\ & \qquad\times\left[Y_{22}(\theta_a,\varphi_a)\,\hat{g}^{(2)}(s)+Y_{2-2}(\theta_a,\varphi_a)\,\hat{g}^{(-2)}(s)\right]\Bigg\}\end{aligned} \qquad (29)$$

where $\zeta_0^2$ is the non-degenerate eigenvalue, and $\zeta_1^2$ and $\zeta_2^2$ are the doubly degenerate eigenvalues; correspondingly, $\omega_{0\pm}$, $\omega_{1\pm}$ and $\omega_{2\pm}$ are the associated frequency doublets. Equation (29) clearly shows the *selective* coupling between the GW amplitudes and the system response $\hat{q}_a(s)$ alluded above. Much like in the *TIGA* distribution, it is also possible to define here the following *mode channels*:

$$y^{(m)}(s) = \frac{2\,\eta^{1/2}}{b_n\,\zeta_m}\sum_{a=1}^{5} Y_{2m}^*(\theta_a,\varphi_a)\,\hat{q}_a(s)\ , \qquad m=-2,\ldots,2 \qquad (30)$$

which yield five quantities directly proportional to the five GW quadrupole amplitudes $\hat{g}^{(m)}(s)$ —this can be easily seen to be a consequence of the symmetry properties of

[4]Specifically, $\zeta_c = 6/5$ for $c = 1,\ldots,5$, and $\zeta_6 = 0$.

the spherical harmonics in a pentagonal distribution. Such configuration can now be advantageously implemented in a GW detector by the following argument.

Imagine we are so lucky that we *know* a GW arrives down the resonators' symmetry axis. Now, evidence of excitation of the $\omega_{0\pm}$ or $\omega_{1\pm}$ frequency components for example is a *strong veto* on General Relativity, as this theory predicts the excitation of *only* the $\pm 2$ modes. It is unrealistic to think of such good fortune in the first place, but a more likely practical situation can also be handled advantageously. Indeed, the fact that different *wave modes* couple to different *detector frequencies* is a very powerful *discrimination* tool; at the same time, the frequency span of the multiplet in a forseeable GW antenna will only be a few tens of Hz, so the *signal spectrum* is likely to be constant over such span, and hence proposed deconvolution techniques [6, 16] comfortably applicable.

In Figure 2 we give a graphical representation of what might be considered an interesting *practical* implementation of a GW antenna based on the just discussed pentagonal transducer layout. It relies on the philosophy of having a *polyhedron*, rather than a sphere, as a suitable approach to the GW spherical antenna, for ease of instrumentation attachement and manipulation [13]; the choice was made having in mind that the polyhedron should be as spherical as possible, whilst having at the same time axes of pentagonal sym-

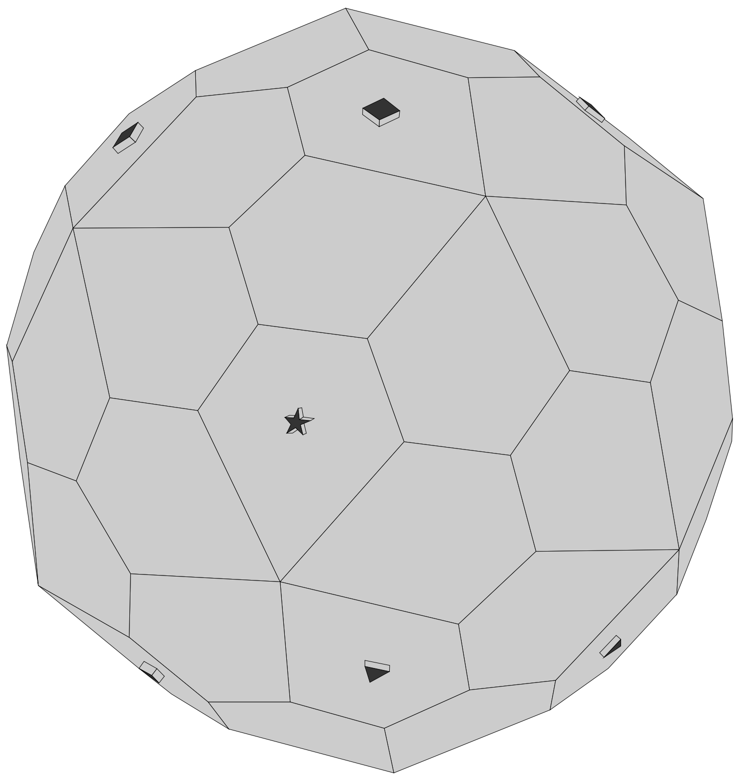


Figure 2. Our proposed polyhedric antenna. Resonators are marked as follows: a *square* for the first quadrupole frequency, a *triangle* for the second, and a *star* for the monopole.

