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MULTIPLIER HOPF ALGEBRAS AND DUALITY

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Abstract. We define a category containing the discrete quantum groups (and hence the discrete groups and the duals of compact groups) and the compact quantum groups (and hence the compact groups and the duals of discrete groups). The dual of an object can be defined within the same category and we have a biduality theorem. This theory extends the duality between compact quantum groups and discrete quantum groups (and hence the one between compact abelian groups and discrete abelian groups).

The objects in our category are multiplier Hopf algebras, with invertible antipode, admitting invariant functionals (integrals), satisfying some extra condition (to take care of the non-abelianness of the underlying algebras). If we start with a multiplier Hopf *-algebra with positive invariant functionals, then also the dual is a multiplier Hopf *-algebra with positive invariant functionals. This makes it possible to formulate this duality also within the framework of C*-algebras.

1. Introduction. Let (A, Δ) be a finite-dimensional Hopf algebra (over $\mathbb{C}$). The dual space A' of A is again a finite-dimensional Hopf algebra if the product and the coproduct are defined in the usual way by

$$(fg)(a) = (f \otimes g)(\Delta(a)), \qquad (\Delta(f))(a \otimes b) = f(ab)$$

whenever $f, g \in A'$ and $a, b \in A$. It is immediately clear from these formulas that conversely, (A, Δ) is the dual of (A', Δ).

This duality generalizes the dualtiy of finite abelian groups. Indeed, given a finite group G, there are two ways to associate a Hopf algebra to G. In the first case, we take the vector space of complex functions on G with pointwise multiplication. In this case, the comultiplication is defined by $(\Delta(f))(p, q) = f(pq)$ whenever f is a complex function on G and $p, q \in G$. In the second case, we take the same vector space of functions on G, but now with the convolution product. Here, the comultiplication is defined by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ when λ_p is the function defined as 1 on p and 0 elsewhere. These two Hopf algebras are

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dual to each other. Moreover, if G is abelian, then the Fourier transform will provide an isomorphism of the second Hopf algebra associated with G to the first one associated with the dual group $\widehat{G}$.

In this note, we present a natural extension of the above dualtity for finite-dimensional Hopf algebras to the infinite dimensional case. Our objects are certain multiplier Hopf algebras (see section 2). The dual of an object will again be an object of the same type. And the original object can be canonically identified with the dual of the dual object (biduality).

Whereas the duality for finite-dimensional Hopf algebras includes the duality of finite abelain groups, the duality of multiplier Hopf algebras that we obtain here, will include the Pontryagin duality between abelian discrete groups and abelian compact groups. In fact, more generally, it will include the duality between disrete quantum groups (as defined e.g. in [3, 11]) and compact quantum groups (as defined in [15, 16]). The main point is that all this is realized within the same category.

It also turns out to be possible to define the quantum double of Drinfel'd within this setting. Therefore, we obtain a class of objects that contains much more than simply the discrete and the compact quantum groups.

One can do harmonic analysis in this framework. The Fourier transform can be defined, the Fourier inversion formula can be proven, we have the Plancherel formula, convolution products, ...

If the original object is a multiplier Hopf *-algebra with a positive left invariant functional, then the dual is also a multiplier Hopf *-algebra and the right invariant functional will be positive. Moreover, in this case, the Drinfel'd double will also be a multiplier Hopf *-algebra with a positive left invariant functional. All this makes it possible to formulate the above duality within the framework of C*-algebras as well.

These notes essentially contain the content of my talk at the Workshop on Quantum Groups and Quantum Spaces (November 1995). Only the main results are given here. Also the necessary definitions are included, but no proofs. These will be published elsewhere (see [12, 13, 14]).

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2. Multiplier Hopf algebras. We first recall the notion of a multiplier Hopf algebra (see [10] for details).

Let A be an associative algebra over $\mathbb{C}$ with a non-degenerate product. Consider the tensor product $A \otimes A$ of A with itself. Let $M(A \otimes A)$ be the multiplier algebra of $A \otimes A$. It is the largest algebra with identity, in which $A \otimes A$ sits as an essential two-sided ideal. If e.g. A is the algebra $K(G)$ of complex functions with finite support in a set G (with pointwise operations), then $A \otimes A$ can be identified with $K(G \times G)$ and $M(A \otimes A)$ with the algebra $C(G \times G)$ of all complex functions on $G \times G$ (again with pointwise operations).

