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SELFGRAVITATING SYSTEMS IN NEWTONIAN THEORY—THE VLASOV–POISSON SYSTEM

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Abstract. We give a review of results on the initial value problem for the Vlasov–Poisson system, concentrating on the main ingredients in the proof of global existence of classical solutions.

1. Introduction. Consider a large ensemble of particles which interact only by the gravitational field which they create collectively, in particular, assume that collisions among the particles are sufficiently rare to be neglected. An example of such an ensemble could be a galaxy where the stars play the role of the particles. We want to describe such an ensemble in a Newtonian setting. Each particle moves on a trajectory determined by Newton's equations of motion

$$\dot{x} = v,\ \dot{v} = -\partial_x U(t, x), \tag{1.1}$$

where $U = U(t, x)$ denotes the gravitational potential of the ensemble; $t \in \mathbb{R}$ denotes time and $x, v \in \mathbb{R}^3$ denote position and velocity respectively. For simplicity we assume that all particles have the same mass, equal to unity. If one describes the matter distribution of the ensemble as a whole by a number density $f = f(t, x, v)$ on phase space then f satisfies the so-called Vlasov equation

$$\partial_t f + v \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0; \tag{1.2}$$

note that the characteristic system of this equation is (1.1). Since the gravitational field is generated collectively by the ensemble itself the Vlasov equation is coupled to Newton's law of gravity

$$\triangle U = 4\pi\rho \tag{1.3}$$

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where the spatial mass density $\rho = \rho(t,x)$ is given by

$$\rho(t,x) = \int f(t,x,v)\,dv. \tag{1.4}$$

The nonlinear system of partial differential equations (1.2), (1.3), (1.4) is known as the Vlasov–Poisson system.

There is a well developed mathematical theory of such systems, and the distinguishing feature of the Vlasov–Poisson system is that there is a global existence and uniqueness theorem for the corresponding Cauchy problem for general initial data. This means that the Vlasov matter model should be useful in describing self-gravitating matter in the context of general relativity, since any singularity which appears in such a relativistic set-up must be due to in some sense genuinely relativistic effects. The present review is intended to supplement the lecture notes of A. Rendall on the Cauchy problem for the Vlasov–Einstein system. For the latter system partial results are available which lend support to the belief that the Vlasov matter model is "well-behaved" when compared to other models such as dust, and we shall show that the Vlasov–Poisson system, which is the Newtonian limit of the Vlasov–Einstein system, is indeed "well-behaved".

To do so we must supplement the system with a boundary condition for the potential. We require that

$$\lim_{x\to\infty} U(t,x) = 0, \tag{1.5}$$

that is, we consider the case of an isolated system. For reasons of exposition we include the so-called plasma physics case, that is, we replace (1.3) by

$$\triangle U = 4\pi\gamma\rho \tag{1.6}$$

where $\gamma = 1$ is the gravitational and $\gamma = -1$ the plasma physics case.

THEOREM 1.1. *For every nonnegative, continuously differentiable, and compactly supported initial datum $\mathring{f} \in C^1_c(\mathbb{R}^6)$ there exists a unique solution $f \in C^1([0,\infty[\times\mathbb{R}^6)$ of the Vlasov–Poisson system with $f(0) = \mathring{f}$.*

This result was first established by Pfaffelmoser [16, 17], simplified versions of the proof are due to Horst [13] and Schaeffer [24], and a proof along quite different lines is due to Lions and Perthame [15]. In the next section we will discuss the main ingredients and a-priori estimates which enter into the proof of this theorem, in particular, we will point out exactly why the same proof covers both the plasma physics and the stellar dynamics cases, a fact which is surprising from a physics point of view. Then we shall give a complete proof of the theorem which is due to Schaeffer [25] and is, in our opinion, the clearest available proof but has not been published previously. Then we will briefly discuss a blow-up result for the so-called relativistic Vlasov–Poisson system, which is due to Glassey and Schaeffer [9] and points to some of the obstacles one may encounter when trying to extend the result to related systems. In the last section we will review an extension of the global existence result to a geometrically different situation, namely the cosmological case.

2. Ingredients in the proof of global existence

2.1. Local existence and continuation of solutions. A necessary first step towards global existence is a local existence result and a criterion which says what kind of bounds one has to establish in order to extend a local solution to a global one. This is the content of the following theorem.

THEOREM 2.1. *For every* $\mathring{f} \in C^1_c(\mathbb{R}^6)$, $\mathring{f} \geq 0$, *there exists a unique solution* $f \in C^1([0,T[\times\mathbb{R}^6)$ *of the Vlasov–Poisson system on a right maximal existence interval* $[0,T[$, $T > 0$, *with* $f(0) = \mathring{f}$. *If*

$$P(t) := \max\Big\{|v| \mid (x,v) \in \mathrm{supp} f(t)\Big\}$$

is bounded on $[0,T[$ *or if* $\|\rho(t)\|_\infty$ *is bounded on* $[0,T[$ *then* $T = \infty$, *that is, the solution is global.*

Throughout these notes $\|\cdot\|_p$ will denote the usual L^p-norm, $1 \leq p \leq \infty$. The proof of this result is implicit in [2] although the result is not explicitly stated there. We do not go into the proof and only mention that the continuation criterion already gives some physically relevant and nontrivial information: If a singularity should form then it cannot form in such a way that only a derivative, say of ρ, blows up, but ρ itself has to become singular. Similar results are known for related systems such as the relativistic Vlasov–Poisson system, the Vlasov-Maxwell system, or the Vlasov–Einstein system with spherical symmetry.

