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GROUPOIDS AND COMPACT QUANTUM GROUPS

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1. Introduction. In this article, we explain how compact quantum groups are related to groupoids.

It is remarkable that in the (algebraic) theory of quantum groups developed by Drinfeld and many others [D, RTF], one gets a deformation of both the Poisson structure and the group structure simultaneously, for a multiplicative Poisson structure on a Lie group. Further studies showed that [So1] the algebra of compact quantum groups is closely related to the underlying singular symplectic foliation [We, LuWe] and that Rieffel's deformation quantization [Ri1, Ri2, Ri3] in this context exhibits very subtle properties [Ri4, Sh2, Sh3, Sh4] with regard to the symplectic foliation.

Parallel to the algebraic theory of quantum groups, Woronowicz initiated an analytical (C*-algebraic) theory of quantum groups [Wo1, Wo2] which successfully provides a general C*-algebraic framework for compact quantum groups. Since Connes' successful use [Co] of groupoid C*-algebras [Re] in the study of foliations, it has been well recognized that groupoid C*-algebras provide a useful tool in studying operator algebras [CuM, MRe, Sh1] (which often arise from geometric objects). We found that Vaksman and Soibelman's result [So1, VSo2] relating the singular symplectic foliation to representations of the algebra of compact quantum groups and quantum spheres can be used to establish a relation between these quantum spaces and some fundamental 'discrete' groupoids [Sh5]. In fact, the algebra of such a quantum space forms the 'core' of the groupoid C*-algebra of a 'discrete' groupoid, and for quantum spheres and quantum $SU(3)$, it is actually equal to some groupoid C*-algebras.

2. Compact matrix quantum groups. In this section, we follow the setting used in [LeSo] to summarize the results about compact matrix quantum groups that we need

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in establishing a connection with groupoid.

For a simple complex Lie group G, we fix a root system Λ with (positive) simple roots $\{\alpha_i\}_{i=1}^r$ for its Lie algebra $\mathfrak{g}$. (Here we use Λ for the root system instead of Δ which will be used for the comultiplication.) There corresponds a Cartan-Weyl basis $\{X_\alpha\}_{\alpha\in\Lambda} \cup \{H_i\}_{i=1}^r$ with $H_i = [X_{\alpha_i}, X_{-\alpha_i}]$ for each i. Let $\mathfrak{k}$ be the real form (i.e. the +1-eigenspace) for the antilinear involution $\omega : \mathfrak{g} \to \mathfrak{g}$ defined by $\omega(X_\alpha) = -X_{-\alpha}$ and $\omega(H_i) = -H_i$ for all $\alpha \in \Lambda$ and $1 \leq i \leq r$. It is easy to see that $\mathfrak{k}$ is the $\mathbb{R}$-linear span of $X_\alpha - X_{-\alpha}$, $iX_\alpha + iX_{-\alpha}$, and iH_i in $\mathfrak{g}$. It is well-known that $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K of G. The pair $(G, K) = (SL(n,\mathbb{C}), SU(n))$ is a fundamental example.

It is known [B-D, So2] that all multiplicative Poisson structures on G and on K are determined (up to an isomorphism) by $p = a\mathbf{r} + v$ with $a \in \mathbb{R}$ and $v \in \mathfrak{h} \wedge \mathfrak{h}$, where

$$\mathbf{r} = \frac{i}{2} \sum_{\alpha\in\Lambda_+} (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in \mathfrak{k} \wedge \mathfrak{k}$$

and $\mathfrak{h}$ is the real Cartan subalgebra linearly spanned by H_i's over $\mathbb{R}$. In fact, $\pi_x := L_x p - R_x p$ gives the Poisson 2-tensor [We] at $x \in K$ where L_x and R_x are the left and right translations by x.

Each fixed p determines a family of quantum groups ${}_pK_q$, $q \geq 1$, (or more precisely, a family of Hopf *-algebras $C({}_pK_q)^\infty$ of regular functions on ${}_pK_q$) which deforms the Poisson structure corresponding to p. In the following, we shall concentrate on the standard case of $K_q = {}_{\mathbf{r}}K_q$ with $p = \mathbf{r}$.

By classifying all irreducible *-representations of $C(K_q)^\infty$ on Hilbert spaces, Soibelman completed $C(K_q)^\infty$ into a type-I C*-algebra $C(K_q)$. On the other hand, starting from a purely C*-algebraic setting, Woronowicz developed a general framework for C*-algebraic (compact) quantum groups [Wo2] and proved the existence of the important invariant Haar functional h which will be discussed later.