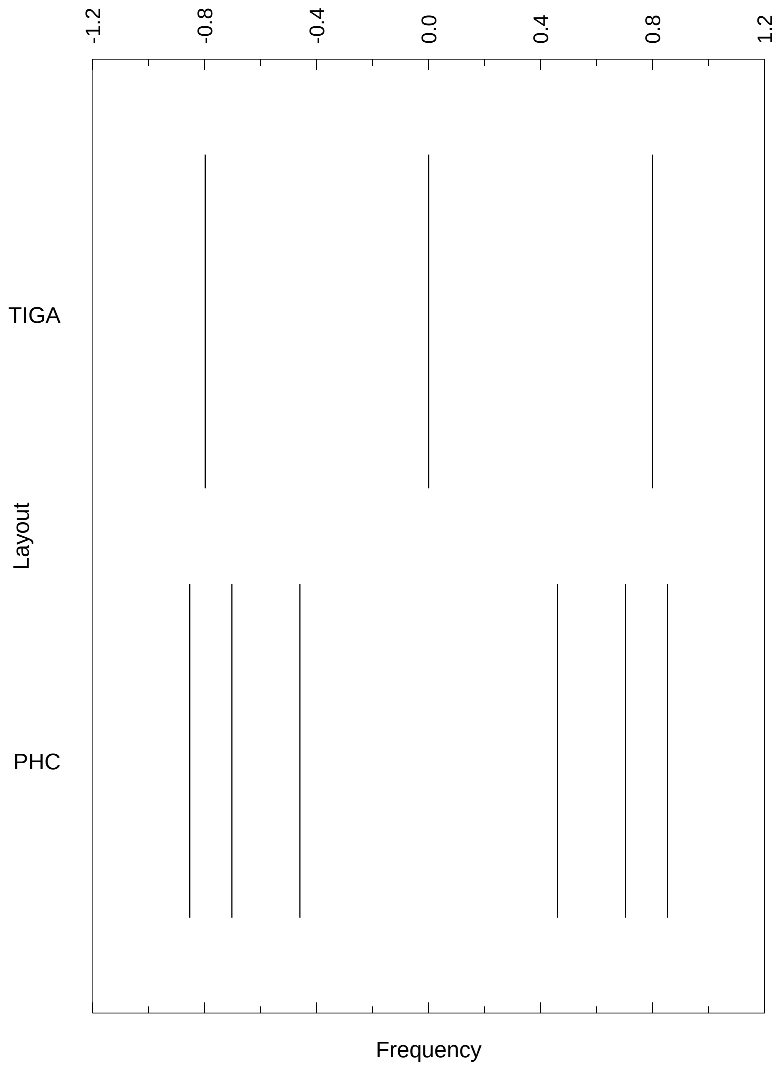



Figure 3. The resonator multiplets in a *TIGA* distribution (above) and in our proposed *PHC* distribution (below). They correspond to the first qudrupole frequency of two identical spheres and identical resonators. The horizontal scale of the plot is the resonance frequency $\omega_{02}$, enhanced by a factor of $\eta^{-1/2}$.

metry. Our polyhedron is called *pentagonal hexacontahedron* [18], has sixty identical faces (irregular pentagons), and is considerably more spherical than the TI [19]. An *inscribed* sphere exists which is tangent to every face at a point, to which a resonator could eventually be linked, thereby accomplishing a perfect simulation of a spherical distribution, i.e., all transducers *equidistant* from the centre.

In Figure 2 we also indicate proposed resonator locations —see caption for details. For example, for the *first* quadrupole resonance $\Omega = \omega_{02}$, it is found that [20]

$$\begin{aligned} \omega_{0\pm} &= \omega_{02}\ \left(1 \pm 0.5755\,\eta^{1/2}\right) \\ \omega_{1\pm} &= \omega_{02}\ \left(1 \pm 0.8787\,\eta^{1/2}\right) \\ \omega_{2\pm} &= \omega_{02}\ \left(1 \pm 1.0668\,\eta^{1/2}\right) \end{aligned} \tag{31}$$

In Figure 3 we display a graphical representation of this frequency multiplet together with the *TIGA* multiplet, for comparison. Of course, the assumption has been made that resonators and sphere are identical in bothe cases. As can be seen, the frequency span of the multiplets is practically the same, with a richer response spectrum in our proposed polyhedron.