2.2. A-priori bounds. The first set of a-priori bounds is based on the conservation of phase space volume. Let $(X,V)(s,t,x,v)$ denote the solution of the characteristic system

$$\dot{x} = v,\ \ \dot{v} = -\partial_x U(s,x)$$

with initial condition $(X,V)(t,t,x,v) = (x,v)$, $t \geq 0$, $x,v \in \mathbb{R}^3$. It is well known that

$$f(t,x,v) = \mathring{f}\big((X,V)(0,t,x,v)\big),\ \ t \geq 0,\ \ x,v \in \mathbb{R}^3,$$

which is simply to say that f solves the Vlasov equation if and only if it is constant along characteristics. In particular, this implies that $f(t)$ is again a nonnegative C^1-function of compact support, and we get the a-priori bound

$$\|f(t)\|_\infty = \|\mathring{f}\|_\infty$$

for free. Actually more is true: Since the right hand side of the characteristic system is divergence-free when viewed as a vector field on phase space it follows that the characteristic flow is volume-preserving,

$$\det \frac{\partial(X,V)}{\partial(x,v)}(s,t,x,v) = 1,$$

a fact known as Liouville's Theorem. Via transformation of variables this immediately implies that

$$\|f(t)\|_p = \left(\int \Big|\mathring{f}\big((X,V)(0,t,x,v)\big)\Big|^p dv\,dx\right)^{1/p} = \|\mathring{f}\|_p$$

for all $p \geq 1$, in particular

$$\|\rho(t)\|_1 = \int f(t,x,v)\,dv\,dx = \|\mathring{f}\|_1$$

which is conservation of mass or charge respectively. Note that all of these bounds hold for both the plasma physics and the gravitational cases and come from the Vlasov equation alone.

Next we discuss a-priori bounds based on conservation of energy. Define the kinetic and potential energy of a solution by

$$E_{\rm kin}(t) := \frac{1}{2}\int v^2 f(t,x,v)\,dv\,dx,$$

$$E_{\rm pot}(t) := -\frac{\gamma}{8\pi}\int |\partial_x U(t,x)|^2 dx = \frac{1}{2}\int U(t,x)\,\rho(t,x)\,dx.$$

It is a simple calculation to check that the total energy of the system is conserved:

$$E_{\rm kin}(t) + E_{\rm pot}(t) = const.$$

If we consider the plasma physics case $\gamma = -1$ the potential energy is nonnegative so that the kinetic energy remains bounded:

$$E_{\rm kin}(t) \leq C,\ t \geq 0; \tag{2.1}$$

constants which depend only on the initial condition $\mathring{f}$ are always denoted by C and may change their value from line to line. We want to use this bound on the kinetic energy to obtain an L^p-bound on $\rho(t)$ with $p > 1$. To do so, observe that

$$\begin{aligned}\rho(t,x) &= \int_{|v|\leq R} f(t,x,v)\,dv + \int_{|v|\geq R} f(t,x,v)\,dv \\ &\leq \frac{4\pi}{3}R^3\|\mathring{f}\|_\infty + \frac{1}{R^2}\int v^2 f(t,x,v)\,dv = C\left(\int v^2 f(t,x,v)\,dv\right)^{3/5}\end{aligned}$$

where we have chosen

$$R = \left(\int v^2 f(t,x,v)\,dv\right)^{1/5}.$$

Taking both sides of this estimate to the power $5/3$ and integrating with respect to x yields the a-priori bound

$$\|\rho(t)\|_{5/3} \leq C. \tag{2.2}$$

So far, both (2.1) and (2.2) hold only for the case $\gamma = -1$. However, as Horst observed in [12], there exists some constant $C > 0$ such that for all $t \in [0,T[$,

$$\left|E_{\rm pot}(t)\right| \leq C E_{\rm kin}(t)^{1/2}. \tag{2.3}$$

Together with conservation of energy this implies that the bound on the kinetic energy and as a consequence also (2.2) hold for the gravitational case $\gamma = 1$ as well. From this point on, the two cases never need to be distinguished again since all a-priori bounds which enter into the global existence proof hold in both cases.

To conclude this subsection we briefly explain how the key estimate (2.3) is obtained: Using Hölder's inequality and the generalized Young's inequality we obtain

$$\begin{aligned}\left|E_{\rm pot}(t)\right| &= \frac{1}{2}\int \frac{\rho(t,x)\,\rho(t,y)}{|x-y|}\,dy\,dx \le \|\rho(t)\|_{6/5}\,\left\|\rho(t) * |\cdot|^{-1}\right\|_6 \\ &\le C\|\rho(t)\|_{6/5}^2\,\left\||\cdot|^{-1}\right\|_{3,w} \le C\|\rho(t)\|_{6/5}^2.\end{aligned}$$

This inequality is known as Sobolev's inequality, cf. [18, p. 31], for the generalized Young's inequality and the definition of the weak L^p-norm $\|\cdot\|_{p,w}$ cf. [18]. Now an interpolation argument analogous to the one proving (2.2) yields the estimate

$$\|\rho(t)\|_{6/5} \le C E_{\rm kin}(t)^{1/4},$$

and combining this with Sobolev's inequality proves the estimate (2.3).