Recall that [Wo1, VSo1] the C*-algebra $C(SU(2)_q)$ is generated by u_{ij}, with $1 \leq i, j \leq 2$, satisfying $u_{22} = u_{11}^*$, $u_{12} = -q^{-1}u_{21}^*$, and $u^*u = uu^* = 1$. An important irreducible (non-faithful) *-representation π_0 of $C(SU(2)_q)$, $q > 1$, on $\ell^2(\mathbb{Z}_\geq)$ is given by

$$\pi_0(u) = \begin{pmatrix} \alpha & -q^{-1}\gamma \\ \gamma & \alpha^* \end{pmatrix}$$

where $\alpha(e_j) = (1 - q^{-2j})^{1/2} e_{j-1}$ and $\gamma(e_j) = q^{-j}e_j$ for $j \geq 0$. Here π_0 is applied to $u = (u_{ij})$ entrywise.

The well-known canonical embedding $\phi_{i_*} : SU(2) \to K$ for the basic triple

$$\{X_{\alpha_i}, X_{-\alpha_i}, H_i\}, \qquad 1 \leq i \leq r,$$

induces a Hopf *-algebra homomorphism $\phi_i : C(K_q)^\infty \to C(SU(2)_q)^\infty$. We call $\pi_i := \pi_0 \circ \phi_i$ the fundamental representations of $C(K_q)^\infty$.

Recall that the Weyl group W of K is a Coxeter group (c.f. [H] and the reference there for details) generated by $\{s_i\}_{i=1}^r$ with $(s_i s_j)^{m_{ij}} = 1$ for $m_{ii} = 1$ and some $m_{ij} \in \{2, 3, 4, 6\}$ if $i \neq j$, where $s_i = s_{\alpha_i}$ is the reflection on $\mathfrak{h}^*$ determined by the root α_i. If $w = s_{i_1}s_{i_2}...s_{i_m}$ is the shortest expansion of w in s_i's, then $s_{i_1}s_{i_2}...s_{i_m}$ is called a reduced expression for

w and $\ell(w) := m$ is the length of w. The Bruhat ordering on W is the partial ordering generated by the relations $w_1 < w_2$ satisfying $s_\alpha w_1 = w_2$ and $\ell(w_1) + 1 = \ell(w_2)$ for some positive root $\alpha \in \Lambda_+$. It is known that there is a unique maximal element in W with respect to the Bruhat ordering.

Soibelman's classification of irreducible *-representations of $C(K_q)^\infty$ (or of $C(K_q)$) can be summarized by the following.

(1) One-dimensional irreducible *-representations τ_t of $C(K_q)^\infty$ are parametrized by $t \in \mathbb{T}^r$, the maximal torus in K.

(2) Irreducible *-representations of $C(K_q)^\infty$ are parametrized by elements (t, w) of $\mathbb{T}^r \times W$. In fact, if $t \in \mathbb{T}^r$ and $w = s_{i_1} s_{i_2} ... s_{i_m}$ is a reduced expression for w, then $(\tau_t \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes ... \otimes \pi_{i_m}) \Delta^m$ is the corresponding irreducible *-representation (independent of the choice of the reduced expression for w), where Δ is the comultiplication on $C(K_q)^\infty$ and Δ^m is defined recursively as $\Delta^k = (\Delta \bigotimes id) \Delta^{k-1}$.

It is an interesting discovery [So1] that the symplectic leaves L in K are in one-to-one correspondence with elements (t, w) of $\mathbb{T}^r \times W$ and hence with the irreducible *-representations π_L of $C(K_q)^\infty$. Indeed if $t \in \mathbb{T}^r$ and $w = s_{i_1} s_{i_2} ... s_{i_m}$ is a reduced expression for w, then the set $t S_{i_1} S_{i_2} ... S_{i_m} \subset K$ is the corresponding symplectic leaf, where $S_i = \phi_{i_*}(S)$ with

$$S = \left\{ \begin{pmatrix} \alpha & -\gamma \\ \gamma & \overline{\alpha} \end{pmatrix} : \alpha \in \mathbb{C},\ |\alpha| < 1,\ \gamma = \sqrt{1 - |\alpha|^2} \right\}$$

the prominent 2-dimensional symplectic leaf in $SU(2)$. Completing $C(K_q)^\infty$ with respect to the norm $||x|| := sup_L ||\pi_L(x)||$, we get the type I C*-algebra $C(K_q)$ [So1].

