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CRITICAL PHENOMENA IN GRAVITATIONAL COLLAPSE

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Abstract. A mini-introduction to critical phenomena in gravitational collapse is combined with a more detailed discussion of how gravity regularizes the "critical spacetimes" that dominate these phenomena.

1. Critical phenomena in gravitational collapse. Initial data for general relativity may or may not form a black hole. In several ways one might compare this to a phase transition, and there is a "critical surface" in superspace (the phase space of GR) separating the two kinds of initial data. Choptuik [1] has explored this surface in a systematic way. For simplicity he took a massless scalar field as his matter model, and allowed only spherically symmetric configurations. (The numerical calculations were still pioneering work, because they involve length scales spanning many orders of magnitude.) To explore the infinite-dimensional phase space, he evolved initial data from a number of one-parameter families of data crossing the critical surface. Let us call that parameter generically p. By a bisection search, Choptuik found a critical value p_* for each family, such that data with $p > p_*$ form a black hole, but not data with $p < p_*$. He then discovered two unforeseen effects:

Scaling: Near the critical surface, on the black-hole side of it ($p > p_*$), where the mass of the black hole final state is small (compared to, for example, the ADM mass), it scales as

$$M \simeq C(p - p_*)^\gamma, \tag{1}$$

where the overall factor C depends on the family, but the "critical exponent" γ is universal between families. For the scalar field matter, $\gamma \simeq 0.37$. I should stress that this expression is invariant under redefinitions of $p \to \bar{p}(p)$ such that $\bar{p}(p)$ is differentiable, with $d\bar{p}/dp \neq 0$ at p_*: To leading order, only C changes under such redefinitions. To clarify this, and to

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avoid speaking in terms of one-parameter families altogether, one can formally introduce coordinates (x_0, x_i) on superspace such that the coordinate surface $x_0 = 0$ is the critical surface. Then, for small positive x_0, the black hole mass would be of the form

$$M \simeq f(x_i)\, x_0^\gamma, \tag{2}$$

and again this form is invariant under coordinate diffeomorphisms such that $x_0 = 0$ remains the critical surface.

Universality: Near the critical surface, on either side of it ($p \simeq p_*$), initial data evolve towards an intermediate asymptotic solution (which I'll call Z_*), which is again universal with respect to initial data. (Clearly this solution cannot be a full-blown attractor, because all data must leave it eventually towards forming either a black hole or towards dispersion. But it is an attractor of co-dimension one, or intermediate asymptotic.) This solution shows self-similarity. In the case of scalar field matter, this self-similarity is discrete, showing up as an "**echoing**" in the logarithm of spacetime scale, with period $\Delta \simeq 3.44$.

The matter model of Choptuik is special in that, even coupled to general relativity, it has no intrinsic scale. (This is equivalent to saying that in geometric units, $c = G = 1$, the action has no dimensionful parameters.) Therefore, from dimensional analysis alone, there can no be no static, star-like solutions, and hence no minimum black hole mass. Generic matter has both an intrinsic scale (or several), and star-like solutions. Astrophysical black holes thus have a minimum mass given by the Chandrasekhar mass. To make infinitesimally small black holes from ordinary matter, one would have to use initial conditions (for example rapid implosion) not arising in astrophysics. The important point is that for any matter there is at least **some** region of superspace where critical phenomena arise. Nevertheless, the interest of critical phenomena does not lie in astrophysics, but in the dynamics of GR (with or without matter). Recent work by Choptuik, Bizon and Chmaj [2] has clarified the role of matter scales. Investigating the spherical collapse of Einstein-Yang-Mills, they find two regimes: one with the usual critical phenomena dominated by a self-similar intermediate attractor ("second-order phase transition"), and one dominated by the Bartnik-McKinnon solution, which is a finite mass, static intermediate attractor, and therefore having a mass gap ("first-order phase transition").