2.3. A Gronwall argument. In order to prove global existence it is sufficient to get a bound on the quantity $P(t)$ introduced in Theorem 2.1. In the following we will derive a Gronwall inequality for this quantity which will not yet be strong enough to yield a global bound on $P(t)$, but it will point towards the possible improvements which in the next section will result in a global bound on $P(t)$. First note that by Hölder's inequality,

$$\begin{aligned}\left|\partial_x U(t,x)\right| &\le \int \frac{\rho(t,y)}{|x-y|^2}dy \le \int_{|x-y|\le R}\frac{\rho(t,y)}{|x-y|^2}\,dy + \int_{|x-y|\ge R}\frac{\rho(t,y)}{|x-y|^2}\,dy \\ &\le \|\rho(t)\|_\infty \int_{|x-y|\le R}\frac{dy}{|x-y|^2} + \|\rho(t)\|_{5/3}\left(\int_{|x-y|\ge R}|x-y|^{-5}dy\right)^{2/5} \\ &= C\,R\,\|\rho(t)\|_\infty + C\,R^{-4/5}\,\|\rho(t)\|_{5/3},\end{aligned}$$

and choosing $R = \|\rho(t)\|_{5/3}^{5/9}\,\|\rho(t)\|_\infty^{-5/9}$ we have the estimate

$$\|\partial_x U(t)\|_\infty \le C\,\|\rho(t)\|_{5/3}^{5/9}\,\|\rho(t)\|_\infty^{4/9}. \tag{2.4}$$

Using the a-priori bound (2.2) and the fact that $\rho(t,x) \le CP(t)^3$ we obtain the Gronwall inequality

$$\begin{aligned}P(t) &\le P(0) + \int_0^t \|\partial_x U(s)\|_\infty ds \le P(0) + C\int_0^t \|\rho(s)\|_\infty^{4/9} ds \\ &\le P(0) + C\int_0^t P(s)^{4/3} ds\end{aligned}$$

which yields only a local-in-time bound on $P(t)$. One way to improve this argument is to observe that an a-priori bound on a higher order L^p-norm of $\rho(t)$ allows for a smaller power of the L^∞-norm of $\rho(t)$ in the estimate (2.4) and thus for a smaller power of $P(s)$ in the Gronwall inequality for P. One easily checks that an a-priori bound for $\|\rho(t)\|_2$ is sufficient to obtain a Gronwall inequality for P which yields a global bound on this quantity. Next one observes that $\|\rho(t)\|_2$ can be bounded by a velocity moment of f of sufficiently high order. This route was taken by Lions and Perthame in [15], where they established an a-priori bound for

$$\int |v|^m f(t,x,v)\,dv\,dx$$

for every $m \in]3,6[$. Pfaffelmoser's approach is based on a different observation: Let $(X,V)(t)$ be a fixed characteristic along which we want to estimate the increase in velocity. Then

$$|V(t) - V(t-\Delta)| \leq \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|y - X(s)|^2} dw\, dy\, ds. \tag{2.5}$$

In the above Gronwall argument we first split x-space to obtain the estimate (2.4) and then split v-space to obtain the estimate (2.2). In essence, Pfaffelmoser's idea is that instead of doing one after the other one should rather split (x,v)-space in (2.5) simultaneously to get a bound on the increase in velocity. The details of Schaeffer's version of this idea are given in the next section.

3. Schaeffer's proof of global existence. For technical reasons we redefine the quantity $P(t)$ and make it nondecreasing:

$$P(t) := \max\Big\{|v| \mid (x,v) \in \mathrm{supp} f(s),\ 0 \leq s \leq t\Big\}.$$

We want to show that this function is bounded on bounded time intervals. Let $(X,V)(t)$ be a fixed characteristic in $\mathrm{supp} f$, that is, $(X,V)(0) \in \mathrm{supp}\mathring{f}$, and take $0 \leq \Delta \leq t < T$. After applying the transformation of variables

$$y = X(s,t,x,v),\ \ w = V(s,t,x,v) \tag{3.1}$$

and using the fact that the characteristic flow is volume preserving we can write (2.5) as

$$|V(t) - V(t-\Delta)| \leq \int_{t-\Delta}^{t} \iint \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dv\, dx\, ds. \tag{3.2}$$

For parameters $0 < p \leq P(t)$ and $r > 0$, which are to be specified later, we split the domain of integration in the last integral into the following sets:

$$M_1 := \Big\{(s,x,v) \in [t-\Delta,t] \times \mathbb{R}^6 \Big| |v| \leq p\ \vee\ |v - V(t)| \leq p\Big\},$$

$$M_2 := \Big\{(s,x,v) \in [t-\Delta,t] \times \mathbb{R}^6 \Big| |v| > p\ \wedge\ |v - V(t)| > p$$
$$\wedge \Big[|X(s,t,x,v) - X(s)| \leq r|v|^{-3}\ \vee\ |X(s,t,x,v) - X(s)| \leq r|v - V(t)|^{-3}\Big]\Big\},$$

$$M_3 := \Big\{(s,x,v) \in [t-\Delta,t] \times \mathbb{R}^6 \Big| |v| > p\ \wedge\ |v - V(t)| > p$$
$$\wedge\ |X(s,t,x,v) - X(s)| > r|v|^{-3}\ \wedge\ |X(s,t,x,v) - X(s)| > r|v - V(t)|^{-3}\Big\}.$$