From the above result, we can talk about symplectic leaf-preserving quantizations of K by K_q and group- (or comultiplication-) preserving quantizations of K by K_q. It is interesting to know that there is no quantization of K by K_q which is simultaneously leaf-preserving and group-preserving [Sh3, Sh4]. On the other hand, surprisingly, Rieffel showed that for ${}_uK_q$ with $u \neq 0$, there does exist such a quantization [Ri4].

3. Groupoids for K_q. It has been well recognized that groupoid C*-algebras provide a very powerful tool to study the structure of concrete C*-algebras like Toeplitz C*-algebras, Wiener-Hopf C*-algebras, etc. For the theory of groupoid C*-algebras, we refer to Renault's book [Re].

Recall that the transformation group groupoid $\mathbb{Z}^m \times \overline{\mathbb{Z}}^m$ (with $\mathbb{Z}^m$ acting on $\overline{\mathbb{Z}}^m$ by translation) when restricted to the positive cone $\overline{\mathbb{Z}}^m_\geq$ gives an important (Toeplitz) groupoid

$$\mathbb{Z}^m \times \overline{\mathbb{Z}}^m|_{\overline{\mathbb{Z}}^m_\geq} := \{(j, k) \in \mathbb{Z}^m \times \overline{\mathbb{Z}}^m_\geq \mid j + k \in \overline{\mathbb{Z}}^m_\geq\}$$

where $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$ and $\overline{Z}_\geq := \{0, 1, 2, 3, ...\} \cup \{+\infty\}$.

Let $s_{i_1} s_{i_2} ... s_{i_N}$ be a reduced expression for the unique maximal element in the Weyl group with respect to the Bruhat ordering. Then all irreducible *-representations of $C(K_q)$ factor through the $\mathbb{T}^r$-family $(\tau_t \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes ... \otimes \pi_{i_N}) \Delta^N$ of representations. The $\mathbb{T}^r$-family $\{\tau_t\}_{t \in \mathbb{T}^r}$ of one-dimensional irreducible *-representations of $C(K_q)$ can be viewed as a C*-algebra homomorphism $\tau : C(K_q) \to C(\mathbb{T}^r) \cong C^*(\mathbb{Z}^r)$. Now it is clear that all irreducible *-representations of $C(K_q)$ factor through the homomorphism $(\tau \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes ... \otimes \pi_{i_N})\Delta^N$. Thus we get the following theorem [Sh5].

THEOREM 1. *$C(K_q)$ can be embedded into*

$$C^*(\mathbb{Z}^r \times \mathbb{Z}^N \times \overline{\mathbb{Z}}^N|_{\overline{\mathbb{Z}}^N_{\geq}}) \subseteq C(\mathbb{T}^r) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^N_{\geq}))$$

by $(\tau \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes ... \otimes \pi_{i_N})\Delta^N$, where $\mathbb{Z}^r$ acts trivially on $\overline{\mathbb{Z}}^N$ and $\mathbb{Z}^N$ acts by translation on $\overline{\mathbb{Z}}^N$.

Let us consider the special case of $G = SL(n+1)$ and $K = SU(n+1)$ with $n \geq 1$, for which $r = n$. The C*-algebra $C(SU(n+1)_q)$ is generated by u_{ij}, $1 \leq i, j \leq n+1$, satisfying $u^*u = uu^* = I$ and some other relations [Wo3, So1].

Irreducible 1-dimensional *-representations of $C(SU(n+1)_q)$ are defined by $\tau_t(u_{ij}) = \delta_{ij}t_j$ for $t \in \mathbb{T}^n$ (with $t_{n+1} = t_1^{-1}t_2^{-1}...t_n^{-1}$), and we set $\tau_{n+1} = \tau : C(SU(n+1)_q) \to C^*(\mathbb{Z}^n)$. There are n fundamental *-representations $\pi_i = \pi_0\phi_i$ with $\phi_i : C(SU(n+1)_q) \to C(SU(2)_q)$ given by $\phi_i(u_{jk}) = u_{j-i+1,k-i+1}$ if $\{j,k\} \subseteq \{i, i+1\}$ and $\phi_i(u_{jk}) = \delta_{jk}$ if otherwise.

