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THE POSITIVE MASS THEOREM FOR ALE MANIFOLDS

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Abstract. We show what extra condition is necessary to be able to use the positive mass argument of Witten [12] on an asymptotically locally euclidean manifold. Specifically we show that the "generalized positive action conjecture" holds if one assumes that the signature of the manifold has the correct value.

1. Preliminaries. The purpose of this section is to explain some not so well known aspects of spin-geometry, for the general background see [8]. Let $\mathrm{Spin}(n)$ be the spin group in n dimensions and let $(\rho, \mathcal{S})$ be the spinor representation. Let (M, g) be a Riemannian spin manifold of dimension $n \geq 3$ with spin structure $\mathrm{Spin}(M, g)$ and let $\mathcal{S}(M, g) = \mathrm{Spin}(M, g) \times_\rho \mathcal{S}$ be the spinor bundle on M associated to $\mathrm{Spin}(M, g)$. We will usually drop the g from the notation.

1.1. *Spin structures and spinors on quotient spaces.* Let Γ be a group acting by orientation preserving isometries on M. An element $\gamma \in \Gamma$ acts on a frame f by $f \mapsto \gamma_* f$. Assume that this action of Γ on the frame bundle lifts to an action on $\mathrm{Spin}(M, g)$, that is we have an action $s \mapsto \tilde{\gamma} s$ which projects to the action $f \mapsto \gamma_* f$. Via the spin representation this defines an action on the spinor bundle where we denote the action of γ by $\tilde{\gamma}$.

Assume that Γ is a discrete group acting without fixed points. Then Γ has a lift if and only if M/Γ is spin. In this case the spin bundle on the quotient is given by

$$\mathrm{Spin}(M/\Gamma) = \mathrm{Spin}(M)/\Gamma$$

and the associated spinor bundle is given by

$$\mathcal{S}(M/\Gamma) = \mathcal{S}(M)/\Gamma.$$

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This means that given a lift the sections of $\mathcal{S}(M/\Gamma)$ are precisely the Γ-periodic sections of $\mathcal{S}(M)$.

1.2. *Comparing spinors for different metrics.* Let g, g' be Riemannian metrics on a manifold M, and define the 'gauge transformation' $A \in \mathrm{End}(TM)$ by

$$g(AX, AY) = g'(X, Y).$$

$$g(AX, Y) = g(X, AY)$$

Because of the first property A will map ON-frames for g' to ON-frames for g, and thus A induces a map $\mathrm{SO}(M, g') \overset{A}{\to} \mathrm{SO}(M, g)$. If M is spin and we choose equivalent spin-structures for g and g' this can be lifted to $\mathrm{Spin}(M, g') \overset{A}{\to} \mathrm{Spin}(M, g)$. A spinor field for g can be viewed as a $\mathrm{Spin}(n)$-equivariant map $\mathrm{Spin}(M, g) \overset{\varphi}{\to} \mathcal{S}$, where $\mathcal{S}$ is the spinor space, so the composition $\varphi \circ A$ is a map $\mathrm{Spin}(M, g') \overset{\varphi}{\to} \mathcal{S}$ which also is $\mathrm{Spin}(n)$-equivariant. This gives the extension of A to a map $\mathcal{S}(M, g') \overset{A}{\to} \mathcal{S}(M, g)$ which respects Clifford multiplication:

$$A(X \cdot \varphi) = (AX) \cdot (A\varphi).$$

Since the metric on the spinor bundle is given by a fixed Hermitean inner product on $\mathcal{S}$, A defines a fibrewise isometry. The above can be collected in a diagram.

$$\begin{array}{ccccc}
\mathrm{Spin}(M, g') & \xrightarrow{A} & \mathrm{Spin}(M, g) & \xrightarrow{\varphi} & \mathcal{S} \\
\downarrow \pi & & \downarrow \pi & & \\
\mathrm{SO}(M, g') & \xrightarrow{A} & \mathrm{SO}(M, g) & &
\end{array}$$