To estimate the contribution of each of these sets to the integral in (3.2) we have to choose the length of the time interval $[t-\Delta,t]$ such that velocities do not change to much on this interval. Recall from the previous section that

$$\|\partial_x U(t)\|_\infty \leq C^* P(t)^{4/3},\ \ t \in [0,T[,$$

for some $C^* > 0$ so if we choose

$$\Delta := \min\Big\{t, \frac{p}{4C^* P(t)^{4/3}}\Big\} \tag{3.3}$$

then

$$\Big|V(s,t,x,v) - v\Big| \leq \Delta C^* P(t)^{4/3} \leq \frac{1}{4}p,\ \ s \in [t-\Delta, t],\ \ x, v \in \mathbb{R}^3. \tag{3.4}$$

The contribution of M_1: For $(s,x,v) \in M_1$ we have, by (3.4),

$$|w| < 2p\ \vee\ |w - V(s)| < 2p.$$

Together with the transformation of variables (3.1) this implies the estimate

$$\int_{M_1} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dv\, dx\, ds \leq \int_{t-\Delta}^{t} \int \frac{\tilde{\rho}(s,y)}{|y - X(s)|^2} dy\, ds,$$

where

$$\tilde{\rho}(s,y) := \int_{|w|<2p \vee |w-V(s)|<2p} f(s,y,w)\, dw \leq Cp^3,$$

and also

$$\|\tilde{\rho}(s)\|_{5/3} \leq \|\rho(s)\|_{5/3} \leq C.$$

Therefore, the estimate (2.4) implies that

$$\int_{M_1} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dv\, dx\, ds \leq Cp^{4/3}\Delta. \tag{3.5}$$

The contribution of M_2: For $(s,x,v) \in M_2$ we have, by (3.4),

$$\begin{aligned} &\frac{1}{2}p < |w| < 2|v|\ \wedge\ \frac{1}{2}p < |w - V(s)| < 2|v - V(t)| \\ &\qquad\qquad\qquad \wedge\ \Big[|y - X(s)| < 8r|w|^{-3}\ \vee\ |y - X(s)| < 8r|w - V(s)|^{-3}\Big]. \end{aligned}$$

Thus by (3.1),

$$\begin{aligned} &\int_{M_2} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dv\, dx\, ds \\ &\qquad \leq \int_{t-\Delta}^{t} \int_{\frac{1}{2}p<|w|<2P(t)} \int_{|y-X(s)|<8r|w|^{-3}} \frac{f(s,y,w)}{|y - X(s)|^2} dw\, dy\, ds \\ &\qquad\quad + \int_{t-\Delta}^{t} \int_{\frac{1}{2}p<|w-V(s)|<2P(t)} \int_{|y-X(s)|<8r|w-V(s)|^{-3}} \frac{f(s,y,w)}{|y - X(s)|^2} dw\, dy\, ds \\ &\qquad \leq Cr \ln \frac{4P(t)}{p} \Delta. \end{aligned} \tag{3.6}$$

To obtain this estimate we used the fact that $\|f(s)\|_\infty = \|\mathring{f}\|_\infty$ and then simply calculated the remaining intergrals, integrating first with respect to y and then with respect to w.

The contribution of M_3: The main idea in estimating the contribution of the set M_3 is to integrate with respect to time first and to use the fact that on M_3 the distance of $X(s,t,x,v)$ from $X(s)$ can be bounded from below linearly in time. Let $(x,v) \in \mathbb{R}^6$ with $|v - V(t)| > p$ and define

$$d(s) := X(s,t,x,v) - X(s),\ \ s \in [t-\Delta, t].$$

We estimate this difference from below by Taylor-expanding it to first order around the point $s_0 \in [t-\Delta, t]$ which is defined by

$$|d(s_0)| = \min\big\{|d(s)|\ |\ t - \Delta \leq s \leq t\big\}.$$

To this end, define

$$\bar{d}(s) := d(s_0) + (s - s_0)\dot{d}(s_0),\ s \in [t - \Delta, t].$$

Then

$$d(s_0) = \bar{d}(s_0),\ \dot{d}(s_0) = \dot{\bar{d}}\ (s_0),$$

and

$$|\ddot{d}(s) - \ddot{\bar{d}}\ (s)| = |\dot{V}(s, t, x, v) - \dot{V}(s)| \le 2\|\partial_x U(s)\|_\infty \le 2C^* P(t)^{4/3}.$$

Therefore,

$$\begin{aligned} |d(s) - \bar{d}(s)| &\le C^* P(t)^{4/3}(s - s_0)^2 \le C^* P(t)^{4/3}\Delta |s - s_0| \\ &\le \frac{1}{4} p\,|s - s_0| < \frac{1}{4}|v - V(t)||s - s_0|. \end{aligned} \tag{3.7}$$