The unique maximal element in the Weyl group of $SU(n+1)$ can be expressed in the reduced form

$$s_1s_2s_1s_3s_2s_1...s_ns_{n-1}...s_2s_1.$$

So $C(SU(n+1)_q)$ can be embedded into

$$C^*(\mathcal{G}^n) \subseteq C^*(\mathbb{Z}^n) \otimes \mathcal{B}(\ell^2(\ \mathbb{Z}^N_{\geq}))$$

by

$$(\tau_{n+1} \otimes \pi_{121321...n(n-1)..21})\Delta^N$$

where $N = n(n+1)/2$,

$$\pi_{i_1i_2...i_m} := \pi_{i_1} \otimes \pi_{i_2} \otimes ... \otimes \pi_{i_m},$$

and $\mathcal{G}^n$ is the groupoid $\mathbb{Z}^n \times \mathbb{Z}^N \times \overline{\mathbb{Z}}^N|_{\overline{\mathbb{Z}}^N_{\geq}}$ with $\mathbb{Z}^n$ acting trivially on $\overline{\mathbb{Z}}^N$, and $\mathbb{Z}^N$ acts by translation on $\overline{\mathbb{Z}}^N$.

With a minor modification, we can study the related quantum $U(n)_q$ in a similar way.

4. Structure theorems for $C(SU(n)_q)$. Applying the above groupoid approach to $SU(n)_q$, we get the following structure theorems for $C(SU(n)_q)$ [Sh5].

For any subset J of $\{1, 2, ..., N\}$, we define

$$X_J := \{k \in \overline{\mathbb{Z}}^N_{\geq} \mid k_i = \infty \text{ if } i \notin J\}$$

an invariant closed subset of the unit space of $\mathcal{G}^n$, called a face of $\overline{\mathbb{Z}}^N_{\geq}$. By restricting the embedded algebra $C(SU(n+1)_q)$ to various faces X_J, we can analyze its algebra structure and get interesting composition sequences of $C(SU(n+1)_q)$.

$C(SU(n+1)_q)$ is determined by $C(SU(n+1)_q)|_{X_J}$ with admissible J only. (J is called admissible if $s_{\iota(j_1)}s_{\iota(j_2)}...s_{\iota(j_m)}$ is a reduced element in the Weyl group where $\iota : \{1, 2, ..., N\} \to \{1, 2, ..., n\}$ is defined by

$$(\iota(1), ..., \iota(N)) = (1, 2, 1, 3, 2, 1, ..., n, n-1..., 1).)$$

In fact, J is admissible if and only if for each $0 \leq k < n$, there is some $0 \leq m_k \leq k+1$ such that

$$J = \left\{ j | \quad \left(\frac{k^2+k}{2}\right) < j \leq \left(\frac{k^2+k}{2}\right) + m_k, \quad \text{for some } 0 \leq k < n \right\}.$$

Let $\mathcal{L}_k$ be the collection of admissible $J \subseteq J_n$ with size $|J| = k$, for $0 \leq k \leq N$, and set $X_k = \cup_{J \in \mathcal{L}_k} X_J$.

THEOREM 2. *The C^*-algebra $C(SU(n+1)_q)$ has the composition sequence*

$$C(SU(n+1)_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq ... \supseteq \mathcal{I}_N \supseteq \mathcal{I}_{N+1} := 0,$$

with

$$\mathcal{I}_k/\mathcal{I}_{k+1} \simeq \bigoplus_{J \in \mathcal{L}_k} C(\mathbb{T}^n) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^k_{\geq}))$$

where $\mathcal{I}_k$ are ideals of $C(SU(n+1)_q)$ such that

$$C(SU(n+1)_q)|_{X_{N-k}} \simeq C(SU(n+1)_q)/\mathcal{I}_k.$$

(Here we use $\mathcal{K}(\ell^2(\mathbb{Z}^0_{\geq})) = \mathbb{C}$.)

Remark. Each $C(\mathbb{T}^n) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^k_{\geq}))$ corresponds to a $\mathbb{T}^n$-family of $2k$-dimensional symplectic leaves.