Critical phenomena have also been found in two one-parameter families of matter models in spherical symmetry (not to be confused with one-parameter families of data): The first family is that of the perfect fluids with $p = k\rho$, k a constant [3, 4, 5]. The other family is that of the constant-curvature, two-dimensional sigma models, characterized by the dimensionless curvature parameter κ [6]. (This family contains the free complex scalar field [7], an inflaton-dilaton model [8], and an axion-dilaton model [9] as special cases.) It is known for the latter family, and likely for the second, that at some value of the parameter κ (and perhaps k) the critical solution switches over from continuous to discrete self-similarity. Moreover, the critical exponent γ depends on the parameter κ or k.

Historically, the second occurrence of critical phenomena was found in the collapse of axisymmetric gravitational waves (time-symmetric initial data, and hence with zero angular momentum) [10]. Because of the much greater numerical difficulty, there are fewer experimental data, and regrettably, there has been no follow-up work so far. Axisymmetric waves are highly interesting in two aspects: They go beyond spherical symmetry, and they are vacuum data. It would also be extremely interesting to see what happens for initial data with angular momentum, because black holes can of course have angular momentum, but it must be smaller than the mass.

2. Self-similarity in GR. The critical solution for each matter model has the property of being self-similar. In hindsight this is simply the form scale-invariance takes in the problem. Self-similarity however takes the novel form of echoing, or discrete self-similarity, in some models, notably Choptuik's scalar field and axisymmetric gravitational waves. Some definitions are necessary:

(Continuous) self-similarity (CSS) (or homotheticity) in a relativistic context [11] is the presence of a vector field χ such that

$$\mathcal{L}_\chi g_{ab} = 2 g_{ab}, \tag{3}$$

where $\mathcal{L}_\chi$ denotes the Lie derivative. In discrete self-similarity (DSS) there exist a diffeomorphism ϕ and a real constant Δ such that, for any integer n,

$$\left(\phi_*\right)^n g_{ab} = e^{2n\Delta} g_{ab}, \tag{4}$$

where ϕ_* is the pull-back of ϕ.

To see what DSS looks like in coordinate terms, we introduce coordinates (τ, x^α), such that if a point p has coordinates (τ, x^α), its image $\phi(p)$ has coordinates $(\tau+\Delta, x^\alpha)$. One can verify that DSS in these coordinates is equivalent to

$$g_{\mu\nu}(\tau, x^\alpha) = e^{2\tau} \tilde{g}_{\mu\nu}(\tau, x^\alpha), \quad \text{where} \quad \tilde{g}_{\mu\nu}(\tau, x^\alpha) = \tilde{g}_{\mu\nu}(\tau+\Delta, x^\alpha) \tag{5}$$

In other words, the DSS acts as a discrete isomorphism on the rescaled metric $\tilde{g}_{\mu\nu}$. τ is intuitively speaking the logarithm of spacetime scale.

In order to clarify the connection between CSS and DSS, one may define a vector field $\chi \equiv \partial/\partial\tau$, although there is no unique χ associated with a given ϕ. The discrete diffeomorphism ϕ is then realized as the Lie dragging along χ by a distance Δ. Clearly, CSS corresponds to DSS for infinitesimally small Δ, and hence for all Δ, and is in this sense a degenerate case of DSS. In this limit, χ becomes unique.

3. Universality and the critical solution. The critical solution dominating critical phenomena for a given matter model (and perhaps choice of symmetry, such as spherical symmetry), has two essential properties. First, it must be self-similar, either CSS or DSS. Secondly, it must have exactly one unstable mode. There are examples of self-similar solutions with more than one unstable mode, with exactly one (i.e. genuine critical solutions), but none with none: the latter would constitute a violation of cosmic censorship in the strongest possible sense, that of a naked singularity arising from generic initial data. The presence of exactly one unstable mode, on the other hand, constitutes a dynamical explanation of the universality of the critical exponent. In this explanation, near-critical data $p \simeq p_*$ (data near the critical surface) are precisely those in which the one unstable mode is initially small. Skimming along the critical surface, they are attracted towards the critical solution, which is either a fixed point (CSS) or a limit cycle