Since by (3.4),

$$|\dot{d}(s_0)| = |V(s_0, t, x, v) - V(s_0)| \ge |v - V(t)| - \frac{1}{2}p > \frac{1}{2}|v - V(t)|,$$

and

$$(s - s_0)d(s_0) \cdot \dot{d}(s_0) \ge 0$$

—the latter estimate follows from the definition of s_0 by distinguishing the cases $s_0 = t - \Delta$, $s_0 \in]t - \Delta, t[$, and $s_0 = t$—, we have for all $s \in [t - \Delta, t]$ the estimate

$$|\bar{d}(s)|^2 \ge \frac{1}{4}|v - V(t)|^2 |s - s_0|^2,$$

and combining this with (3.7) finally implies that the estimate

$$|d(s)| \ge \frac{1}{4}|v - V(t)||s - s_0| \tag{3.8}$$

holds for all $s \in [t - \Delta, t]$ and $(x, v) \in \mathbb{R}^6$ with $|v - V(t)| > p$. Now define the functions

$$\sigma_1(\xi) := \begin{cases} \xi^{-2} & , \xi > r|v|^{-3} \\ \left(r|v|^{-3}\right)^{-2} & , \xi \le r|v|^{-3} \end{cases},$$

$$\sigma_2(\xi) := \begin{cases} \xi^{-2} & , \xi > r|v - V(t)|^{-3} \\ \left(r|v - V(t)|^{-3}\right)^{-2} & , \xi \le r|v - V(t)|^{-3} \end{cases}.$$

The definition of M_3, the fact that the functions σ_i are nonincreasing and the estimate (3.8) imply that

$$|d(s)|^{-2} 1_{M_3}(s, x, v) \le \sigma_i(|d(s)|) \le \sigma_i\left(\frac{1}{4}|v - V(t)||s - s_0|\right)$$

for $i = 1, 2$ and $s \in [t - \Delta, t]$; 1_M denotes the characteristic function of the set M. We can now estimate the time integral in the contribution of M_3 in the following way:

$$\begin{aligned} \int_{t-\Delta}^{t} |d(s)|^{-2} 1_{M_3}(s, x, v) ds &\le 8\,|v - V(t)|^{-1} \int_0^\infty \sigma_i(\xi)\, d\xi \\ &= 16 |v - V(t)|^{-1} \begin{cases} r^{-1}|v|^3 & , i = 1 \\ r^{-1}|v - V(t)|^3 & , i = 2 \end{cases}, \end{aligned}$$

and since this estimate holds for both $i = 1$ and $i = 2$ we have

$$\int_{t-\Delta}^{t} |d(s)|^{-2} 1_{M_3}(s,x,v) ds \leq 16 r^{-1} |v - V(t)|^{-1} \min\{|v|^3, |v - V(t)|^3\} \leq 16 r^{-1} |v|^2.$$

Therefore,

$$\int_{M_3} \frac{f(t,x,v)}{|X(s,t,x,v) - X(s)|^2} dv\, dx\, ds \leq \iint f(t,x,v) \int_{t-\Delta}^{t} |d(s)|^{-2} 1_{M_3}(s,x,v)\, ds\, dv\, dx$$
$$\leq C r^{-1} \iint v^2 f(t,x,v)\, dv\, dx \leq C r^{-1} \tag{3.9}$$

by the boundedness of the kinetic energy. Combining the estimates (3.5), (3.6), (3.9) we obtain the estimate

$$|V(t) - V(t-\Delta)| \leq C\Big(p^{4/3} + r \ln \frac{4P(t)}{p} + r^{-1}\Delta^{-1}\Big)\, \Delta$$
$$\leq C\Big(p^{4/3} + r \ln \frac{4P(t)}{p} + r^{-1} p^{-1} P(t)^{4/3}\Big)\, \Delta\ .$$

Now we choose our parameters p and r in such a way that all three terms in the sum on the right hand side of this estimate become equal, which is the case if

$$p = P(t)^{4/11},\ r = P(t)^{16/33} \ln^{-1/2} \frac{4P(t)}{p},$$

and we conclude that

$$|V(t) - V(t-\Delta)| \leq C P(t)^{16/33} \ln^{1/2} P(t)\, \Delta;$$

without loss of generality we assume that $P(t) \geq 1$ so that $p \leq P(t)$, otherwise we can replace $P(t)$ by $P(t) + 1$. Thus, for any $\epsilon > 0$ there exists a constant $C > 0$ such that

$$|V(t) - V(t-\Delta)| \leq C P(t)^{16/33+\epsilon} \Delta. \tag{3.10}$$

It is now tempting to replace the difference in V by the corresponding difference in P, divide by Δ, let $\Delta \to 0$, and use the resulting differential inequality for $P(t)$ to derive a bound for $P(t)$. However, Δ is defined in (3.3) as a function of t, so we cannot let it go to zero and have to argue with the difference inequality (3.10) directly. Since $P(t)$ is nondecreasing there exists a unique $T^* \in]0, T]$ such that

$$\Delta(t) = t,\ t < T^*,\ \Delta(t) = \frac{1}{4C^*} P(t)^{-32/33},\ t \geq T^*.$$

Clearly, (3.10) implies that

$$|V(t) - V(0)| \leq C P(t)^{16/33+\epsilon} t,\ t < T^*.$$

Let $t \geq T^*$ and define $t_0 := t$ and $t_{i+1} := t_i - \Delta(t_i)$ as long as $t_i \geq T^*$. Since

$$t_i - t_{i+1} = \Delta(t_i) \geq \Delta(t_0)$$

there exists $k \in \mathbb{N}$ such that

$$t_k < T^* \leq t_{k-1} < \cdots < t_0 = t.$$

Repeated application of (3.10) yields

$$|V(t) - V(0)| \leq |V(t) - V(t_k)| + |V(t_k) - V(0)|$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{k} |V(t_{i-1}) - V(t_i)| + C t_k P(t)^{16/33+\epsilon} \\
&\leq C P(t)^{16/33+\epsilon} \left(t_k + \sum_{i=1}^{k} (t_{i-1} - t_i) \right) \\
&= C P(t)^{16/33+\epsilon} t.
\end{aligned}
$$

By definition of P this implies that

$$P(t) \leq P(0) + C P(t)^{16/33+\epsilon} t$$

so that for any $\delta > 0$ there exists a constant $C > 0$ such that

$$P(t) \leq C(1+t)^{33/17+\delta},\ t \in [0, T[,$$

and by Theorem 2.1 the proof is complete.

