1. Introduction. In this article, we explain how compact quantum groups are related to groupoids.

It is remarkable that in the (algebraic) theory of quantum groups developed by Drinfeld and many others [D, RTF], one gets a deformation of both the Poisson structure and the group structure simultaneously, for a multiplicative Poisson structure on a Lie group. Further studies showed that [So1] the algebra of compact quantum groups is closely related to the underlying singular symplectic foliation [We, LuWe] and that Rieffel’s deformation quantization [Ri1, Ri2, Ri3] in this context exhibits very subtle properties [Ri4, Sh2, Sh3, Sh4] with regard to the symplectic foliation.

Parallel to the algebraic theory of quantum groups, Woronowicz initiated an analytical (C*-algebraic) theory of quantum groups [Wo1, Wo2] which successfully provides a general C*-algebraic framework for compact quantum groups. Since Connes’ successful use [Co] of groupoid C*-algebras [Re] in the study of foliations, it has been well recognized that groupoid C*-algebras provide a useful tool in studying operator algebras [CuM, MRe, Sh1] (which often arise from geometric objects). We found that Vaksman and Soibelman’s result [So1, VSo2] relating the singular symplectic foliation to representations of the algebra of compact quantum groups and quantum spheres can be used to establish a relation between these quantum spaces and some fundamental ‘discrete’ groupoids [Sh5]. In fact, the algebra of such a quantum space forms the ‘core’ of the groupoid C*-algebra of a ‘discrete’ groupoid, and for quantum spheres and quantum SU(3), it is actually equal to some groupoid C*-algebras.

2. Compact matrix quantum groups. In this section, we follow the setting used in [LeSo] to summarize the results about compact matrix quantum groups that we need...
in establishing a connection with groupoid.

For a simple complex Lie group $G$, we fix a root system $\Delta$ with (positive) simple roots $\{\alpha_i\}_{i=1}^r$ for its Lie algebra $\mathfrak{g}$. (Here we use $\Lambda$ for the root system instead of $\Delta$ which will be used for the comultiplication.) There corresponds a Cartan-Weyl basis $\{X_{\alpha}\}_{\alpha \in \Lambda} \cup \{H_i\}_{i=1}^r$ with $H_i = [X_{\alpha_i}, X_{-\alpha_i}]$ for each $i$. Let $\mathfrak{k}$ be the real form (i.e. the +1-eigenspace) for the antilinear involution $\omega : \mathfrak{g} \to \mathfrak{g}$ defined by $\omega(X_{\alpha}) = -X_{-\alpha}$ and $\omega(H_i) = -H_i$ for all $\alpha \in \Lambda$ and $1 \leq i \leq r$. It is easy to see that $\mathfrak{k}$ is the $\mathbb{R}$-linear span of $X_{\alpha} - X_{-\alpha}, iX_{\alpha} + iX_{-\alpha}$, and $iH_i$ in $\mathfrak{g}$. It is well-known that $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup $K$ of $G$. The pair $(G, K) = (SL(n, \mathbb{C}), SU(n))$ is a fundamental example.

It is known [B-D, So2] that all multiplicative Poisson structures on $G$ and on $K$ are determined (up to an isomorphism) by $p = ar + v$ with $a \in \mathbb{R}$ and $v \in \mathfrak{k} \wedge \mathfrak{h}$, where

$$r = \frac{i}{2} \sum_{\alpha \in \Lambda_+} (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in \mathfrak{k} \wedge \mathfrak{h}$$

and $\mathfrak{h}$ is the real Cartan subalgebra linearly spanned by $H_i$'s over $\mathbb{R}$. In fact, $\pi_x := L_x p - R_x p$ gives the Poisson 2-tensor [We] at $x \in K$ where $L_x$ and $R_x$ are the left and right translations by $x$.

Each fixed $p$ determines a family of quantum groups $pK_q$, $q \geq 1$, (or more precisely, a family of Hopf $^*$-algebras $C_p(K_q)$ of regular functions on $pK_q$) which deforms the Poisson structure corresponding to $p$. In the following, we shall concentrate on the standard case of $K_q = rK_q$ with $p = r$.