We will now look at the relation between the canonical covariant derivatives for (M, g) and (M, g'). Let ∇ and ∇' be the Levi-Civita connections for g and g', to be able to compare ∇ and ∇' on the frame and spin bundles for g we define a connection $\overline{\nabla}$ by

$$\overline{\nabla} X = A(\nabla' A^{-1} X). \tag{1}$$

The connection $\overline{\nabla}$ is metric with respect to g and has torsion

$$\begin{aligned} \overline{T}(X, Y) &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \\ &= -((\nabla'_X A) A^{-1} Y - (\nabla'_Y A) A^{-1} X). \end{aligned} \tag{2}$$

Expressing the covariant derivative in terms of the Lie bracket and the metric we get

$$2g(\overline{\nabla}_X Y - \nabla_X Y, Z) = g(\overline{T}(X, Y), Z) - g(\overline{T}(X, Z), Y) - g(\overline{T}(Y, Z), X). \tag{3}$$

Next we compare $\nabla, \overline{\nabla}$ when lifted to the spinor bundle $\mathcal{S}(M, g)$. Let $\{e_i\}$ be a local orthonormal frame for g, and let $\{\sigma_\alpha\}$ be the corresponding local orthonormal frame of the spinor bundle. Denote by $\omega_{ij}, \overline{\omega}_{ij}$ the connection one-forms for $\nabla, \overline{\nabla}$ defined with respect to $\{e_i\}$,

$$\omega_{ij} = g(\nabla e_i, e_j)$$

$$\overline{\omega}_{ij} = g(\overline{\nabla} e_i, e_j),$$

then the covariant derivatives of $\varphi = \varphi^\alpha \sigma_\alpha$ are given by [8, Thm 4.14]

$$\nabla \varphi = d\varphi^\alpha \otimes \sigma_\alpha + \frac{1}{2} \sum_{i<j} \omega_{ij} \otimes e_i e_j \varphi,$$

$$\overline{\nabla}\varphi = d\varphi^\alpha \otimes \sigma_\alpha + \frac{1}{2}\sum_{i<j} \overline{\omega}_{ij} \otimes e_i e_j \varphi$$

and hence the difference between $\overline{\nabla}$ and ∇ acting on φ is

$$\overline{\nabla}\varphi - \nabla\varphi = \frac{1}{2}\sum_{i<j}(\overline{\omega}_{ij} - \omega_{ij}) \otimes e_i e_j \varphi. \tag{4}$$

Using (2) and (3) we can estimate

$$|(\overline{\omega}_{ij} - \omega_{ij})(e_k)| \leq C|A^{-1}||\nabla' A|.$$

We have proved the following lemma

Lemma 1.1. *Let Y be a vectorfield and let φ be a spinor (w.r.t the g spin bundle), then*

$$|\overline{\nabla}Y - \nabla Y| \leq C|A^{-1}||\nabla' A||Y|, \tag{5}$$

$$|\overline{\nabla}\varphi - \nabla\varphi| \leq C|A^{-1}||\nabla' A||\varphi| \tag{6}$$

and

$$|\overline{D}\varphi - D\varphi| \leq C|A^{-1}||\nabla' A||\varphi|, \tag{7}$$

where $D, \overline{D}$ are the Dirac operators associated to the connections $\nabla, \overline{\nabla}$. ❑

2. Asymptotically locally euclidean manifolds. We are going to study manifolds with ends asymptotic to a flat cone $\mathbb{R}^n/\Gamma$, they are called asymptotically locally euclidean or ALE. We use a definition basically as in [3] since we will refer to that paper for analytical results.

Definition 2.1. A complete Riemannian manifold (M, g) is called *ALE* with group Γ if

1. Γ is a finite group of isometries of $\mathbb{R}^n$ acting freely outside the origin.
2. There is a compact set C and a diffeomorphism between $M \setminus C$ and $(\mathbb{R}^n \setminus B)/\Gamma$ where B is a ball around the origin in $\mathbb{R}^n$. This diffeomorphism gives a specific set of "coordinates at infinity".
3. On the end the metric g and the flat metric g_0 on $\mathbb{R}^n$ are uniformly equivalent.
4. Using the coordinates at infinity the difference between g and g_0 satisfies
$$g - g_0 \in W^{1,q}_{-d}(S^2T^*M)$$
where $W^{1,q}_{-d}$ is a weighted Sobolev space as defined next and $d > 0$ is called the *order* of (M, g).