Remarks. The proof above yields an explicit bound on $P(t)$ and thus also on $\|\rho(t)\|_\infty$. Using slightly more sophisticated estimates one can obtain

$$P(t) \leq C(1+t)^{1+\delta},\ \|\rho(t)\| \leq C(1+t)^{3+\delta},\ t \geq 0,$$

for any $\delta > 0$. This refinement is due to Horst [13].

With one exception all the properties of the Vlasov–Poisson system which were exploited in the above proof have a clear physical interpretation: we made use of the conservation of energy and—in various forms—of the conservation of phase space volume. The one exception is the estimate (2.3) whose physical content is not apparent to us but without which the proof would go through only in the plasma physics case.

Due to (2.3) the above proof goes through and gives the same bounds for both the plasma physics and the gravitational cases. The latter fact is unsatisfactory, since in the plasma physics case the particles repulse each other—in the case of several particle species with charges of different sign this is true at least on the average—and thus the density should decay as $t \to \infty$. Results in this direction have been obtained in [14], where it has been shown that $\|\rho(t)\|_{5/3}$ decays. Together with other decay estimates in [14] this can be used to improve the bound on $\|\rho(t)\|_\infty$: In the plasma physics case this quantity can grow at most as t^2, cf. [21].

In the gravitational case and under the assumption of spherical symmetry both $P(t)$ and $\|\rho(t)\|_\infty$ are bounded uniformly in t, cf. [2]. If the initial datum satisfies a certain smallness condition then $P(t)$ is again bounded uniformly in t and the density decays: $\|\rho(t)\|_\infty \leq C t^{-3}$, cf. [1]

4. Limitations to global existence. In this section we want to show that the global existence proof can not necessarily be carried over to closely related systems. Consider the following, so-called relativistic Vlasov–Poisson system:

$$\partial_t f + \frac{v}{\sqrt{1+v^2}} \cdot \partial_x f - \partial_x U \cdot \partial_v f = 0,$$

$$\triangle U = 4\pi\rho,\ \rho(t,x) = \int f(t,x,v)\, dv.$$

Here the particles move according to the special relativistic equations of motion

$$\dot{x} = \frac{v}{\sqrt{1+v^2}},\ \dot{v} = -\partial_x U(t,x).$$

The rest mass of the particles as well as the speed of light is set to unity, and v should now be interpreted as the momentum of the particles. Since the particles still interact via the non-relativistic gravitational field determined by the Poisson equation, this system is neither Galilei- nor Lorentz-invariant, it can be viewed as in some sense intermediate between the Vlasov–Poisson and the Vlasov–Einstein system.

If one examines the a-priori bounds from the previous section for the case of the relativistic Vlasov–Poisson system one finds that the bounds based on the characteristics remain valid, also conservation of energy still holds, with

$$E_{\text{kin}}(t) := \int \sqrt{1+v^2}\, f(t,x,v)\, dv\, dx$$

defining the kinetic energy, but it is no longer possible to bound the potential energy by some root of the kinetic energy so that there is no longer an a-priori bound for the latter. Actually, not only does the proof for global existence no longer go through but for certain initial data the solutions blow-up in finite time:

THEOREM 4.1. *Let $\mathring{f} \in C_c^1(\mathbb{R}^6)$, $\mathring{f} \geq 0$, be spherically symmetric, i. e., $\mathring{f}(Ax,Av) = \mathring{f}(x,v)$ for all $x, v \in \mathbb{R}^3$ and $A \in \mathrm{SO}\,(3)$. If the total energy of $\mathring{f}$ is negative then the corresponding solution of the relativistic Vlasov–Poisson system blows up in finite time.*

This result is due to Glassey and Schaeffer [9], and we sketch the main idea of the proof: First one can check by direct computation that every solution satisfies the so-called dilation identity

$$\frac{d}{dt}\int x\cdot v\, f(t,x,v)\, dv\, dx = E(t) - \int \frac{1}{\sqrt{1+v^2}} f(t,x,v)\, dv\, dx,$$

where $E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t)$ denotes the total energy of the system, which is conserved. In passing we note that the dilation identity is related to the following scaling invariance of the system: For every $\lambda > 0$ the transformation

$$f_\lambda(t,x,v) = \lambda^2 f(\lambda t, \lambda x, v)$$

turns solutions f of the system into new solutions f_λ. Integration of the dilation identity yields the estimate

$$\int x\cdot v\, f(t,x,v)\, dv\, dx \leq \int x\cdot v\, \mathring{f}(x,v)\, dv\, dx + E(0)t.$$