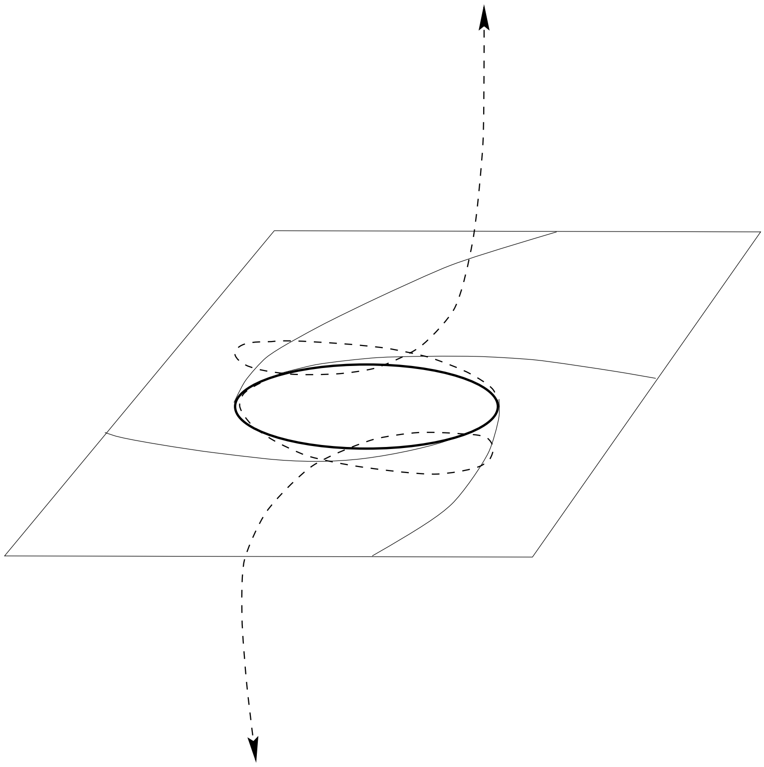

Figure 1: The phase space picture for discrete self-similarity. The plane represents the critical surface. (In reality this is a hypersurface of co-dimension one in an infinite-dimensional space.) The circle (fat unbroken line) is the limit cycle representing the critical solution. The thin unbroken curves are spacetimes attracted to it. The dashed curves are spacetimes repelled from it. There are two families of such curves, labeled by one periodic parameter, one forming a black hole, the other dispersing to infinity. Only one member of each family is shown.

(DSS), until the growing mode eventually takes over and ejects the trajectory, either towards black hole formation or towards dispersion, in a unique manner. The solution has forgotten from which initial data it came, up to one parameter (roughly speaking, the number of echos, or the time spent on the intermediate asymptotic), which depends on the initial amplitude of the growing mode and which will eventually determine the black hole mass. Fig. 1 illustrates this behavior for DSS.

A given spacetime does not correspond to a unique trajectory in superspace, because it can be sliced in different ways. In reverse, a point in superspace corresponds to a unique spacetime, but not to a unique trajectory. In other words, there is no preferred time-evolution flow on superspace, and therefore Fig. 1 is true only for a fixed slicing condition, something extraneous to GR. A possible solution is to demand that the preferred time evolution (choice of lapse and shift) should correspond to a "renormalisation group flow", or change of scale. This is the case for example if one takes τ as defined above as the time variable. The analogy with critical phenomena in statistical mechanics would then be deeper: superspace corresponds to the space of Hamiltonians, and the preferred time evolution to the renormalisation group flow.

In constructing the critical solution one has to proceed in two steps. Imposing self-similarity together with certain regularity conditions, one obtains a non-linear hyperbolic eigenvalue problem, which may have a solution. If it does, in a second step one has to calculate the spectrum of its linear perturbations and check that there is exactly one growing mode.

In the following I restrict myself to spherical symmetry. The regularity conditions just mentioned are imposed in two places. One is the center of spherical symmetry. There one imposes the absence of a conical singularity in the 3-metric, and that all fields be either even or odd in r, depending on their being vector or scalar under rotations. The other set of regularity conditions arises at, roughly speaking $r = -t$, and needs to be discussed in more detail.