Definition 2.2. Let (M, g) be an ALE manifold and let V be a vector bundle with a connection. Let r be a positive function extending the background radial coordinate on the end. The weighted Sobolev spaces $W^{k,q}_\delta(V)$ are defined as the completion of the smooth compactly supported sections of V with respect to the norm $||\cdot||_{k,q,\delta}$ defined by

$$||f||^q_{k,q,\delta} = \sum_{j=0}^k \int_M ||\nabla^j f||^q r^{-q(\delta-j)-n} dx.$$

Remark 2.3. If $n > kq$ then $f \in W^{k,q}_{\delta}(V)$ implies $||f|| = o(r^{\delta})$.

If $\Gamma = \{1\}$ we say that M is asymptotically euclidean or AE. The motivating examples of ALE manifolds come from the study of "gravitational instantons", noncompact Ricci-flat four-manifolds. Kronheimer has classified the four dimensional hyper-Kähler ALE manifolds, see [6, 7].

2.1. *The mass.* From general relativity comes the following definition of the mass of an ALE manifold.

Definition 2.4. The *mass* of an asymptotically locally euclidean manifold (M, g) is defined by

$$m(M, g) = \lim_{r\to\infty} \int_{S_r} (\partial_i g_{ij} - \partial_j g_{ii})\partial_j \lrcorner d\,\mathrm{vol}$$

if the limit exists. Here i, j refer to the coordinates at infinity and S_r is the sphere of radius r in these coordinates.

Following [3] we assume the following "mass decay conditions" hold.

Assumption 2.5. *1. $g - g_0 \in W^{2,q}_{-d}(S^2T^*M)$ for some $q > n$ and $d \geq \frac{n}{2} - 1$,*
2. $s(g) \in L^1(M)$, where s is the scalar curvature.

And we get the following proposition.

Proposition 2.6. *From the mass decay conditions it follows that mass is well-defined, not depending on the coordinates at infinity. If $d > n - 2$ then the mass vanishes.*

It was first shown by Schoen and Yau that an asymptotically euclidean manifold with non-negative scalar curvature has non-negative mass, and that if such a manifold has vanishing mass it has to be flat $\mathbb{R}^n$. It was also conjectured that a similar positive mass theorem would hold for ALE manifolds, but LeBrun found counterexamples to the conjecture [9]. In this paper we will see that the positive mass theorem does hold for ALE spin manifolds if one also assumes that the signature of the manifold takes the correct value.

3. ALE from curvature decay. If the curvature of a manifold has sufficiently fast decay and the volume grows to the same order as in euclidean space the manifold will be ALE, by assuming even faster decay of the Ricci tensor the mass decay condition will be satisfied. This follows from the main theorem in [2].

Theorem 3.1. *Let (M, g) be a complete manifold and let ρ be the distance from a fixed point. Suppose that the sectional curvature and the Ricci curvature decay asymptotically as*

$$|K| \leq C\rho^{-(2+\epsilon)}$$

$$|\,\mathrm{Ric}\,| \leq C\rho^{-(2+\mu)}$$

for some $\mu \geq \epsilon > 0$. Also assume that asymptotically the volume of the balls $B_R = \{\rho \leq R\}$ grows as

$$\mathrm{vol}(B_R) \geq VR^n$$

for some $V > 0$. Then the manifold is ALE as in definition 2.1. If $\mu > \frac{n}{2} - 1$ then point 1. *of assumption 2.5 holds. If $\mu > n-1$ then point* 2. *holds as well and the mass vanishes.*

We will formulate the main theorems in this paper as concerning ALE manifolds as in definition 2.1. Using the above theorem one sees that they apply under the geometrically more natural assumptions of curvature decay and volume growth. In [5] it is shown that one can do without the volume growth assumption if one instead assumes that the end has finite fundamental group and non-trivial tangent bundle. However, this excludes the four dimensional case which will be the focus of this paper.