Next observe that

$$\begin{aligned}\frac{d}{dt}\int x^2\sqrt{1+v^2}\, f(t,x,v)\, dv\, dx &= 2\int x\cdot v f\, dv\, dx - \int x^2 \partial_x U\cdot j\, dx\\ &\leq 2\int x\cdot v\, \mathring{f}\, dv\, dx + 2E(0)t - \int x^2\partial_x U\cdot j\, dx\end{aligned}$$

where

$$j(t,x) = \int \frac{v}{\sqrt{1+v^2}} f(t,x,v)\, dv$$

denotes the mass current density. Due to the spherical symmetry

$$|\partial_x U(t,x)| = \frac{4\pi}{r^2}\int_0^r s^2 \rho(t,s)\,ds, \tag{4.1}$$

where $r = |x|$, and the estimate

$$x^2 |\partial_x U(t,x)| \le x^2 \frac{M}{x^2} = M$$

holds, where $M = \|\mathring{f}\|_1$ denotes the total mass of the system. Since in addition

$$\int |j(t,x)|\,dx \le \int \rho(t,x)\,dx = M$$

we obtain the estimate

$$\frac{d}{dt}\int x^2 \sqrt{1+v^2}\, f(t,x,v)\,dv\,dx \le 2\int x\cdot v\, \mathring{f}\,dv\,dx + 2E(0)t + M^2,$$

and after another integration in time this implies that

$$\int x^2\sqrt{1+v^2}\, f(t,x,v)\,dv\,dx \le \int x^2 \sqrt{1+v^2}\,\mathring{f}\,dv\,dx + \left(2\int x\cdot v\,\mathring{f}\,dv\,dx + M^2\right)t + E(0)t^2.$$

Since the left hand side of this estimate is nonnegative and $E(0)$ is negative by assumption this gives an explicit upper bound on the lifetime of the solution, and the proof is complete.

There is a simple argument which shows where the solution blows up. The first thing to note is that a result analogous to Theorem 2.1 also holds for the relativistic Vlasov–Poisson system. In particular, ρ has to blow up in the L^∞-norm. Now assume there exists an arbitrarily small radius $\epsilon > 0$ such that ρ is bounded on $[0,T[\times B_\epsilon(0)$. By (4.1) this yields the estimates

$$|\partial_x U(t,x)| \le \frac{M}{r^2} \le \frac{M}{\epsilon^2},\ r \ge \epsilon,$$

and

$$|\partial_x U(t,x)| \le \frac{4\pi}{3} r \sup\Big\{\rho(t,x) \mid (t,x) \in [0,T[\times B_\epsilon(0)\Big\},\ r \le \epsilon,$$

i. e., $\partial_x U$ is uniformly bounded on the existence interval of the solution which immediately implies that $P(t)$ is bounded on $[0,T[$, a contradiction. Thus we have the following corollary to the theorem above:

Corollary 4.2. *Let $\mathring{f}$ be as in Theorem 4.1. Then*

$$\sup\Big\{\rho(t,x) \mid t \in [0,T[,\ |x| \le \epsilon\Big\} = \infty$$

for every $\epsilon > 0$.

The solution therefore blows up at the center of symmetry. This should be compared with a corresponding result for the Vlasov–Einstein system, which says that if a spherically symmetric solution developes a singularity, the first one will be at the center, cf. A. Rendall's lecture notes.

5. Extensions of the global existence result. We have seen above that the global existence result for the Vlasov–Poisson system cannot be extended to the relativistic Vlasov–Poisson system, at least not in the gravitational case, which is the one of interest here. For the plasma physics case of the relativistic Vlasov–Poisson system as well as for the Vlasov–Maxwell system the question of global existence of classical solutions in three dimensions is still open, partial results are known for small initial data and related situations, cf. [19] and the references therein.

A different direction into which one may wish to extend Theorem 1.1 is to take collisional effects into account. One way to do this is to include so-called Fokker–Planck terms into the Vlasov equation which can be considered as a linear approximation to the Boltzmann collision term. For the resulting Vlasov–Fokker–Planck–Poisson system Bouchut has proven global existence of classical solutions, based on the proof by Lions and Perthame for the Vlasov–Poisson system, cf. [7].

So far, all the results which we discussed referred to the case of an isolated system, that is, the boundary condition (1.5) has always been assumed. However, in the gravitational case one can also consider cosmological solutions. For a nonnegative, compactly supported function $H \in C^1_c(\mathbb{R})$ we set

$$f_0(t,x,v) := H\left(\left|a(t)v - \dot{a}(t)x\right|^2\right)$$

where a is a positive function to be determined later. Then

$$\rho_0(t,x) = \int f_0(t,x,v)\,dv = a^{-3}(t)\int H(v^2)\,dv,$$

and, after normalizing

$$\int H(v^2)\,dv = 1,$$

we obtain the homogeneous mass density

$$\rho_0(t) = a^{-3}(t),\ t \geq 0.$$

A solution of the Poisson equation with this source term is

$$U_0(t,x) := \frac{2\pi}{3}a^{-3}(t)\,x^2,\ t \geq 0,\ x \in \mathbb{R}^3,$$

and it remains to determine the function a in such a way that f_0 satisfies the Vlasov equation with force term

$$\partial_x U_0(t,x) = \frac{4\pi}{3}a^{-3}(t)\,x.$$

A short computation shows that this is the case if a is a solution of the differential equation

$$\ddot{a} + \frac{4\pi}{3}a^{-2} = 0, \tag{5.1}$$

which is the equation of radial motion in the gravitational field of a point mass. We assume that $a(0) = 1$ and $\dot{a}(0) > 0$. The solution of (5.1) exists on some right maximal interval $[0, T_a[$ where $T_a = \infty$ if

$$E_a := \frac{1}{2}\dot{a}(t)^2 - \frac{4\pi}{3}a^{-1}(t) \geq 0,$$

i. e., the energy, a conserved quantity along solutions of (5.1), is nonnegative. We want to consider solutions of the Vlasov–Poisson system of the form

$$f = f_0 + g,\ \rho = \rho_0 + \sigma,\ U = U_0 + W,$$

with the perturbations g, σ, W being spatially periodic. To this end it is useful to perform the following transformation of variables:

$$\tilde{x} = a^{-1}(t)\,x, \quad \tilde{v} = v - a^{-1}(t)\dot{a}(t)\,x, \quad \tilde{g}(t,\tilde{x},\tilde{v}) = g(t,x,v). \tag{5.2}$$