By classifying all irreducible $^*$-representations of $C(K_q) \otimes \infty$ on Hilbert spaces, Soibelman completed $C(K_q) \otimes \infty$ into a type-I C*-algebra $C(K_q)$. On the other hand, starting from a purely C*-algebraic setting, Woronowicz developed a general framework for C*-algebraic (compact) quantum groups [Wo2] and proved the existence of the important invariant Haar functional $h$ which will be discussed later.

Recall that [Wo1, VS01] the C*-algebra $C(SU(2)_q)$ is generated by $u_{ij}$, with $1 \leq i, j \leq 2$, satisfying $u_{22} = u_{11}^*$, $u_{12} = -q^{-1}u_{21}^*$, and $u^*u = uu^* = 1$. An important irreducible (non-faithful) $^*$-representation $\pi_0$ of $C(SU(2)_q)$, $q > 1$, on $L^2(\mathbb{R})$ is given by

$$\pi_0(u) = \begin{pmatrix} \alpha & q^{-1} \gamma \\ \gamma & \alpha^* \end{pmatrix}$$

where $\alpha(e_j) = (1 - q^{-2}i)^{1/2}e_{j-1}$ and $\gamma(e_j) = \gamma^{-j}e_j$ for $j \geq 0$. Here $\pi_0$ is applied to $u = (u_{ij})$ entrywise.

The well-known canonical embedding $\phi_i : SU(2) \to K$ for the basic triple

$$\{X_{\alpha_i}, X_{-\alpha_i}, H_i\}, \quad 1 \leq i \leq r,$$

induces a Hopf $^*$-algebra homomorphism $\phi_i : C(K_q) \otimes \infty \to C(SU(2)_q) \otimes \infty$. We call $\pi_i := \pi_0 \circ \phi_i$ the fundamental representations of $C(K_q) \otimes \infty$.

Recall that the Weyl group $W$ of $K$ is a Coxeter group (c.f. [H] and the reference there for details) generated by $\{s_i\}_{i=1}^r$ with $(s_i s_j)_{m_{ij}} = 1$ for $m_{ii} = 1$ and some $m_{ij} \in \{2, 3, 4, 6\}$ if $i \neq j$, where $s_i = s_{\alpha_i}$ is the reflection on $\mathfrak{h}^*$ determined by the root $\alpha_i$. If $w = s_{i_1} s_{i_2} \ldots s_{i_m}$ is the shortest expansion of $w$ in $s_i$’s, then $s_{i_1} s_{i_2} \ldots s_{i_m}$ is called a reduced expression for
w and \( \ell(w) := m \) is the length of \( w \). The Bruhat ordering on \( W \) is the partial ordering generated by the relations \( w_1 < w_2 \) satisfying \( s_\alpha w_1 = w_2 \) and \( \ell(w_1) + 1 = \ell(w_2) \) for some positive root \( \alpha \in \Lambda_+ \). It is known that there is a unique maximal element in \( W \) with respect to the Bruhat ordering.

Soibelman’s classification of irreducible *-representations of \( C(K_q) \) (or of \( C(K_q) \)) can be summarized by the following:

1. One-dimensional irreducible *-representations \( \pi_i \) of \( C(K_q) \) are parametrized by \( t \in \mathbb{T}^r \), the maximal torus in \( K \).

2. Irreducible *-representations of \( C(K_q) \) are parametrized by elements \( (t, w) \) of \( \mathbb{T}^r \times W \). In fact, if \( t \in \mathbb{T}^r \) and \( w = s_{i_1} s_{i_2} \ldots s_{i_m} \) is a reduced expression for \( w \), then \( (\pi_i \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes \ldots \otimes \pi_{i_m}) \Delta^m \) is the corresponding irreducible *-representation (independent of the choice of the reduced expression for \( w \)), where \( \Delta \) is the comultiplication on \( C(K_q) \) and \( \Delta^m \) is defined recursively as \( \Delta^k = (\Delta \otimes \text{id}) \Delta^{k-1} \).