4. Spinors and the Lichnerowicz formula. The Lichnerowicz formula is the Bochner formula for the Dirac operator D relating D^2 to the connection Laplacian. We need an integrated version for manifolds with boundary which is derived as follows. Fix a spinorfield φ and define a vectorfield L by

$$\langle L, X\rangle = \langle (\nabla_X + X\cdot D)\varphi, \varphi\rangle.$$

Integrating the divergence of L over a manifold M with boundary ∂M we get the Lichnerowicz formula

$$\int_M (\frac{s}{4}|\varphi|^2 + |\nabla\varphi|^2 - |D\varphi|^2) = \int_{\partial M} \langle (\nabla_\nu + \nu D)\varphi, \varphi\rangle \tag{8}$$

where ν is the outward normal of the boundary. Using the Lichnerowicz formula Witten found a simple proof of the positive mass theorem. The important observation he made was that if one has an AE (or ALE) manifold and a spinor φ which is constant (i.e. parallell) with respect to the flat background metric in the coordinates at infinity then the boundary integrals

$$\int_{S_r} \langle (\nabla_\nu + \nu D)\varphi, \varphi\rangle$$

tend to a constant times $m(M, g)$ as $r \to \infty$. More precisely on the end φ is on the form $A\varphi_0$ where A is the gauge-transformation between g and the flat background metric (see section 1.2) and φ_0 is a constant spinor with respect to the flat background metric.

4.1. *Constant spinors on $\mathbb{R}^n/\Gamma$.* To be able to use Witten's positive mass argument on an ALE manifold we need to know when there are parallell spinors on the flat background cones for the end. Denote $\mathbb{R}^n \setminus \{0\}$ by $\mathbb{R}^n_*$ and let $\Gamma \subset \mathrm{SO}(n)$ be a finite group acting freely on $\mathbb{R}^n_*$. We need to know when there are parallel spinors on the quotient $\mathbb{R}^n_*/\Gamma$. There are natural trivializations $\mathrm{SO}(\mathbb{R}^n_*) = \mathbb{R}^n_* \times \mathrm{SO}(n)$, $\mathrm{Spin}(\mathbb{R}^n_*) = \mathbb{R}^n_* \times \mathrm{Spin}(n)$ and $\mathcal{S}(\mathbb{R}^n_*) = \mathbb{R}^n_* \times \mathcal{S}$. In these trivializations $\gamma \in \Gamma$ acts on $\mathrm{SO}(\mathbb{R}^n_*)$ as

$$(x, f) \overset{\gamma}{\to} (\gamma(x), \gamma f).$$

Assume that $\mathbb{R}^n_*/\Gamma$ is spin. Then by the discussion in Section 1.1 there is a bijective lift of Γ to a subgroup $\widetilde{\Gamma} \subset \mathrm{Spin}(n)$, which specifies the action of Γ on the spin and spinor bundles of $\mathbb{R}^n_*$, $\tilde{\gamma} \in \widetilde{\Gamma}$ acts as

$$(x, s) \overset{\gamma}{\to} (\gamma(x), \tilde{\gamma}s),$$

$$(x, \varphi) \overset{\gamma}{\to} (\gamma(x), \rho(\tilde{\gamma})\varphi),$$

The sections of $\mathcal{S}(\mathbb{R}^n_*/\Gamma)$ are naturally identified with the Γ-periodic sections of $\mathcal{S}(\mathbb{R}^n_*)$. Since being parallel is a local condition on the spinor the parallel spinors on $\mathbb{R}^n_*/\Gamma$ are precisely given by the parallel spinors on $\mathbb{R}^n_*$ which are Γ-periodic. The parallel spinors on the quotient thus correspond precisely to the elements of $\mathcal{S}$ which are fixed by the spin-representation of the lifted group $\widetilde{\Gamma}$, they depend both on Γ and via the choice of lift the spin structure on the quotient. This proves the following Proposition.

PROPOSITION 4.1. *Let Γ be a subgroup of* $\mathrm{SO}(n)$ *acting freely on $\mathbb{R}^n_*$ and let $\widetilde{\Gamma}$ be a bijective lift to* $\mathrm{Spin}(n)$*, then the parallell spinors on $\mathbb{R}^n_*/\Gamma$ with the spin-structure defined by $\widetilde{\Gamma}$ correspond one-to-one to the $\varphi \in \mathcal{S}$ which are fixed by the spinor representation of $\widetilde{\Gamma}$.* ❑

5. The positive mass theorem. We now come to a general positive mass theorem for ALE manifolds. The first part of the theorem is the statement that $m(M,g)$ is non-negative, the second and more interesting part concerns the case when the mass vanishes.