The general spherically symmetric solution of the wave equation in flat space is

$$\phi(r,t) = r^{-1}\,[f(r-t) + g(r+t)], \tag{6}$$

with f and g arbitrary functions. Furthermore, ϕ is DSS (in flat space) if $\phi(r,t) = \phi(e^{\Delta}r, e^{\Delta}t)$ for some Δ. One easily derives that the general DSS solution in flat space is of the form

$$\phi(r,t) = (1+z^{-1})F[\tau + \ln(1+z)] + (1-z^{-1})G[\tau + \ln(1-z)], \quad \tau \equiv \ln t, \quad z \equiv r/t, \tag{7}$$

where F and G are now periodic (with period Δ), but otherwise arbitrary functions.

With the exception of $\phi = 0$, a DSS solution can only be regular at either $r = 0$ (for $F = G$), or at $r = t$ (for $G = 0$), or at $r = -t$ (for $F = 0$). All DSS solutions (except the zero solution) are singular at the point $(r = 0, t = 0)$. Coupling the wave equation to GR in spherical symmetry changes the dynamics, but not the degrees of freedom. (There are no spherically symmetric gravitational waves.)

Surprisingly, the presence of gravity acts as a regulator. In the presence of gravity, there is (at least) one DSS solution which is regular at both $r = 0$ and at the past light cone of $(r = 0, t = 0)$ (the generalization of $r = -t$ to curved spacetime) in the sense of being analytic. It is even regular, in a weaker sense, at the future light cone. $(r = 0, t = 0)$ remains singular, in the sense of a curvature singularity. This solution is precisely the intermediate attractor dominating near-critical collapse. It is found by numerically solving a non-linear hyperbolic eigenvalue problem, with periodicity in some coordinate τ, and regularity now imposed at both $r = 0$ and the past light cone [12, 13].

4. Calculation of the critical exponent. The calculation of the critical exponent is such a nice piece of dimensional analysis [3, 4, 14] that I simply must sketch it here. For simplicity of notation I assume CSS of the critical solution, the more generic DSS case is given in the paper [13]. Mathematically, the calculation that follows (for CSS) is identical with that of the critical exponent governing the correlation length near the critical point in statistical mechanics [15].

Let Z stand for the variables of the problem in a first-order formulation, in spacetime coordinates and matter variables adapted to the problem [12, 13]. $Z(r)$ is an element of the phase space, and $Z(r,t)$ a solution. The self-similar solution is of the form $Z(r,t) = Z_*(r/t)$. (I won't spell out what Z stands for or how r and t are defined.) In the echoing region, where Z_* dominates, we linearize around it. To linear order, the solution must be of the form

$$Z(r,t) \simeq Z_* \left(\frac{r}{t}\right) + \sum_{i=1}^{\infty} C_i(p)(-t)^{\lambda_i} \delta_i Z \left(\frac{r}{t}\right). \tag{8}$$

Here, the general form of the linear perturbations follows from the form of the background solution Z_*. Their coefficients C_i depend in a complicated way on the initial data, and hence on p. If Z_* is a critical solution, by definition there is exactly one λ_i with negative real part (in fact it is purely real), say λ_1. As $t \to 0$, all other perturbations vanish, and in the following we consider this limit, and retain only the perturbation with $i = 1$. Furthermore, by definition the critical solution corresponds to $p = p_*$, so we must have $C_1(p_*) = 0$. Linearizing around p_*, we obtain

$$\lim_{t \to 0} Z(r,t) \simeq Z_* \left(\frac{r}{t}\right) + \frac{dC_1}{dp}(p - p_*)(-t)^{\lambda_1} \delta_1 Z \left(\frac{r}{t}\right). \tag{9}$$