In addition to the set of 5 transducers tuned to $\omega_{02}$, *another set* of five resonators, tuned to the *second* quadrupole frequency, $\omega_{12}$, and located symmetrically in the 'southern hemisphere', could be attached to the sphere, too. An *eleventh* resonator, tuned to the *first monopole* frequency, $\omega_{00}$, and placed at an arbitrary position, could finally be added as well. Such an altogether 11 transducer configuration would take advantage of the large sphere GW absorption cross section at its second quadrupole mode [10], and would therefore constitute a rather *complete GW detector* of its own. Also, it just requires 5 transducers rather than 6 for each quadrupole mode sensed.

**4. The suspended sphere.** Theory so far has been made on the rather idealised assumption that the GW antenna maintains a perfectly spherical symmetry. This however is not quite what one will find in actual practice, as any earth based detector will have to be *suspended* in a suitable platform in order to isolate it from local sources of noise. But no matter how such suspension system is implemented in real life, it will have as an unavoidable consequence the *breaking of spherical symmetry*, thence (at least partial) removal of eigenfrequency degeneracy occurring. Perturbative calculations can be made to quantitatively solve the problem, but I would not like to go into any details of those procedures here. Rather, I will only take up the case when suspension is maintained by a wedge at the end of a rod which passes through a diametral bore practiced across the sphere. This is the way the *LSU* people have implemented their prototype *TIGA*, and on which reliable experimental data are available [14]. The purpose of this section is only to persuade the reader of the power of the theoretical analysis presented above, as backed by the remarkable matching of its predictions to real measurements.

The nature of the suspension by a diametral bore is such that the otherwise five-fold degenerate quadrupole sphere frequencies $\omega_{n2}$ split up into 5 non-degenerate frequencies $\omega_{n2m}$, $m = -2,...,2$, though the difference between members with equal $|m|$ is rather small due to the cylindrical symmetry maintained by the bore. This means that a somewhat more complicated analysis of the general equation (23) needs to be done in this case than had been done earlier. It however turns out to be feasible thanks to the fact that the amount of *relative* frequency splitting caused by symmetry breaking *is of order* $\eta^{1/2}$, and this enables the application of a perturbative approach with marked similarities with the one followed in the degenerate case. I would not like to go into the technical details of how this is done, but only report on its final results.