It is an interesting discovery [So1] that the symplectic leaves \( L \) in \( K \) are in one-to-one correspondence with elements \( (t, w) \) of \( \mathbb{T}^r \times W \) and hence with the irreducible *-representations \( \pi_i \) of \( C(K_q) \). Indeed if \( t \in \mathbb{T}^r \) and \( w = s_{i_1} s_{i_2} \ldots s_{i_m} \) is a reduced expression for \( w \), then the set \( tS_{i_1} S_{i_2} \ldots S_{i_m} \subset K \) is the corresponding symplectic leaf, where \( S_i = \phi_{i_1}(S) \) with

\[
S = \left\{ \left( \alpha \begin{pmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{pmatrix} \right) : \alpha \in \mathbb{C}, \ |\alpha| < 1, \ \gamma = \sqrt{1 - |\alpha|^2} \right\}
\]

the prominent 2-dimensional symplectic leaf in \( SU(2) \). Completing \( C(K_q) \) with respect to the norm \( ||x|| := \sup_{xL} ||\pi(x)\|| \), we get the type I \( \text{C}^* \)-algebra \( C(K_q) [\text{So1}] \).

From the above result, we can talk about symplectic leaf-preserving quantizations of \( K \) by \( K_q \) and group- (or comultiplication-) preserving quantizations of \( K \) by \( K_q \). It is interesting to know that there is no quantization of \( K \) by \( K_q \) which is simultaneously leaf-preserving and group-preserving [Sh3, Sh4]. On the other hand, surprisingly, Rieffel showed that for \( uK_q \) with \( u \neq 0 \), there does exist such a quantization [Ri4].

3. Groupoids for \( K_q \). It has been well recognized that groupoid \( \text{C}^* \)-algebras provide a very powerful tool to study the structure of concrete \( \text{C}^* \)-algebras like Toeplitz \( \text{C}^* \)-algebras, Wiener-Hopf \( \text{C}^* \)-algebras, etc. For the theory of groupoid \( \text{C}^* \)-algebras, we refer to Renault’s book [Re].

Recall that the transformation group groupoid \( \mathbb{Z}^m \times \mathbb{Z}^m \) (with \( \mathbb{Z}^m \) acting on \( \mathbb{Z}^m \) by translation) when restricted to the positive cone \( \mathbb{Z}^m_{\geq 0} \) gives an important (Toeplitz) groupoid

\[
\mathbb{Z}^m \times \mathbb{Z}^m_{\geq 0} := \{(j, k) \in \mathbb{Z}^m \times \mathbb{Z}^m_{\geq 0} \mid j + k \in \mathbb{Z}^m_{\geq 0} \}
\]

where \( \mathbb{Z} = \mathbb{Z} \cup \{+\infty\} \) and \( \mathbb{Z}^m_{\geq 0} := \{0, 1, 2, 3, \ldots \} \cup \{+\infty\} \).

Let \( s_{i_1} s_{i_2} \ldots s_{i_N} \) be a reduced expression for the unique maximal element in the Weyl group with respect to the Bruhat ordering. Then all irreducible *-representations of \( C(K_q) \) factor through the \( \mathbb{T}^r \)-family \( (\tau \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes \ldots \otimes \pi_{i_N}) \Delta^N \) of representations. The \( \mathbb{T}^r \)-family \( \{\tau_k\}_{k \in \mathbb{T}^r} \) of one-dimensional irreducible *-representations of \( C(K_q) \) can be viewed as a \( \text{C}^* \)-algebra homomorphism \( \tau : C(K_q) \to C(\mathbb{T}^r) \cong C^*(\mathbb{Z}^r) \). Now it is
clear that all irreducible *-representations of $C(K_q)$ factor through the homomorphism $(\tau \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes \ldots \otimes \pi_{i_n})\Delta^N$. Thus we get the following theorem [Sh5].

**Theorem 1.** $C(K_q)$ can be embedded into $C^*(\mathbb{Z}^r \times \mathbb{Z}^N |_{\mathbb{Z}_2^r}) \subseteq C(\mathbb{T}^r) \otimes B(\ell^2(\mathbb{Z}^N_2))$ by $(\tau \otimes \pi_{i_1} \otimes \pi_{i_2} \otimes \ldots \otimes \pi_{i_n})\Delta^N$, where $\mathbb{Z}^r$ acts trivially on $\mathbb{Z}^N$ and $\mathbb{Z}^N$ acts by translation on $\mathbb{Z}^N$.

Let us consider the special case of $G = SL(n+1)$ and $K = SU(n+1)$ with $n \geq 1$, for which $r = n$. The C*-algebra $C(SU(n+1)_q)$ is generated by $u_{ij}$, $1 \leq i, j \leq n + 1$, satisfying $u^*u = uu^* = I$ and some other relations [Wo3, So1].