2.1. Definition. A *comultiplication* on A is a homomorphism $\Delta : A \to M(A \otimes A)$ such that

i) $(a \otimes 1)\Delta(b)$ and $\Delta(a)(1 \otimes b)$ belong to $A \otimes A$ for all $a, b \in A$,
ii) $(a \otimes 1 \otimes 1)(\Delta \otimes \iota)(\Delta(b)(1 \otimes c)) = (\iota \otimes \Delta)((a \otimes 1)\Delta(b))(1 \otimes 1 \otimes c)$ for all $a, b, c \in A$ (where ι denotes the identity map on A).

Remark that for algebras with identity, i) is automatic and ii) gives the usual coassociativity law $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

The following example will show that this notion is a natural generalization of the notion of a coproduct to the case of algebras without identity.

2.2. Example. Let G be a group and let, as before, A be the algebra $K(G)$ of complex functions with finite support in G with pointwise operations. Define $\Delta : A \to M(A \otimes A)$ by $(\Delta(f))(p, q) = f(pq)$ whenever $p, q \in G$. Then

$$((f \otimes 1)\Delta(g))(p, q) = f(p)g(pq), \qquad (\Delta(f)(1 \otimes g))(p, q) = f(pq)g(q)$$

for all $f, g \in A$ and $p, q \in G$. Then, it is easy to see that condition i) of 2.1 is satisfied. The coassociativity follows from the associativity of the product in G.

Remark that in this example, $\Delta(A) \not\subseteq A \otimes A$ when G is not finite. This is the reason why we have to work with the multiplier algebra $M(A \otimes A)$ here.

We see that, in this example, the linear maps from $A \otimes A$ to itself, defined by

$$f \otimes g \to (f \otimes 1)\Delta(g), \qquad f \otimes g \to \Delta(f)(1 \otimes g)$$

are bijective because they are dual to the maps $(p, q) \to (p, pq)$ and $(p, q) \to (pq, q)$ from $G \times G$ to itself. In fact, this is equivalent with the group property of the product in G.

This brings us to the following definition.

2.3. Definition. Let A be an algebra over $\mathbb{C}$ (with non-degenerate product) and let Δ be a comultiplication on A. We say that (A, Δ) is a *multiplier Hopf algebra* if the linear maps, defined from $A \otimes A$ to itself by

$$a \otimes b \to (a \otimes 1)\Delta(b), \qquad a \otimes b \to \Delta(a)(1 \otimes b),$$

are bijective.

We have the following results.

2.4. Proposition. *If (A, Δ) is a multiplier Hopf algebra, then there is a unique linear map $\epsilon : A \to \mathbb{C}$ (the counit) satisfying*

$$(\epsilon \otimes \iota)(\Delta(a)(1 \otimes b)) = ab, \qquad (\iota \otimes \epsilon)((a \otimes 1)\Delta(b)) = ab$$

for all $a, b \in A$. This map is a homomorphism. There is also a unique linear map S from A to the multiplier algebra $M(A)$ of A (the antipode) satisfying

$$m(S \otimes \iota)(\Delta(a)(1 \otimes b)) = \epsilon(a)b, \qquad m(\iota \otimes S)((a \otimes 1)\Delta(b)) = \epsilon(b)a$$

for all $a, b \in A$. Here, m denotes the product considered as a linear map from $M(A) \otimes A$ or $A \otimes M(A)$ to A. The antipode is an anti-homomorphism.

As an immediate consequence, we get the following.

2.5. Proposition. *If* (A, Δ) *is a multiplier Hopf algebra and if* A *has an identity, then it is a Hopf algebra.*

In fact, also conversely, any Hopf algebra is a multiplier Hopf algebra (the antipode is used to obtain the inverses of the maps in definition 2.3).

As for Hopf algebras, the antipode need not be invertible. If however $S(A) \subseteq A$ and S is bijective, we call the multiplier Hopf algebra regular. If A is abelian or coabelian, regularity is automatic. If A is a *-algebra and Δ is a *-homomorphism, then we call (A, Δ) a multiplier Hopf *-algebra. Also in this case, regularity is automatic.