THEOREM 5.1. *Let (M,g) be a spin ALE-manifold with group Γ and non-negative scalar curvature. Suppose that the spin structure on the ALE end is equivalent to to the spin structure on $\mathbb{R}^n_*/\Gamma$ defined by a lift $\widetilde{\Gamma}$ of Γ. If $\widetilde{\Gamma}$ fixes a spinor $u \in \mathcal{S}$ then $m(M,g) \geq 0$. If $m(M,g) = 0$ (for instance if the order $d > n-2$) then the space of parallel spinors on (M,g) is isomorphic to the subspace of $\mathcal{S}$ fixed by $\widetilde{\Gamma}$.*

Remark 5.2. If $\Gamma = \{1\}$ we have the classical positive mass theorem. In this case $m = 0$ implies that there is $\dim(\mathcal{S})$-dimensional space of parallel spinors so the spinor bundle is flat and the manifold is isometric to $\mathbb{R}^n$.

Remark 5.3. The existence of parallel spinors gives a strong restriction on the holonomy of the manifold, see the papers by Wang [10, 11]. For instance if M is simply connected then M is either flat $\mathbb{R}^n$ or one of the following cases hold (N is the dimension of the space of parallel spinors on M)

1. $N = 2$, M has dimension $n = 2m$ and holonomy $\mathrm{SU}(m)$,
2. $N = m+1$, M has dimension $n = 4m$ and holonomy $\mathrm{Sp}(m)$,
3. $N = 1$, $n = 8$, holonomy $\mathrm{Spin}(7)$,
4. $N = 1$, $n = 7$, holonomy G_2.

PROPOSITION 5.4 ([3]). *Suppose (M,g) is an ALE manifold satisfying assumption (2.5) and having non-negative scalar curvature. Then the Dirac operator is an isomorphism from $W^{2,q}_{-\eta}(\mathcal{S})$ to $W^{1,q}_{-\eta-1}(\mathcal{S})$ for $0 < \eta < n-1$.*

Proof of the theorem. Let ψ be a parallel spinor on $\mathbb{R}^n_*/\Gamma$, f be a cut-off function for the end and A be the gauge transformation between g and the flat metric on the end. Define the spinor $\varphi_0 = fA\psi$ on (M,g). It follows from (7) and our assumption 2.5 that $D\varphi_0 \in W^{1,q}_{-\frac{n}{2}}(\mathcal{S})$ so the equation $D\varphi_1 = D\varphi_0$ has a unique solution $\varphi_1 \in W^{2,q}_{-\frac{n}{2}+1}(\mathcal{S})$. Set $\varphi = \varphi_0 - \varphi_1$, then $D\varphi = 0$ and since φ_1 vanishes at infinity φ is asymptotic to φ_0. When plugging φ into the Lichnerowicz identity we get

$$0 \leq \int_M \left(\frac{s}{4}|\varphi|^2 + |\nabla\varphi|^2\right) = \lim_{r\to\infty} \int_{S_r} \langle(\nabla_\nu + \nu D)\varphi, \varphi\rangle = c(n)m(M,g),$$

the calculation for the last line can be found in [3]. This shows that $m \geq 0$ and if $m = 0$ we see that $\nabla\varphi = 0$. This procedure produces different parallel spinors φ if we start with different ψ, and the theorem follows. ❑

6. Four dimensions. We now consider the four dimensional case, in four dimensions there are isomorphisms

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$

where the projection $\mathrm{Spin}(4) \to \mathrm{SO}(4)$ takes (p, q) to the map $v \to pvq^*$ (quaternion multiplication) and the spinor representation is $\mathrm{SU}(2) \times \mathrm{SU}(2)$ acting on $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^- = \mathbb{C}^2 \oplus \mathbb{C}^2$.