Irreducible 1-dimensional *-representations of $C(SU(n+1)_q)$ are defined by $\tau_t(u_{ij}) = \delta_{ij}t_j$ for $t \in \mathbb{T}^n$ (with $t_{n+1} = t_1^{-1}t_2^{-1}\ldots t_n^{-1}$), and we set $\tau_{n+1} = \tau : C(SU(n+1)_q) \rightarrow C^*(\mathbb{Z}^N)$. There are $n$ fundamental *-representations $\pi_i = \pi_0\phi_i$ with $\phi_i : C(SU(n+1)_q) \rightarrow C(SU(2)_q)$ given by $\phi_i(u_{jk}) = u_{j-i+1,k-i+1}$ if $\{j,k\} \subseteq \{i,i+1\}$ and $\phi_i(u_{jk}) = \delta_{jk}$ if otherwise.

The unique maximal element in the Weyl group of $SU(n+1)$ can be expressed in the reduced form $s_1s_2s_1s_3s_2s_1\ldots s_n s_{n-1} \ldots s_2s_1$. So $C(SU(n+1)_q)$ can be embedded into $C^*(\mathcal{G}^n) \subseteq C^*(\mathbb{Z}^n) \otimes B(\ell^2(\mathbb{Z}^N_2))$

by

$$(\tau_{n+1} \otimes \pi_{121321\ldots(n+1-2)}\Delta^N)$$

where $N = n(n+1)/2$, $\pi_{i_1i_2\ldots i_m} := \pi_{i_1} \otimes \pi_{i_2} \otimes \ldots \otimes \pi_{i_m}$, and $\mathcal{G}^n$ is the groupoid $\mathbb{Z}^n \times \mathbb{Z}^N \times \mathbb{Z}^N |_{\mathbb{Z}^N_2}$ with $\mathbb{Z}^n$ acting trivially on $\mathbb{Z}^N$, and $\mathbb{Z}^N$ acts by translation on $\mathbb{Z}^N$.

With a minor modification, we can study the related quantum $U(n)_q$ in a similar way.

**4. Structure theorems for $C(SU(n)_q)$.** Applying the above groupoid approach to $SU(n)_q$, we get the following structure theorems for $C(SU(n)_q)$ [Sh5].

For any subset $J$ of $\{1,2,\ldots,N\}$, we define

$X_J := \{k \in \mathbb{Z}^N | k_i = \infty \text{ if } i \notin J\}$

an invariant closed subset of the unit space of $\mathcal{G}^n$, called a face of $\mathbb{Z}^N$. By restricting the embedded algebra $C(SU(n+1)_q)$ to various faces $X_J$, we can analyze its algebra structure and get interesting composition sequences of $C(SU(n+1)_q)$.

$C(SU(n+1)_q)$ is determined by $C(SU(n+1)_q)|_{X_J}$ with admissible $J$ only. ($J$ is called admissible if $s_{i_1}(J_1)s_{i_2}(J_2)\ldots s_{i_m}(J_m)$ is a reduced element in the Weyl group where $: \{1,2,\ldots,N\} \rightarrow \{1,2,\ldots,n\}$ is defined by

$$(i(1),\ldots,i(N)) = (1,2,1,3,2,1,\ldots,n,n-1,\ldots,1).$$
In fact, \( J \) is admissible if and only if for each \( 0 \leq k < n \), there is some \( 0 \leq m_k \leq k + 1 \) such that
\[
J = \left\{ j \left| \frac{k^2 + k}{2} < j \leq \frac{k^2 + k}{2} + m_k, \text{ for some } 0 \leq k < n \right. \right\}.
\]

Let \( X_k \) be the collection of admissible \( J \subseteq J_n \) with size \( |J| = k \), for \( 0 \leq k \leq N \), and set \( X_k = \bigcup_{J \in L_k} X_J \).