THEOREM 3. *For $C(SU(n)_{q,m}) := C(SU(n+1)_q)|_{X_J}$ with $J = \{1, 2, ..., N-m+1\}$, there are short exact sequences*

$$\begin{gathered}
0 \to \mathcal{A} \otimes \mathcal{K} \to C(SU(n)_{q,n}) \to C(SU(n)_{q,n+1}) \to 0 \\
0 \to \mathcal{A} \otimes \mathcal{K} \to C(SU(n)_{q,n-1}) \to C(SU(n)_{q,n}) \to 0 \\
... \\
0 \to \mathcal{A} \otimes \mathcal{K} \to C(SU(n)_{q,1}) \to C(SU(n)_{q,2}) \to 0
\end{gathered}$$

with

$$\mathcal{A} = C(\mathbb{T}) \bigotimes C(SU(n)_q) \simeq C(SU(n+1)_{q,n+1})$$

and $C(SU(n)_{q,1}) = C(SU(n+1)_q)$. Furthermore, there is an element

$$1 \otimes T \in C(SU(n)_{q,m}) \subseteq C(SU(n)_{q,n+1}) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^{n-m+1}_{\geq}))$$

such that T is a Fredholm operator with index 1.

These short exact sequences correspond to the classical fibration of $SU(n+1)$ over $\mathbb{C}P(n)$ by fibres $U(n)$.

COROLLARY 4. *The C^*-algebra $C(SU(n+1)_q)$ has the composition sequence*

$$C(SU(n+1)_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq ... \supseteq \mathcal{I}_n \supseteq \mathcal{I}_{n+1} := 0,$$

with

$$\mathcal{I}_k/\mathcal{I}_{k+1} \simeq C(U(n)_q) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^k_{\geq}))$$

for $k > 0$ and $\mathcal{I}_0/\mathcal{I}_1 \simeq C(U(n)_q) \cong C(\ T) \otimes C(SU(n)_q)$.

Corollary 5. *The C^*-algebra $C(SU(n+1)_q)$ has the composition sequence*

$$C(SU(n+1)_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq ... \supseteq \mathcal{I}_{(n+1)!} := 0,$$

with

$$\mathcal{I}_k/\mathcal{I}_{k+1} \simeq C(\mathbb{T}^n) \otimes \mathcal{K}$$

for $k > 0$ and $\mathcal{I}_0/\mathcal{I}_1 \simeq C(\mathbb{T}^n)$.

5. Quantum spheres. Similar structure theorems have also been obtained for quantum spheres by using this groupoid approach [Sh5].

Recall that Nagy [N] considered quantum homogeneous spaces $M_q = H_q\backslash K_q$ defined by

$$C(M_q) = \{f \in C(K_q) : (\Phi \otimes id)(\Delta f) = 1 \otimes f\}$$

where H is a closed subgroup of K and $\Phi : C(K_q) \to C(H_q)$ is the quantization of the embedding homomorphism from H into K. With $(K,H) = (SU(n+1), SU(n))$, we get (odd-dimensional) quantum spheres $S_q^{2n+1} = SU(n)_q\backslash SU(n+1)_q$.

Proposition 6.

$$C(S_q^{2n+1}) = C^*(\{u_{n+1,m} |\, 1 \leq m \leq n+1\})$$

and $\tau_{n+1} \otimes \pi_n \otimes \pi_{n-1} \otimes ... \otimes \pi_1$ gives an embedding of $C(S_q^{2n+1})$ in $C^*(\mathcal{F}^n)$, where $\mathcal{F}^n = \mathbb{Z} \times (\mathbb{Z}^n \times \overline{\mathbb{Z}}^n|_{\overline{\mathbb{Z}}^n_{\geq}})$ and $\tau_{n+1}(u_{n+1,m}) = (\delta_{n+1,m})\delta_1 \in C^*(\mathbb{Z})$.

Proposition 7. *There is a short exact sequence*

$$0 \to C(\mathbb{T}) \otimes \mathcal{K} \to C(S_q^{2k+1}) \to C(S_q^{2k-1}) \to 0$$

for $k \geq 1$ with $C(S_q^1) \simeq C(\mathbb{T})$ such that $C(S_q^{2k+1})$ contains an element $1 \otimes T \in C(\mathbb{T}) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^k_{\geq}))$ where T is a Fredholm operator with index 1.

Corollary 8. *The C^*-algebra $C(S_q^{2n+1})$ has the composition sequence*

$$C(S_q^{2n+1}) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq ... \supseteq \mathcal{I}_n \supseteq \mathcal{I}_{n+1} := 0,$$

with

$$\mathcal{I}_k/\mathcal{I}_{k+1} \simeq C(\mathbb{T}) \otimes \mathcal{K}(\ell^2(\mathbb{Z}^k))$$

for $k > 0$ and $\mathcal{I}_0/\mathcal{I}_1 \simeq C(\mathbb{T})$.