If we compute the system satisfied by g, σ, W in these transformed variables and afterwards drop the tildas we obtain the following version of the Vlasov–Poisson system which governs the time evolution of deviations from the homogeneous state:

$$\partial_t g + \frac{1}{a} v\cdot\partial_x g - \frac{1}{a}\Big(\partial_x W + \dot{a} v\Big)\cdot\partial_v g = 2aH'(a^2v^2)\,v\cdot\partial_x W, \tag{5.3}$$

$$\triangle W = 4\pi a^2\sigma, \tag{5.4}$$

$$\sigma(t,x) = \int g(t,x,v)\,dv. \tag{5.5}$$

If we define $Q := [0,1]^3$, $S := Q\times\mathbb{R}^3$, and

$$C^1_{p,c}(S) := \Big\{h\in C^1(\mathbb{R}^6)\mid h(x+\alpha,v) = h(x,v),\ x,v\in\mathbb{R}^3,\ \alpha\in\mathbb{Z}^3,$$
$$\exists u_0>0:\ h(x,v)=0,\ |v|\geq u_0\Big\}$$

then the following extension of Theorem 1.1 to the cosmological case holds, cf. [23]:

Theorem 5.1. *For every initial datum* $\mathring{g}\in C^1_{p,c}(S)$ *with* $\int_S \mathring{g} = 0$ *and* $\mathring{g}(x,v)+H(v^2)\geq 0$ *for* $x,v\in\mathbb{R}^3$ *there exists a unique solution* $g\in C^1([0,T_a[\times\mathbb{R}^6)\cap C([0,T_a[;C^1_{p,c}(S))$ *of the system* (5.3), (5.4), (5.5) *with* $g(0)=\mathring{g}$. *In particular, the solution is global, if* $E_a\geq 0$.

We do not give a proof of this result here but indicate how the a-priori bounds have to be modified in the cosmological case. First the fact that the Vlasov equation (5.3) for g is now inhomogeneous implies that g is no longer constant along characteristics, but is given by

$$g(t,z) = \mathring{g}\big((X,V)(0,t,x,v)\big) + H\big(V^2(0,t,x,v)\big) - H\big(a^2(t)v^2\big),\ t\geq 0,\ (x,v)\in\mathbb{R}^6,$$

and this implies the a-priori estimate

$$\|g(t)\|_\infty \leq \|\mathring{g}\|_\infty + 2\|H\|_\infty.$$

The solutions of the characteristic system

$$\dot{x} = \frac{1}{a(s)}v,\ \dot{v} = -\frac{1}{a(s)}\Big(\partial_x W(s,x) + \dot{a}(s)v\Big).$$

do no longer define a volume-preserving flow, but

$$\det\frac{\partial(X,V)}{\partial(x,v)}(s,t,x,v) = \left(\frac{a(t)}{a(s)}\right)^3,$$

and this implies that

$$\|g(t)\|_1 \leq \big(\|\mathring{g}\|_1 + 2\big)\,a^{-3}(t).$$

Thus, the a-priori bounds on the L^1- and L^∞-norm of $g(t)$ still hold. Next we define

$$E_{\mathrm{kin}}(t) := \frac{1}{2}\int_S v^2 f(t,x,v)\,dv\,dx,\ \ E_{\mathrm{pot}}(t) := \frac{1}{2}\int_Q W(t,x)\sigma(t,x)\,dx,$$

where $f = f_0 + g$, a function which is nonnegative as opposed to g. A short computation shows that

$$a^5(t)E(t) = E(0) + \int_0^t \dot{a}(s)a^4(s)E_{\mathrm{pot}}(s)ds,\ \ t \geq 0,$$

where $E := E_{\mathrm{kin}} + E_{\mathrm{pot}}$ denotes the total energy. This identity serves as the replacement for the conservation of energy and can be used in connection with the analogue of Horst's estimate (2.3) to obtain the bounds

$$E_{\mathrm{kin}}(t) \leq Ca^{-3}(t),\ \ \|\sigma(t)\|_{5/3} \leq Ca^{-9/5}(t)$$

provided $E_a \geq 0$, and

$$E_{\mathrm{kin}}(t) \leq Ca^{-5}(t),\ \ \|\sigma(t)\|_{5/3} \leq Ca^{-3}(t)$$

provided $E_a < 0$. Starting from these a-priori bounds the global existence proof of Schaeffer can now be adapted to the cosmological case. For the details we refer to [23].

Final remark. We have restricted our review to the question of global existence of solutions to the initial value problem for the Vlasov–Poisson system in three dimensions, but we want at least to mention that there are many results on other aspects of this system such as the existence and properties of stationary solutions [4], stability and instability of stationary solutions [5, 6, 10, 11, 20, 22], asymptotic properties of solutions [3], and numerical simulation schemes [8]. Further references can be found in the cited articles.

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