**Theorem 2.** The \( C^* \)-algebra \( C(SU(n+1)_q) \) has the composition sequence
\[
C(SU(n+1)_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \ldots \supseteq \mathcal{I}_N \supseteq \mathcal{I}_{N+1} := 0,
\]
with
\[
\mathcal{I}_k/\mathcal{I}_{k+1} \simeq \bigoplus_{J \in L_k} C(\mathbb{T}^n) \otimes K(\ell^2(\mathbb{Z}_2^k))
\]
where \( \mathcal{I}_k \) are ideals of \( C(SU(n+1)_q) \) such that
\[
C(SU(n+1)_q)|_{X_{N-k}} \simeq C(SU(n+1)_q)/\mathcal{I}_k.
\]
(Here we use \( K(\ell^2(\mathbb{Z}_2^0)) = \mathbb{C} \).)

**Remark.** Each \( C(\mathbb{T}^n) \otimes K(\ell^2(\mathbb{Z}_2^k)) \) corresponds to a \( \mathbb{T}^n \)-family of \( 2k \)-dimensional symplectic leaves.

**Theorem 3.** For \( C(SU(n,q)_m) := C(SU(n+1)_q)|_{X_J} \) with \( J = \{1,2,\ldots,N-m+1\} \), there are short exact sequences
\[
0 \rightarrow A \otimes K \rightarrow C(SU(n,q,m)) \rightarrow C(SU(n,q,n+1)) \rightarrow 0
\]
\[
0 \rightarrow A \otimes K \rightarrow C(SU(n,q,n-1)) \rightarrow C(SU(n,q,n)) \rightarrow 0
\]
\[
\ldots
\]
\[
0 \rightarrow A \otimes K \rightarrow C(SU(n,q,1)) \rightarrow C(SU(n,q,2)) \rightarrow 0
\]
with
\[
A = C(\mathbb{T}) \bigotimes C(SU(n)_q) \simeq C(SU(n+1,q,n+1))
\]
and \( C(SU(n,q,1)) = C(SU(n+1,q)) \). Furthermore, there is an element
\[
1 \otimes T \in C(SU(n,q,m)) \subseteq C(SU(n,q,n+1)) \otimes \mathcal{B}(\ell^2(\mathbb{Z}_2^{n-m+1}))
\]
such that \( T \) is a Fredholm operator with index 1.

These short exact sequences correspond to the classical fibration of \( SU(n+1) \) over \( \mathbb{C}P(n) \) by fibres \( U(n) \).

**Corollary 4.** The \( C^* \)-algebra \( C(SU(n+1)_q) \) has the composition sequence
\[
C(SU(n+1)_q) = \mathcal{I}_0 \supseteq \mathcal{I}_1 \supseteq \ldots \supseteq \mathcal{I}_n \supseteq \mathcal{I}_{n+1} := 0,
\]
with
\[
\mathcal{I}_k/\mathcal{I}_{k+1} \simeq C(U(n)_q) \otimes K(\ell^2(\mathbb{Z}_2^k))
\]
for \( k > 0 \) and \( \mathcal{I}_0/\mathcal{I}_1 \simeq C(U(n)_q) \cong C(T) \otimes C(SU(n)_q) \).
Corollary 5. The C*-algebra \( C(SU(n + 1)_q) \) has the composition sequence
\[
C(SU(n + 1)_q) = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{n+1} : = 0,
\]
with
\[
I_k/I_{k+1} \simeq C(T^n) \otimes K
\]
for \( k > 0 \) and \( I_0/I_1 \simeq C(T^n) \).

5. Quantum spheres. Similar structure theorems have also been obtained for quantum spheres by using this groupoid approach [Sh5].

Recall that Nagy [N] considered quantum homogeneous spaces \( M_q = H_q \backslash K_q \) defined by
\[
C(M_q) = \{ f \in C(K_q) : (\Phi \otimes id)(\Delta f) = 1 \otimes f \}
\]
where \( H \) is a closed subgroup of \( K \) and \( \Phi : C(K_q) \rightarrow C(H_q) \) is the quantization of the embedding homomorphism from \( H \) into \( K \). With \( (K, H) = (SU(n + 1), SU(n)) \), we get (odd-dimensional) quantum spheres \( S_q^{2n+1} = SU(n)_q \backslash SU(n + 1)_q \).