This form holds over a range of t, that is, is an approximate solution. Now we extract Cauchy data by picking one particular value of t within that range, namely t_p defined by

$$\frac{dC_1}{dp}(p - p_*)(-t_p)^{\lambda_1} \equiv \epsilon, \tag{10}$$

where ϵ is some constant $\ll 1$, so that at t_p the linear approximation is still valid. (The suffix p indicates that t_p depends on p.) At sufficiently small t, the linear perturbation Z_1 has grown so that the linear approximation breaks down. Later on a black hole forms. The crucial point is that we need not follow this evolution in detail. It is sufficient to note that the Cauchy data at $t = t_p$ depend on r only in the combination r/t_p, namely

$$Z(r,t_p) \simeq Z_* \left(\frac{r}{t_p}\right) + \epsilon\, \delta_1 Z \left(\frac{r}{t_p}\right). \tag{11}$$

As furthermore the field equations do not have an intrinsic scale, it follows that the solution based on those data must be **exactly** of the form

$$Z(r,t) = f\left(\frac{r}{t_p}, \frac{t - t_p}{t_p}\right) \tag{12}$$

throughout, even when the black hole forms and perturbation theory breaks down, and still after it has settled down and the solution no longer depends on t. (This solution holds only for $t > t_p$, because in its initial data we have neglected the perturbation modes with $i > 1$, which are growing, not decaying, towards the past.) Because the black hole mass has dimension length, it must be proportional to t_p, the only length scale in the solution,

$$M \propto t_p \propto (p - p_*)^{-\frac{1}{\lambda_1}}, \tag{13}$$

and we have found the critical exponent.

When the critical solution is DSS, the scaling law is modified [13]. On the straight line relating $\ln M$ to $\ln(p - p_*)$ a periodic wiggle of small amplitude is superimposed. This wiggle is again universal with respect to families of initial data, and there is only one free parameter for each family to be adjusted, corresponding to a shift of the wiggly line in the $\ln M$ direction. (No separate adjustment in the $\ln(p-p_*)$ direction is required.)

5. Gravity as a regularizer of self-similar solutions. I now come back to the critical solution, which by definition is DSS and regular. As I said before, DSS is imposed as periodicity in a coordinate system of the form (5) adapted to the problem, and boundary conditions at $r=0$ arise in a straightforward manner, from the necessity of avoiding a conical singularity.

From the form (5) of the metric it is easy to see that the curvature blows up as $\tau \to -\infty$. Furthermore, this singularity is a "point" in, for example, the following sense. Let $(\tau_1, \zeta_1, \theta_1, \varphi_1)$ and $(\tau_2, \zeta_2, \theta_2, \varphi_2)$ be two points. (Here τ is the coordinate defined by equation (5), θ and φ are the Euler angles adapted to the spherical symmetry, and ζ is a choice of remaining coordinate, roughly speaking $\ln(r/t)$.) Their geodesic distance vanishes as e^τ as $\tau_1 \to \tau_2 \to -\infty$, for any values of $(\zeta_1, \theta_1, \varphi_1)$ and $(\zeta_2, \theta_2, \varphi_2)$.

The past light cone of this singularity is the equivalent of $r=-t$ in curved spacetime. It is called the past self-similarity horizon. We use the remaining freedom in the coordinate system (5) to label this light cone $\zeta = 0$. From the analogy with the general CSS solution (7) of the wave equation in flat space one would assume that any solution regular at $r=0$ is singular here, showing an infinite number of oscillations of the form $\phi \sim F(\ln\zeta)$ (with F periodic) as $\zeta \to 0$.

But this is not so, and I now show why [13]. In flat space, the inward and outward traveling modes are $\phi' + \dot\phi$ and $\phi' - \dot\phi$. Let us call their curved-space equivalents X_+ and X_-. It is X_- that we expect to be singular at $\zeta = 0$. The equation for X_-, to leading order in ζ, is

$$X_{-,\zeta} = \frac{A(\tau)X_- - X_{-,\tau} + C(\tau)}{\zeta B(\tau)}, \tag{14}$$

where the coefficients A, B and C are constructed from the other fields and are therefore periodic in τ.