6. Haar functional. In [Wo2], Woronowicz proved the existence of the important invariant Haar functional on compact matrix quantum groups. Using the groupoid approach, we can give an explicit description of the Haar functional h_n on $C(SU(n)_q)$ [Sh5].

Theorem 9. *The unique invariant functional h_{n+1} on $C(SU(n+1)_q)$ is the restriction of the state $f_{\xi^{(n)}}$ of $C^*(\mathcal{G}^n)$ given by the regular representation on $\ell^2(\mathcal{G}^n|_{\mathbb{Z}^N_{\geq}})$ and*

$$\xi^{(n)} = (\prod_{i=1}^{n}(1 - q^{-2(n+1-i)})^{-i/2})$$

$$\sum_{w\in\mathbb{Z}_{\geq}^{N}} q^{-\Sigma(n+1-[i])w_i}\cdot\delta_{(0,0,w)}\in\ell^2(\mathcal{G}^n|_{\mathbb{Z}_{\geq}^{N}}),$$

i.e. $h_{n+1}=f_{\xi^{(n)}}\circ(\tau_{n+1}\otimes\pi_{J_n})$, *where* $[i]\geq 0$ *is defined by* $i=[i]+k(k+1)/2$ *and* $[i]\leq k+1$.

This result can be used to prove the known facts that [N] h_{n+1} is a faithful state on $C(SU(n+1)_q)$, and that $K_i(C(S_q^{2n+1}))\simeq Z$ for both $i=0$ and 1 [VSo2].

7. Subquotient groupoids. It is an interesting question whether $C(S_q^{2n+1})$ (or $C(SU(n)_q)$) is really a groupoid C*-algebra of some groupoid instead of a C*-subalgebra of some groupoid C*-algebra.

It is not difficult to see that the answer is affirmative for the following two simple but fundamental examples.

(1) $C(SU(2)_q)\simeq C^*(\mathfrak{G}_1)$ where

$$\mathfrak{G}_1=\{(\mathfrak{m},\mathfrak{j},\mathfrak{k})\mid \text{ if } \mathfrak{k}=\infty, \text{ then } \mathfrak{m}=\mathfrak{o}\}$$

is a subgroupoid of the groupoid $\mathbb{Z}\times(\mathbb{Z}\times\overline{\mathbb{Z}}|_{\overline{\mathbb{Z}}_{\geq}})$, with

$$u_{11}=\sum_{0<j\leq\infty}(1-q^{-2j})^{1/2}\delta_{(0,-1,j)}$$

and $u_{21}=\sum_{0\leq j<\infty}q^{-j}\delta_{(1,0,j)}$.

(2) Podles' quantum sphere $C(S_{\mu c}^2)\simeq C^*(\mathfrak{G}')$, $c>0$, where

$$\mathfrak{G}'=\{(\mathfrak{j},\mathfrak{j},\mathfrak{k}_1,\mathfrak{k}_2)\mid\ \mathfrak{k}_1=\infty \text{ or } \mathfrak{k}_2=\infty\}$$

is a subgroupoid of the groupoid $\mathbb{Z}^2\times\overline{\mathbb{Z}}^2|_{\overline{\mathbb{Z}}_{\geq}^2}$.

It turns out that the answer is also affirmative for odd-dimensional quantum spheres S_q^{2n+1} and for quantum $SU(3)_q$.

Define a subquotient groupoid $\mathfrak{F}_\mathfrak{n}$ of $\mathcal{F}^n=\mathbb{Z}\times(\mathbb{Z}^n\times\overline{\mathbb{Z}}^n|_{\overline{\mathbb{Z}}_{\geq}^n})$ as follows. Let

$$\widetilde{\mathfrak{F}_\mathfrak{n}}:=\{(m,j,k)\in\mathcal{F}^n|\ k_i=\infty\Longrightarrow$$

$$j_i=m-j_1-j_2-...-j_{i-1} \text{ and } j_{i+1}=...=j_n=0\}$$

be a subgroupoid of $\mathcal{F}^n$. Define $\mathfrak{F}_\mathfrak{n}:=\widetilde{\mathfrak{F}_\mathfrak{n}}/\sim$ where $\sim$ is the equivalence relation generated by

$$(m,j,k)\sim(m,j,k_1,...,k_i=\infty,\infty,...,\infty)$$

for all (m,j,k) with $k_i=\infty$ for an $1\leq i\leq n$.