Proposition 6.
\[
C(S_q^{2n+1}) = C^*(\{ u_{n+1,m} | 1 \leq m \leq n + 1 \})
\]
and \( \tau_{n+1} \otimes \pi_n \otimes \pi_{n-1} \otimes \ldots \otimes \pi_1 \) gives an embedding of \( C(S_q^{2n+1}) \) in \( C^*(F^n) \), where \( F^n = \mathbb{Z} \times (\mathbb{Z}^n \times \mathbb{T}^n_{\mathbb{T}^n}) \) and \( \tau_{n+1}(u_{n+1,m}) = (\delta_{n+1,m}) \delta_1 \in C^*(\mathbb{Z}) \).

Proposition 7. There is a short exact sequence
\[
0 \rightarrow C(T) \otimes K \rightarrow C(S_q^{2k+1}) \rightarrow C(S_q^{2k-1}) \rightarrow 0
\]
for \( k \geq 1 \) with \( C(S_q^1) \simeq C(T) \) such that \( C(S_q^{2k+1}) \) contains an element \( 1 \otimes T \in C(T) \otimes \mathcal{B}(\ell^2(\mathbb{Z}^k)) \) where \( T \) is a Fredholm operator with index 1.

Corollary 8. The C*-algebra \( C(S_q^{2n+1}) \) has the composition sequence
\[
C(S_q^{2n+1}) = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_n \supseteq I_{n+1} : = 0,
\]
with
\[
I_k/I_{k+1} \simeq C(T) \otimes K(\ell^2(\mathbb{Z}^k))
\]
for \( k > 0 \) and \( I_0/I_1 \simeq C(T) \).

6. Haar functional. In [Wo2], Woronowicz proved the existence of the important invariant Haar functional on compact matrix quantum groups. Using the groupoid approach, we can give an explicit description of the Haar functional \( h_n \) on \( C(SU(n)_q) \)[Sh5].

Theorem 9. The unique invariant functional \( h_{n+1} \) on \( C(SU(n + 1)_q) \) is the restriction of the state \( f_\xi(n) \) of \( C^*(G^n) \) given by the regular representation on \( \ell^2(G^n|_{\mathbb{Z}^n}) \) and
\[
\xi^{(n)} = \left( \prod_{i=1}^{n} (1 - q^{-2(n+1-i)})^{-1/2} \right)
\]
and $u$ is a subgroupoid of the groupoid $Z$, i.e. $h$.

This result can be used to prove the known facts that $[N] h_{n+1}$ is a faithful state on $C(SU(n+1)_q)$, and that $K_i(C(SU^{2n+1}_q)) \simeq Z$ for both $i = 0$ and $1$ [VSo2].

7. Subquotient groupoids. It is an interesting question whether $C(SU^{2n+1}_q)$ (or $C(SU(n)_q)$) is really a groupoid $C^*$-algebra of some groupoid instead of a $C^*$-subalgebra of some groupoid $C^*$-algebra.

It is not difficult to see that the answer is affirmative for the following two simple but fundamental examples.

1. $C(SU(2)_q) \simeq C^* (\mathcal{G}_1)$ where

$$\mathcal{G}_1 = \{ (m,j,\ell) \mid \text{if } \ell = \infty, \text{then } m = 0 \}$$

is a subgroupoid of the groupoid $Z \times (Z \times Z), with

$$u_{11} = \sum_{0 < j < \infty} (1 - q^{-2j})^{1/2} \delta_{(1,0,j)}$$

and $u_{21} = \sum_{0 < j < \infty} q^{-j} \delta_{(1,0,j)}$.

2. Podles’ quantum sphere $C(SU^{n+1}_q) \simeq C^* (\mathcal{G}'), c > 0$, where

$$\mathcal{G}' = \{ (j, j_1, j_2) \mid j_1 = \infty \text{ or } j_2 = \infty \}$$

is a subgroupoid of the groupoid $Z^2 \times Z^2$.

It turns out that the answer is also affirmative for odd-dimensional quantum spheres $S^{2n+1}_q$ and for quantum $SU(3)_q$.

Define a subquotient groupoid $\mathcal{F}_n$ of $\mathcal{F}_n := Z \times (Z^n \times Z^2)$ as follows. Let

$$\mathcal{F}_n := \{ (m,j,k) \in \mathcal{F}_n \mid k_i = \infty \Rightarrow j_i = m - j_1 - j_2 - ... - j_{i-1} \text{ and } j_{i+1} = ... = j_n = 0 \}$$

be a subgroupoid of $\mathcal{F}_n$. Define $\tilde{\mathcal{F}}_n := \mathcal{F}_n/ \sim$ where $\sim$ is the equivalence relation generated by

$$(m,j,k) \sim (m,j,k_1,...,k_i = \infty, \infty, ..., \infty)$$

for all $(m,j,k)$ with $k_i = \infty$ for an $1 \leq i \leq n$.