3. Invariant functionals. Let (A, Δ) be a regular multiplier Hopf algebra.

For any linear functional ω on A and any element a in A, we can define elements $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ in $M(A)$ by e.g.

$$((\omega \otimes \iota)\Delta(a))b = (\omega \otimes \iota)(\Delta(a)(1 \otimes b)).$$

3.1. Definition. A linear functional φ on A is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for alle $a \in A$. A linear functional ψ on A is called *right invariant* if $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all $a \in A$.

If a (non-zero) left invariant functional φ exists, then it is unique (up to a scalar) and it is faithful (in the sense that $\varphi(ax) = 0$ for all x implies $a = 0$ and similarly, if $\varphi(xa) = 0$ for all x, then $a = 0$). Similarly for the right invariant functional. And of course, one is related with the other by applying the antipode.

Invariant functionals do not always exist. However, they do exist in many important cases. One can prove the existence of such functionals if A is finite-dimensional (finite quantum groups - see [1, 6, 8]), if A is semi-simple (discrete quantum groups - see [3, 11, 12]) and if A is a *-algebra with 1 and admits a C*-norm for which Δ is continuous (compact quantum groups - see [9, 15, 16]). In the general case, it is possible to formulate necessary and sufficient conditions on the comultiplication for non-trivial invariant functionals to exist.

Now assume that φ is a non-zero left invariant functional and that ψ is a non-zero right invariant functional.

The uniqueness leads to the following result.

3.2. Proposition. *There is an invertible element* δ *in* $M(A)$ *such that*

$$(\varphi \otimes \iota)\Delta(a) = \varphi(a)\delta, \qquad (\iota \otimes \psi)\Delta(a) = \psi(a)\delta^{-1}$$

for all a *in* A. *Moreover*, $\varphi(S(a)) = \varphi(a\delta)$ *for all* $a \in A$. *Similarly* $\psi(S^{-1}(a)) = \psi(a\delta^{-1})$ *for all* a.

This δ is the equivalent of the modular function on a locally compact (non-unimodular) group, relating the left and right Haar measures. This 'modular element' also behaves like the modular function. We have

$$\Delta(\delta) = \delta \otimes \delta, \quad \epsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}$$

(to be made precise by extending the maps Δ, ϵ and S to $M(A)$ in the obvious way).

It follows from the relation $\varphi(S(a)) = \varphi(a\delta)$ that the two sets of linear functionals

$$\{\varphi(a\,\cdot\,) \mid a \in A\}, \qquad \{\psi(a\,\cdot\,) \mid a \in A\}$$

are equal. The same is true for the sets

$$\{\varphi(\,\cdot\,a) \mid a \in A\}, \qquad \{\psi(\,\cdot\,a) \mid a \in A\}.$$

However, we need these four sets to be equal and then we will take this set of functionals as our dual. Only when they are all equal, this dual will be again a multiplier Hopf algebra.

This brings us to the following definition.

3.3. Definition. We say that φ satisfies the *weak K.M.S. condition* if for all $a \in A$ there is a $b \in A$ such that $\varphi(ax) = \varphi(xb)$ for all x in A.

With this condition, it will follow that the four sets of functionals above are all the same.

The name K.M.S. comes from physics (Kubo-Martin-Schwinger condition) and it is very common in the theory of operator algebras. In this algebraic setting, the weak K.M.S. condition gives an automorphism σ of A such that φ is σ-invariant and $\varphi(ab) = \varphi(b\sigma(a))$ for all a, b. It is a technical condition to deal with the problems arising from the non-commutativity of A.

Of course, in the abelian case, it is automatic. But also when A is finite-dimensional, it is always fulfilled, as well as in the other cases mentioned before (the discrete and the compact quantum groups). This is not so difficult to show (see [12]).

There are also a number of easy relations between σ and the other invariants Δ, ϵ, S, φ, ψ obtained so far. In [12], examples are worked out in detail to illustrate these different objects and the relations among them. In fact, remark that there is one more constant of the system: As $\varphi \circ S^2$ is again left invariant, we have a number $\tau \in \mathbb{C}$ such that $\varphi(S^2(a)) = \tau\varphi(a)$ for all $a \in A$.