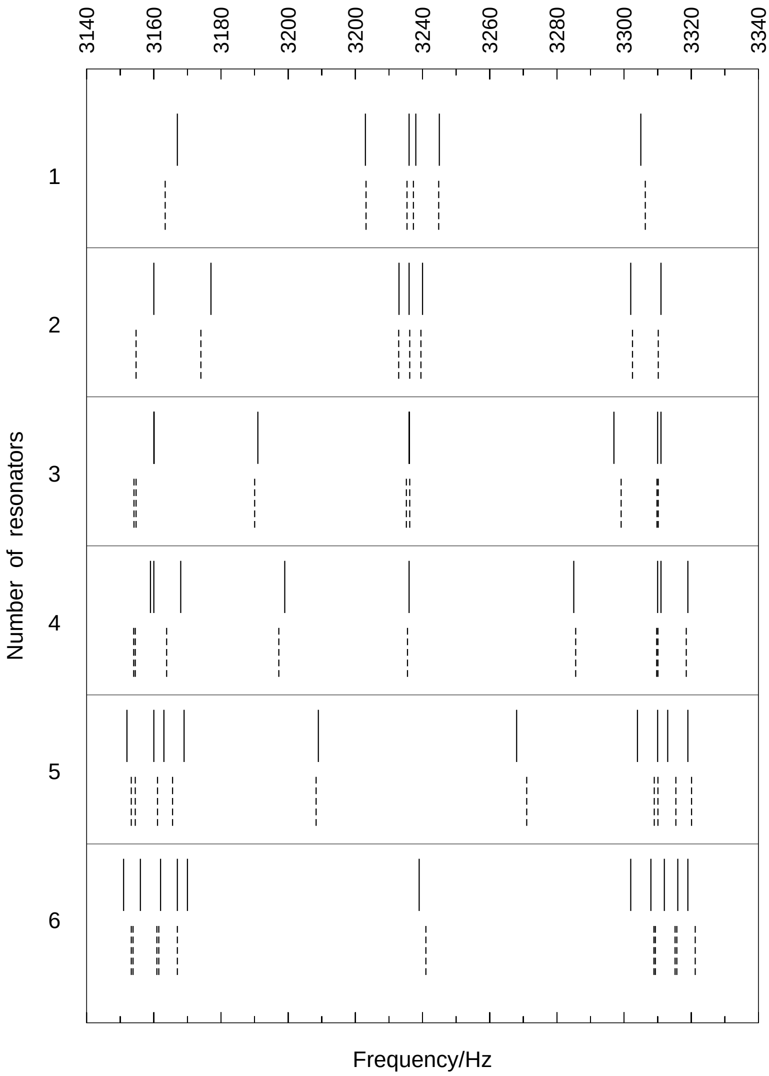



Figure 4. The frequency multiplets of a *suspended* *TIGA*. Solid slashes correspond to *measured* frequencies [14], and broken slashes to *theoretically* calculated ones. As can be seen, matching between both sets is remarkably good for all six resonator distributions.

These are summarised in Figure 4, where we see a graphical representation of the frequency spectrum of a sphere with resonators in the vicinity of the first quadrupole resonance of the former. The graphic displays the result of the theoretical calculations for system parameters corresponding to real *TIGA* prototype values, as given in [14], along with the values of the actually *measured* frequencies. As we see, coincidence between theory and practice is outstanding for all the resonator configurations studied: the worst discrepancy is only a few parts in $10^4$. I consider this a strong indication that the theoretical model described above is the correct one.

**5. Conclusion.** I have presented here a summary of the main aspects of a rather rigorous *theoretical* model of spherical GW antenna, both regarding its sensitivity parameters (cross sections) and the problem of motion sensing by means of a set of radial resonators attached to its surface. The development of such model contributes, I believe, to a more thorough understanding of the whole problem of this kind of GW detector than was available to date.

Beyond this, however, a new propoposal for transducer layout has emerged out of our theoretical considerations which I think may be worth considering as an interesting "brother" of the *TIGA* layout, for several reasons. For example, the fact that different *mode channels* see corresponding GW amplitudes at *different* member frequencies of the detector multiplet can be used to great advantage in any signal deconvolution techniques, and this is of much theoretical value for GW physics as such. Also, the *pentagonal hexacontahedron* is a more spherical polyhedron than the truncated icosahedron, and has more *identical* faces which enable mounting of more resonators at positions *equidistant* from its centre; as already argued, a rather complete GW antenna can be accomplished on the basis of this intrinsic capability. The fact that *fewer* sensors per quadrupole mode are needed in our proposed detector should also be considered a potential simplification of the scheme.

It could be objected at this point that the *TIGA* has the unpaired virtue of having *isotropic* sensitivity. While this is true in principle, one may not forget that such symmetry will be *broken* in any earth based implementation of a spherical antenna due to suspension requirements. If our proposed resonator set should be made to have the suspension axis as its own axis of pentagonal symmetry, then optimum benefit would be naturally obtained from the system's inherent features.

A major problem I have not touched upon here is that of *noise*. While this deserves a thorough analysis of its own —currently underway— I can presently forsee no signal to noise ratio penalty in this configuration relative to *TIGA* or others. The reason for this *conjecture* is that all mode channels are combinations of *all* five resonators' amplitudes, which in turn extract energy for their excitation from the incoming GW. Now, if the reasonable assumption is made that the spectrum of that GW is *flat* over the relatively narrow frequency span of the detector multiplet (a few tens of Hz in e.g. a 3 metre diameter alluminum sphere), then the *same* amount of energy will be available at each frequency, and hence the same signal to noise ratio at every member of the multiplet. This is however a somewhat "hand waving" argument which needs to be more thoroughly substantiated.

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[19] For example, its volume is 1.057 times that of its inscribed sphere, while the truncated icosahedron is 1.153 times less voluminous than its circumscribed sphere; this means sphericity is a factor of almost 3 better for our polyhedron.

[20] The chosen distribution has the property that the frequency spacing between members of the associated multiplet is the most *even* compatible with the polyhedron face orientations.