THEOREM 10. $C(S_q^{2n+1})\simeq C^*(\mathfrak{F}_\mathfrak{n})$.

From this, we get the following known result [VSo2].

COROLLARY 11. *The C*-algebra* $C(S_q^{2n+1})$ *is independent of* q.

The key technical point in proving that $C(SU(3)_q)$ is a groupoid C*-algebra is based on Lance's [La] (or Woronowicz's [Wo4]) result that there exists some isometry

$$v:\ell^2(\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{N})\to\ell^2(\mathbb{Z}\times\mathbb{N}\times\mathbb{Z}\times\mathbb{Z})$$

such that

$$v(\Delta x)v^* = x \otimes 1$$

for $x \in C(SU(2)_q) \subset \mathcal{B}\left(\ell^2(\mathbb{Z} \times \mathbb{Z}_{\geq})\right)$. Modifying this result, we can show that

Proposition 12. *There is some isometry*

$$w : \ell^2(\mathbb{Z}_{\geq} \times \mathbb{Z}_{\geq}) \to \ell^2(\mathbb{Z} \times \mathbb{Z})$$

such that

$$w((\pi_1 \otimes \pi_1)\Delta(x))w^* = (\tau \otimes \pi_1)\Delta(x)$$

for $x \in C(SU(3)_q)$.

As far as the algebra $C(SU(3)_q)$ is concerned, the face $X_{\{1,3\}}$ corresponding to $\pi_1 \otimes \pi_1$ is 'degenerate' and is 'dominated' by the face $X_{\{1,2\}}$ corresponding to $\pi_1 \otimes \pi_2$. This combined with the above composition sequence for $C(SU(3)_q)$ helps to build a quotient groupoid $\mathfrak{G}_2$ from a subgroupoid $\widetilde{\mathfrak{G}}_2$ of $\mathcal{G}^2 = \mathbb{Z}^2 \times (Z^3 \times \overline{\mathbb{Z}}^3|_{\overline{\mathbb{Z}}^3_{\geq}})$ where

$$\begin{aligned}\widetilde{\mathfrak{G}}_2 := \{ \quad & (m, j, k) \in \mathcal{G}^2 \mid m_1 + m_2 + m_3 = 0, \\ & k_1 = \infty \Longrightarrow j_1 = m_1 - m_2, \\ & k_2 = \infty \Longrightarrow j_2 = m_1 - m_3 - j_1, \\ & k_3 = \infty \Longrightarrow j_3 = m_2 - m_3 + j_1 - j_2 \ \}.\end{aligned}$$

We use the following notations: $r \wedge s := \min\{r, s\}$, $r \vee s := \max\{r, s\}$,

$$\psi(r, s) := (r - s) \vee 0 - (s - r) \vee 0,$$

and $\eta(r) := (r, 0)$ if $r > 0$ and $\eta(r) := (0, -r)$ if $r < 0$. Define $\mathfrak{G}_2 := \widetilde{\mathfrak{G}}_2/\sim$, the quotient groupoid given by the equivalence relation $\sim$ generated by

$$\begin{aligned}&(m, j_1, j_2, j_3, k_1, k_2 = \infty, k_3) \sim \\ &\quad (m, \tilde{j}_1, j_2 + j_1 - \tilde{j}_1, \tilde{j}_3, k_1 \wedge k_3, k_2 = \infty, k_1 \wedge k_3)\end{aligned}$$

where

$$\begin{aligned}\left(\tilde{j}_1, \tilde{j}_3\right) \quad : \quad &= ((k_1 + l_1) \wedge (k_3 + l_3) - k_1 \wedge k_3)(1, 1) \\ &+ \eta(\psi(k_1 + l_1, k_3 + l_3) - \psi(k_1, k_3)).\end{aligned}$$

Theorem 13. $C(SU(3)_q) \cong C^*(\mathfrak{G}_2)$.

From this, we get a result of Nagy.

Corollary 14. $C(SU(3)_q)$ *as a* C^**-algebra is independent of* q.

It is interesting to note that S. Wang [Wa] has proved that for $n \geq 2$, $C(SU(n)_q)$ with different q's are not isomorphic as Hopf C*-algebras.

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