4. Duality. Again, let (A, Δ) be a regular multiplier Hopf algebra. Now also assume that there are non-trivial invariant functionals satisfying the weak K.M.S. condition. In this section, we will construct the dual $(\widehat{A}, \widehat{\Delta})$. It will again be a multiplier Hopf algebra in the same category. And the dual of $(\widehat{A}, \widehat{\Delta})$ will again be (A, Δ).

Choose a non-zero left invariant functional φ and a non-zero right invariant functional ψ.

4.1. Notation. Set $\widehat{A} = \{\varphi(a\,\cdot\,) \mid a \in A\}$.

We have seen that elements in $\widehat{A}$ can also be represented in three other forms, namely $\varphi(\,\cdot\,a)$, $\psi(a\,\cdot\,)$ and $\psi(\,\cdot\,a)$. In general, these elements a will all be different but in any case, they are uniquely determined.

4.2. Proposition. *If we let* $(\omega_1\omega_2)(x) = (\omega_1 \otimes \omega_2)(\Delta(x))$ *whenever* $\omega_1, \omega_2 \in \widehat{A}$, *then* $\widehat{A}$ *becomes an associative algebra with a non-degenerate product.*

The main point is that, if $\omega_1 = \varphi(a_1\,\cdot\,)$ and $\omega_2 = \varphi(a_2\,\cdot\,)$, then

$$(\omega_1\omega_2)(x) = (\varphi \otimes \varphi)((a_1 \otimes a_2)\Delta(x))$$

for all x and if we write

$$a_1 \otimes a_2 = \sum_i (p_i \otimes 1)\Delta(q_i),$$

we find that $\omega_1\omega_2 = \varphi(c\,\cdot\,)$ where $c = \sum \varphi(p_i)q_i$.

4.3. Proposition. *We can define a comultiplication $\widehat{\Delta}$ on $\widehat{A}$ such that*

$$(\widehat{\Delta}(\omega_1)(1 \otimes \omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)((x \otimes 1)\Delta(y)),$$
$$((\omega_1 \otimes 1)\widehat{\Delta}(\omega_2))(x \otimes y) = (\omega_1 \otimes \omega_2)(\Delta(x)(1 \otimes y)),$$

where $\omega_1, \omega_2 \in \widehat{A}$ and $x, y \in A$. Again, $(\widehat{A}, \widehat{\Delta})$ will be a regular multiplier Hopf algebra.

To prove this result, we really must use the four different representations of elements in $\widehat{A}$. Of course, we also must use that (A, Δ) itself is a regular multiplier Hopf algebra. And as expected, the counit on $\widehat{A}$ is given by evaluation in 1 and the antipode on $\widehat{A}$ is just the adjoint of the antipode on A.

4.4. Proposition. *Define functionals $\widehat{\varphi}$ and $\widehat{\psi}$ on $\widehat{A}$ by*

$$\widehat{\varphi}(\omega) = \epsilon(a) \quad \text{when } \omega = \psi(a\,\cdot\,),$$
$$\widehat{\psi}(\omega) = \epsilon(a) \quad \text{when } \omega = \varphi(\,\cdot\,a).$$

Then $\widehat{\varphi}$ and $\widehat{\psi}$ are respectively left and right invariant functionals on $(\widehat{A}, \widehat{\Delta})$ and they satisfy the weak K.M.S. property.

Because we obtain a dual object of the same kind, we can consider the dual of $(\widehat{A}, \widehat{\Delta})$ again and the following lemma essentially indicates that this dual is again (A, Δ).

4.5. Lemma. *If $\omega \in \widehat{A}$ and $\omega = \varphi(\,\cdot\,a)$, then $\widehat{\psi}(\omega_1\omega) = \omega_1(S^{-1}(a))$ for all $\omega_1 \in \widehat{A}$.*

We can summarize these results in the following duality theorem.