Suppose that $\Gamma \subset \mathrm{SO}(4)$ has a lift to $\widetilde{\Gamma} \subset \mathrm{Spin}(4)$ such that for all $\tilde{\gamma} \in \widetilde{\Gamma}$, $\tilde{\gamma} \cdot u = u$ where $0 \neq u \in \mathcal{S}$. Then the same holds for the parts u_+, u_-, by a choice of orientation we may assume $u_- \neq 0$. This means that for all $\tilde{\gamma}$ the part in the second SU(2) factor, $\tilde{\gamma}_-$, has one eigenvalue equal to one and since the determinant is one we must have $\tilde{\gamma}_- = Id$. So $\mathcal{S}^-$ is fixed by $\widetilde{\Gamma}$ and $\widetilde{\Gamma}$ is a subgroup of the first SU(2) factor, which also means that $\Gamma \subset \mathrm{Sp}(1) = \mathrm{SU}(2) \subset \mathrm{SO}(4)$. The same reasoning gives that every finite subgroup of SU(2) acts freely on the sphere. We conclude;

Proposition 6.1. *If Γ is a finite subgroup of* SU(2) *then with the above choice of orientation, the spinors φ_u with $u \in \mathcal{S}^-$ give parallel spinors on the quotient $((\mathbb{R}^4_*)/\Gamma$ provided we choose the spin structure*

$$\mathrm{Spin}(\mathbb{R}^4_*/\Gamma) = \mathrm{Spin}(\mathbb{R}^4_*)/\widetilde{\Gamma}$$

defined by the lift $\gamma \to \tilde{\gamma} = (\gamma, Id)$. Except for reversing orientation these are the only cases allowing parallel spinors.

So we can only find asymptotically parallel spinors on an ALE four manifold if the fundamental group of the locally Euclidean end is a finite subgroup of SU(2). The following groups are up to conjugation all finite subgroups of SU(2) ([13, Thm. 2.6.7]).

- $\mathbf{A}_n$: The cyclic group of order n generated by $z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where $\zeta = e^{2\pi i/n}$.
- $\mathbf{D}^*_n$: The binary dihedral group of order $4n$, this consists of $\{z^a, jz^a\}_{a=0}^{2n-1}$ where $z = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ with $\zeta = e^{2\pi i/2n}$ and $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- $\mathbf{T}^*$: The binary tetrahedral group.
- $\mathbf{O}^*$: The binary octahedral group.
- $\mathbf{I}^*$: The binary icosahedral group.

For these groups there is a canonical lift to Spin(4) and an associated canonical spin structure on the end which we call the trivial spin structure. Any other lift of Γ to Spin(4) is given up to conjugation by an element $\kappa \in \mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ as follows

$$\Gamma \ni \gamma \to \tilde{\gamma} = \kappa(\gamma)(\gamma, Id) \in \mathrm{Spin}(4).$$

The elements of $\mathrm{Hom}(\Gamma, \mathbb{Z}_2)$ are

- $\mathbf{A}_n$: $\kappa_0 = 1$ and if n is even κ_1 defined by $\kappa_1(z) = -1$.

$\mathbf{D}_n^*$: κ_{pq}, $p, q = 0, 1$ defined by $\kappa_{pq}(z) = (-1)^p$ and $\kappa_{pq}(j) = (-1)^q$.
$\mathbf{T}^*$: 1.
$\mathbf{O}^*$: $\kappa_0 = 1$ and κ_1 which is nontrivial.
$\mathbf{I}^*$: 1.

We will now see that the spin structure on an ALE end can be detected by the signature of the manifold. Let M_r be an ALE manifold M with the end cut off at distance r. So M_r is a compact manifold with boundary and as $r \to \infty$ the boundary ∂M_r will, if we rescale to constant volume, approach the spherical space-form S^3/Γ. To relate the spinstructure at the boundary to the signature we use the Atiyah–Patodi–Singer index theorem to compute the index of the Dirac operator with the Atiyah–Patodi–Singer boundary condition and the relative signature $\sigma(M_r, \partial M_r)$. This signature is independent of r if r is large enough and we will denote it by just $\sigma(M)$.

$$\operatorname{ind}(D) = \int_{M_r} \widehat{A} + \int_{\partial M_r} T\widehat{A} - \eta_D(\partial M_r), \tag{9}$$

$$\sigma(M) = \int_{M_r} L + \int_{\partial M_r} TL - \eta_\sigma(\partial M_r), \tag{10}$$

where $\widehat{A}$,L are the A-roof genus and the Hirzebruch L-genus, $T\widehat{A}$,TL are their transgressions and $\eta_\sigma(\partial M_r)$ and $\eta_D(\partial M_r)$ are the eta-invariants of the Signature- and the Dirac-operator on the boundary. In four dimensions the Hirzebruch genus and the A-roof genus are both proportional to the first Pontrjagin class,

$$-8\widehat{A} = \frac{1}{3}p_1 = L$$

and

$$-8T\widehat{A} = TL$$

so we can cancel the integrals and get

$$\sigma(M) + 8\operatorname{ind}(D) = -\eta_\sigma(\partial M_r) - 8\eta_D(\partial M_r). \tag{11}$$