This approximate equation admits an exact general solution, namely

$$X_- = X_-^{\rm inhom}(\tau) + X_-^{\rm hom}(\zeta, \tau). \tag{15}$$

The particular inhomogeneous solution $X_-^{\rm inhom}$ is defined as the unique solution of

$$AX_-^{\rm inhom} - X_{,\tau}^{\rm inhom} + C = 0 \tag{16}$$

with periodic boundary conditions. This solution exists and is unique, unless the average value of A vanishes. The general homogeneous solution $X_-^{\rm hom}$ is of the form

$$X_-^{\rm hom} = \zeta^{\frac{A_0}{B_0}}\, e^{\int A\, - \frac{A_0}{B_0}\int B}\, F\left[\tau + \frac{\int B - \ln|\zeta|}{B_0}\right]. \tag{17}$$

where A_0 is the average value of the periodic function A, and $\int A$ is its principal function *after the average value has been subtracted*, so that $\int A$ is by definition also periodic. The periodic function F depends on the initial data for the equation (14). I have not given the expressions for A, B, and C here, but in flat space A and B vanish. In the critical solution, which is far from flat, A_0 is negative and B_0 is positive. The exponent of ζ,

A_0/B_0, is negative for the critical or neighboring solutions. In consequence, X_-^{hom} only has two alternatives, it either blows up at $\zeta = 0$, or it is analytic there (for $F \equiv 0$). To impose regularity at $\zeta = 0$, it suffices therefore to impose

$$A(\tau)X_- - X_{-,\tau} + C(\tau) = 0 \tag{18}$$

at $\zeta = 0$. As it happens, there is locally just one solution which obeys this condition, as well as the other boundary conditions.

As I have said, A vanishes in flat space. But then the equation (18) has no solution X_- *with periodic boundary conditions*, because the average value of C does not vanish. But the presence of the term proportional to X_- changes the character of the equation quantitatively, and a solution always exists. (In the limit as A vanishes, this solution blows up.)

We have now found the solution in the past light cone of the singularity. The data on the past light cone then determine the solution up to the future light cone, also called the future self-similarity horizon. (We can go no further because the future light cone of the singularity is a Cauchy horizon.) Calculating this maximal extension numerically requires two more nontrivial changes of coordinate system, giving rise to fresh eigenvalue problems. We arrange the final coordinate patch so that once more the future light cone is a coordinate line. Now it is X_+ which is potentially singular. Its equation is of the same form (14), where X_+ replaces X_-, and ζ has been redefined so that $\zeta = 0$ is now the future light cone. Now, however, there is no freedom left to adjust any data in order to set $F \equiv 0$, and in fact F does not vanish with the data we have in hand. So the solution cannot be analytic at the future light cone. In contrast to the past light cone, however, both A_0 and B_0 are positive. This means that X_+^{hom}, with an infinite number of oscillations as $\zeta = 0$ is approached, is present, but vanishes at $\zeta = 0$ as a (small) positive power of ζ. X_+ exists at $\zeta = 0$, but $\partial X_+/\partial\zeta$ does not. From the Einstein equations, which I have not given here, it follows that the metric and all its first derivatives exist, but not some of its second derivatives. Nevertheless these particular second derivatives cancel out of all components of the Riemann tensor, so that the Riemann tensor exists (but not some of its first derivatives).

Fig. 2 summarizes the global situation, with one angular coordinate suppressed. We have spherically symmetric, discretely self-similar spacetime, with a single point-like singularity. The self-similarity corresponds roughly speaking to periodicity in the logarithm of the distance from the singularity. The past light cone of the singularity is totally regular, in no way distinguished from other spacetime points. The future light cone, or Cauchy horizon, is perhaps as regular as one can expect, with the scalar matter field C^0, the metric C^1 and the Riemann tensor C^0. In particular it carries well-defined null data (which are of course self-similar), and there exists a regular, self-similar (and of course, non-unique) extension of the spacetime inside the Cauchy horizon, with only the horizon itself of limited differentiability. Surprisingly, the null data on the horizon are very small, so that the extension can be made almost flat and empty. The situation is similar in the other two cases where the critical solution has been calculated up to the Cauchy horizon [7, 9]. The spacetime there is CSS, but also almost flat at the horizon. A limiting case arises in the closed-form solution of Roberts [16], where the null data on the horizon