4.6. Theorem. *If (A, Δ) is a regular multiplier Hopf algebra with non-trivial invariant functionals that are weakly K.M.S., then there exists a dual $(\widehat{A}, \widehat{\Delta})$ which is again a regular multiplier Hopf algebra with non-trivial invariant functionals satisfying the weak K.M.S. property. The dual of $(\widehat{A}, \widehat{\Delta})$ is canonically isomorphic with (A, Δ).*

This duality extends the duality between discrete and compact quantum groups (such as defined in [3, 11] and [15, 16]). In fact, it can be extended to a bigger class of objects (discrete type and compact type, see [12]). On the other hand, if we restrict to special cases, we get the Pontryagin duality between disrete abelian and compact abelian groups, the Tannaka-Krein duality for compact groups, ... and of course also the duality for finite-dimensional Hopf algebras that we described in the introduction.

Let us also briefly mention that it is possible to do harmonic analysis in this setting. The Fourier transform of $a \in A$ can be e.g. defined as $\varphi(a\,\cdot\,)$, the convolution product by $a * b = \sum \varphi(p_i)q_i$ when $a \otimes b = \sum (p_i \otimes 1)\Delta(q_i)$ (see the calculation after proposition 4.2), etc.

The multiplier Hopf algebras (A, Δ) and $(\widehat{A}, \widehat{\Delta})$ are paired in the sense that the multiplication of one algebra gives the comultiplication on the other one (see proposition 4.3). Just as for dual pairs of Hopf algebras, this provides the natural framework to construct the quantum double of Drinfel'd ([2, 7]).

If we make this construction for the pair $(A, \widehat{A})$, we get again a multiplier Hopf algebra with non-trivial invariant functionals satisfying the weak K.M.S. property. This is yet another indication that this setting is very well adapted for studying duality. And of course, the quantum double also gives highly non-trivial examples.

5. The C*-algebra framework. It will not be possible to cover e.g. the complete duality for locally compact abelian groups within a purely algebraic setting. A topological theory will be needed. In fact, the C*-algebra approach is the most natural one to consider here. Recently, such a theory was developed ([4]).

So, our algebraic framework will not cover all cases, but it has the advantage of being fairly simple and yet still rich enough to exhibit many interesting features, some of them not present in the classical situation.

Our duality theory also behaves very nicely with respect to the involutive structure. If we start with a multiplier Hopf *-algebra (A, Δ), then the dual $\widehat{A}$ will also be a *-algebra. The *-operation on $\widehat{A}$ is defined in the usual way by $\omega^*(x) = \omega(S(x)^*)^-$ whenever $\omega \in \widehat{A}$ and $x \in A$. Also $(\widehat{A}, \widehat{\Delta})$ is a multiplier Hopf *-algebra. If the left invariant functional φ is positive (i.e. $\varphi(a^*a) \geq 0$ for all a), then also the right invariant functional $\widehat{\psi}$ on the dual is positive.

It can be shown that in this case, when representing the algebra A on a Hilbert space, it is always represented by bounded operators. In particular, this is the case for the GNS-representation of A induced by φ. The comultiplication Δ turns out to be continuous for the C*-norm on A in this representation. In fact, there is also a maximal C*-norm on A and also for this norm, the comultiplication is continuous. It follows that in general, A can be completed to a C*-algebra B (in at least two ways) and that Δ extends to a comultiplication on this C*-algebra (now a *-homomorphism of B into the C*-multiplier algebra $M(B \overline{\otimes} B)$ of the appropriate C*-tensor product of B with itself). The fact that there are different compatible C*-norms is similar to the case of the group C*-algebra for non-amenable groups.

Also the quantum double is again a multiplier Hopf *-algebra in this case and if the left invariant functional is positive on A, the also the left invariant functional on the quantum double will be positive. Therefore, both the quantum double and its dual will, also within the framework of C*-algebras, give non-trivial examples of non-compact, non-discrete locally compact quantum groups. In this way, our theory gives an essential contribution, also to the C*-algebra approach to quantum groups (the locally compact quantum groups).

Finally, it is our belief, that a more general theory can be developed, close to our algebraic one, by making the correct topological modifications. If this turns out to be possible, it will be interesting to see how all this relates with the theory recently developed in [4].

Added in proof. Recently, we discovered that the weak K.M.S.-condition for invariant functionals on regular multiplier Hopf algebras, as defined in this paper, is automatically satisfied (see [12]).

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