Theorem 10. $C(SU^{2n+1}_q) \simeq C^*(\mathcal{F}_n)$.

From this, we get the following known result [VSo2].

Corollary 11. The $C^*$-algebra $C(SU^{2n+1}_q)$ is independent of $q$.

The key technical point in proving that $C(SU(3)_q)$ is a groupoid $C^*$-algebra is based on Lance’s [La] (or Woronowicz’s [Wo4]) result that there exists some isometry

$$v : \ell^2 (Z \times N \times Z) \to \ell^2 (Z \times N \times Z \times Z)$$
such that
\[ v(\Delta x)v^* = x \otimes 1 \]
for \( x \in C(SU(2)_q) \subset \mathcal{B}(\ell^2(\mathbb{Z} \times \mathbb{Z})) \). Modifying this result, we can show that

**Proposition 12.** There is some isometry
\[ w : \ell^2(\mathbb{Z} \times \mathbb{Z}) \to \ell^2(\mathbb{Z} \times \mathbb{Z}) \]
such that
\[ w((\pi_1 \otimes \pi_1)\Delta(x))w^* = (\tau \otimes \pi_1)\Delta(x) \]
for \( x \in C(SU(3)_q) \).

As far as the algebra \( C(SU(3)_q) \) is concerned, the face \( X_{1,2} \) corresponding to \( \pi_1 \otimes \pi_1 \) is ‘degenerate’ and is ‘dominated’ by the face \( X_{1,2} \) corresponding to \( \pi_1 \otimes \pi_2 \). This combined with the above composition sequence for \( C(SU(3)_q) \) helps to build a quotient groupoid \( \mathfrak{G}_2 \) from a subgroupoid \( \tilde{\mathfrak{G}}_2 \) of \( G^2 = \mathbb{Z}^2 \times (\mathbb{Z}^3 \times \mathbb{Z}^3) \) where
\[
\tilde{\mathfrak{G}}_2 := \left\{ (m, j, k) \in G^2 \mid m_1 + m_2 + m_3 = 0, \atop k_1 = \infty \Rightarrow j_1 = m_1 - m_2, \atop k_2 = \infty \Rightarrow j_2 = m_1 - m_3 - j_1, \atop k_3 = \infty \Rightarrow j_3 = m_2 - m_3 + j_1 - j_2 \right\}.
\]

We use the following notations: \( r \wedge s := \min\{r, s\} \), \( r \vee s := \max\{r, s\} \),
\[ \psi(r, s) := (r - s) \vee 0 - (s - r) \vee 0, \]
and \( \eta(r) := (r, 0) \) if \( r > 0 \) and \( \eta(r) := (0, -r) \) if \( r < 0 \). Define \( \mathfrak{G}_2 := \tilde{\mathfrak{G}}_2/\sim \), the quotient groupoid given by the equivalence relation \( \sim \) generated by
\[
(m, j_1, j_2, j_3, k_1, k_2 = \infty, k_3) \sim
(m, \tilde{j}_1, \tilde{j}_2 + j_1 - \tilde{j}_1, k_1 \wedge k_3, k_2 = \infty, k_3 \wedge k_3)
\]
where
\[
(\tilde{j}_1, \tilde{j}_2) := ((k_1 + l_1) \wedge (k_3 + l_3) - k_1 \wedge k_3)(1, 1)
+ \eta(\psi(k_1 + l_1, k_3 + l_3) - \psi(k_1, k_3)).
\]

**Theorem 13.** \( C(SU(3)_q) \cong C^*(\mathfrak{G}_2) \).

From this, we get a result of Nagy.

**Corollary 14.** \( C(SU(3)_q) \) as a \( C^* \)-algebra is independent of \( q \).

It is interesting to note that S. Wang [Wa] has proved that for \( n \geq 2 \), \( C(SU(n)_q) \) with different \( q \)-s are not isomorphic as Hopf \( C^* \)-algebras.

**References**


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