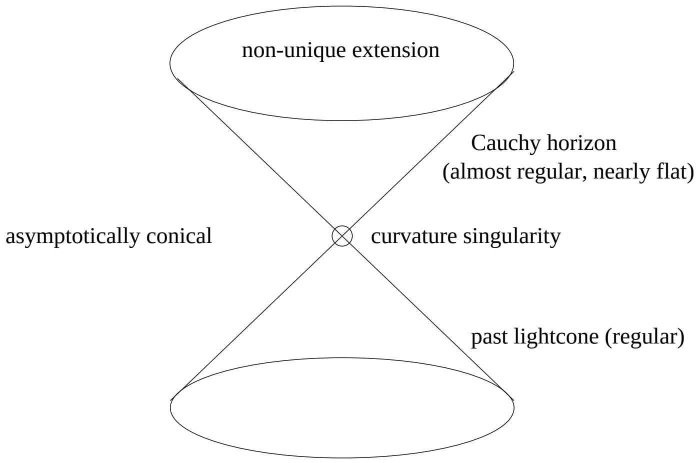

Figure 2: The global structure of the critical spacetimes. One dimension in spherical symmetry has been suppressed.

vanish exactly, so that a possible extension inside the light cone is flat empty space. (In the Roberts solution, the past light cone also carries zero data, in contrast to the other examples, where the past light cone is very far from flat.) Finally I should add that at large spacelike distances from the singularity, spacetime becomes asymptotically conical, with a constant defect solid angle.

Of course, all these global considerations are not directly relevant to critical collapse, where the spacetime asymptotes the critical spacetime in some bounded region inside the past light cone. Therefore, neither the singularity nor the asymptotically conical region, nor the Cauchy horizon appear.

6. Conclusions. Looking back on what has become a small industry in the past two years, I think that the dynamical mechanism of universality, scaling and echoing is now understood in an intuitive way. Furthermore we can calculate the echoing period Δ and critical exponent γ as nonlinear eigenvalues, in a manner distinct from that of fine-tuning data in numerical experiment.

Critical phenomena have also given new material to the study of cosmic censorship. There are self-similar spacetimes (spherically symmetric, except for one axisymmetric example) with a naked singularity that have an infinity of decaying linear perturbation modes opposed to only one increasing perturbation mode. This still implies cosmic censorship in the sense that a generic perturbation, which will contain some small fraction of that unstable mode, destroys the naked singularity (either by forming a horizon or by avoiding a singularity altogether, depending on the sign of the mode amplitude). But we are very close to a violation, in that we only have to set one mode out of an infinity of modes to zero to get the naked singularity. A single generic parameter in the initial data provides a sufficient handle to do this. (In the notation above, we only have to set the one amplitude C_1 equal to zero, and if C_1 depends in some way on p, we can do this by adjusting p.) Therefore by arbitrary fine-tuning of one generic parameter in the initial data one can obtain asymptotically flat spacetimes in which a region of arbitrarily large curvature is visible to an observer at infinity.

There are a number of important open questions. Are there additional unstable perturbations of the critical solution among the non-spherical modes? Are there critical solutions in axisymmetry, or even lower symmetry? Is DSS more generic than CSS? How do the charge and angular momentum of the black hole scale as one fine-tunes initial data with charge and angular momentum, with the aim of making a black hole of infinitesimal mass (and not caring about charge and angular momentum)? Is there a preferred flow on superspace which would complete the analogy with the renormalisation group flow in statistical mechanics? Is there even statistical physics hidden somewhere? (Very, very unlikely, but who would have thought it of black holes pre-1974?) Can someone give an existence proof for the critical solutions, which so far have only been constructed numerically? Can one prove that they must have the limited differentiability (metric C^1 etc.) at the Cauchy horizon which I have described, not more or less?

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