Now if $s \geq 0$ the index of the Dirac-operator on ∂M_r vanishes for r large enough. For suppose we have a harmonic spinor φ satisfying the Atiyah–Patodi–Singer boundary condition and we plug it into the Lichnerowicz formula (8). Then the left-hand side

$$\int_M \left(\frac{s}{4}|\varphi|^2 + |\nabla\varphi|^2\right)$$

is explicitly non-negative. The boundary integral in (8) can be written as

$$\int_{\partial M} \langle (\nabla_\nu + \nu D)\varphi, \varphi\rangle = \int_{\partial M} \langle \nu\widetilde{D}\varphi, \varphi\rangle + \frac{1}{2}\operatorname{tr}(h)\langle\varphi, \varphi\rangle$$

where $\widetilde{D}$ is the Dirac-operator of the induced metric on the boundary and h is the second fundamental form, $h(X, Y) = g(\nu, \nabla_X Y)$. The Atiyah–Patodi–Singer boundary condition tells us that the first term is non-positive and using (2.5) one sees that for r large the trace $\operatorname{tr}(h)$ is negative since it is then sufficiently close to $-(n-1)/r$, which is the corresponding trace in the Euclidean case.

So both sides in the Lichnerowicz formula must vanish and we see from the left-hand side that $\nabla\varphi = 0$ and from the right-hand side that $\varphi = 0$ on the boundary. We conclude that $\varphi = 0$ and $\operatorname{ind}(D) = 0$.

Since the left-hand side of (11) is a topological invariant we can choose any metric to compute the right-hand side, we choose one for which the boundary is isometric to the spherical space-form S^3/Γ. Then

$$\sigma(M) = -\eta_\sigma(S^3/\Gamma) - 8\eta_D(S^3/\Gamma). \tag{12}$$

One can now compute the eta-invariants for S^3/Γ explicitly [4] and η_D involves the spin structure via κ. The details of the computation of the eta-invariants are described in [1]. We summarize the result for the finite subgroups of SU(2) in Table 1.

Table 1: Eta invariants and signature

Group	Spin structure	η_σ	η_D	$\sigma(M_r)$
$\mathbf{A}_n$	κ_0	$\frac{(n-1)(n-2)}{3n}$	$\frac{n^2-1}{12n}$	$-n+1$
	κ_1 (for n even)		$-\frac{n^2+2}{12n}$	1
$\mathbf{D}^*_n$	κ_{00}	$\frac{2n^2+1}{6n}$	$\frac{4n^2+12n-1}{48n}$	$-n-2$
	κ_{01}		$\frac{4n^2-1}{48n}$	$-n$
	κ_{10}		$-\frac{2n^2-12n+1}{48n}$	-2
	κ_{11}		$-\frac{2n^2+1}{48n}$	0
$\mathbf{T}^*$		$\frac{49}{36}$	$\frac{167}{288}$	-6
$\mathbf{O}^*$	κ_0	$\frac{121}{72}$	$\frac{383}{576}$	-7
	κ_1		$-\frac{49}{576}$	-1
$\mathbf{I}^*$		$\frac{361}{180}$	$\frac{1079}{1440}$	-8

The above discussion proves the following version of the positive mass theorem.

Theorem 6.2. *Let (M, g) be a four-dimensional spin ALE-manifold with group Γ and non-negative scalar curvature. Suppose $\sigma(M)$ takes the value corresponding to the trivial spinstructure on the end, that is*

Γ	$\sigma(M)$
$\mathbf{A}_n$	$-n+1$
$\mathbf{D}^*_n$	$-n-2$
$\mathbf{T}^*$	-6
$\mathbf{O}^*$	-7
$\mathbf{I}^*$	$-8.$

Then $m(M, g) \geq 0$ and if $m(M, g) = 0$ the manifold is hyper-Kähler.

The conclusion is that the manifold has to be one of the manifolds constructed in [6].

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