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DIFFERENTIAL CALCULUS ON 'NON-STANDARD' (h-DEFORMED) MINKOWSKI SPACES

JOSÉ A. DE AZCÁRRAGA

Departamento de Física Teórica and IFIC
Centro Mixto Universidad de Valencia-CSIC
E-46100-Burjassot (Valencia), Spain
E-mail: azcarrag@evalvx.ific.uv.es

FRANCISCO RODENAS

Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
E-46071 Valencia, Spain
E-mail: rodenas@evalvx.ific.uv.es

Abstract. The differential calculus on 'non-standard' h-Minkowski spaces is given. In particular it is shown that, for them, it is possible to introduce coordinates and derivatives which are simultaneously hermitian.

1. Introduction. We review first the properties of the two deformed Minkowski spaces [1] associated with the 'Jordanian' or 'non-standard' h-deformation, $SL_h(2)$, of $SL(2,C)$ [2, 3, 4]. The $GL_h(2)$ deformation is defined as the associative algebra generated by the entries a, b, c, d of a matrix M, the commutation properties of which may be expressed by an 'FRT' [5] equation, $R_{12}M_1M_2 = M_2M_1R_{12}$, in which R is the triangular solution of the Yang-Baxter equation

$$R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \ \hat{R}_h \equiv \mathcal{P}R_h = \begin{bmatrix} 1 & -h & h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{bmatrix} \ , \ \mathcal{P}R_h\mathcal{P} = R_h^{-1} \ . \quad (1)$$

The commutation relations of the group algebra generators in M are

$$\begin{array}{lll} [a,b] = h(\xi - a^2) \quad , & [a,c] = hc^2 \quad , & [a,d] = hc(d-a) \quad , \\ [b,c] = h(ac+cd) \quad , & [b,d] = h(d^2-\xi) \quad , & [c,d] = -hc^2 \quad ; \end{array} \quad (2)$$

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$$\xi \equiv det_h M = ad - cb - hcd \quad . \tag{3}$$

Setting $\xi = 1$ reduces $GL_h(2)$ to $SL_h(2)$. The matrix $\hat{R}_h$ has two eigenvalues (1 and -1) and a spectral decomposition in terms of a rank three projector P_{h+} and a rank one projector P_{h-}

$$\hat{R}_h = P_{h+} - P_{h-} \quad , \qquad P_{h\pm}\hat{R}_h = \pm P_{h\pm} \quad , \tag{4}$$

$$P_{h+} = \frac{1}{2}(I + \hat{R}_h) \quad , \quad P_{h-} = \frac{1}{2}(I - \hat{R}_h) = \frac{1}{2}\begin{bmatrix} 0 & h & -h & -h^2 \\ 0 & 1 & -1 & -h \\ 0 & -1 & 1 & h \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{5}$$

The deformed determinant ξ in (3) may be then expressed as

$$(det_h M)P_{h-} := P_{h-}M_1M_2 \quad , \quad (det_h M^{-1})P_{h-} = M_2^{-1}M_1^{-1}P_{h-} \quad , \tag{6}$$
$$det_h M^{-1} = (det_h M)^{-1} \quad , \quad (det_h M^\dagger)P_{h-}^\dagger = M_2^\dagger M_1^\dagger P_{h-}^\dagger \quad .$$

The following relations have an obvious equivalent in the undeformed case:

$$\epsilon_h M^t \epsilon_h^{-1} = M^{-1} \, , \, \epsilon_h = \begin{pmatrix} h & 1 \\ -1 & 0 \end{pmatrix} , \, \epsilon_h^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix} , \, P_{h-\,ij,kl} = \frac{-1}{2}\epsilon_{h\,ij}\epsilon^{-1}_{h\,kl} \, . \tag{7}$$

As in the standard q-case [6], a 'quantum h-plane' may be defined for $GL_h(2)$. The deformed h-plane associated with $GL_h(2)$ is the associative algebra generated by two elements $(x, y) \equiv X$, the commutation properties of which are given by [3, 4] $xy = yx + hy^2$ ($R_h X_1 X_2 = X_2 X_1$). These commutation relations are preserved by the transformations $X' = MX$. This invariance statement, suitably extended to apply to the case of deformed Minkowski spaces, provides the essential ingredient for a classification of the deformations of the Lorentz group [7] and of the associated Minkowski algebras [1] (see also [8]; we shall not consider here deformations governed by a dimensionful parameter).

Deformed 'groups' related with different values of $h \in C$ are equivalent and their R_h matrices are related by a similarity transformation. Thus, from now on, we shall take $h \in R$.

2. h-deformed Lorentz groups. The determination of a complete set of deformations of the Lorentz group (see [7, 1]) requires replacing (see [9, 10, 11]) the $SL(2, C)$ matrices A in $K' = AKA^\dagger$ ($K = K^\dagger = \sigma_\mu x^\mu$) by the generator matrix M of a deformation of $SL(2, C)$, and the characterization of all possible commutation relations among the generators (a, b, c, d) of M and (a^*, b^*, c^*, d^*) of $M^\dagger$.

In particular, for the deformed Lorentz groups associated with $SL_h(2)$, the R-matrix form of these commutation relations may be expressed by

$$\begin{array}{ll} R_h M_1 M_2 = M_2 M_1 R_h \quad , & M_1^\dagger R^{(2)} M_2 = M_2 R^{(2)} M_1^\dagger \quad , \\ M_2^\dagger R^{(3)} M_1 = M_1 R^{(3)} M_2^\dagger \quad , & R_h^\dagger M_1^\dagger M_2^\dagger = M_2^\dagger M_1^\dagger R_h^\dagger \quad , \end{array} \tag{8}$$

where $R^{(3)\,\dagger} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ ('reality' condition for $R^{(3)}$). The consistency of these relations is assured if $R^{(3)}$ satisfies the '*mixed* Yang-Baxter-like' equation (see [12, 13] in this respect)

$$R_{h\,12}R_{13}^{(3)}R_{23}^{(3)} = R_{23}^{(3)}R_{13}^{(3)}R_{h\,12} \quad . \tag{9}$$

This equation, considered as an 'FRT' equation, indicates that $R^{(3)}$ is a representation of $GL_h(2)$, $(M_{ij})_{\alpha\beta} = R^{(3)}_{i\alpha,j\beta}$. Thus, $R^{(3)}$ may be seen as a matrix in which its 2×2 blocks satisfy among themselves the same commutation relations as the entries of M,

$$R^{(3)} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \sim M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} . \tag{10}$$

As a result, the problem of finding all possible Lorentz h-deformations is equivalent to finding all possible $R^{(3)}$ matrices with 2×2 block entries satisfying (2) such that $\mathcal{P}R^{(3)}\mathcal{P} = R^{(3)\dagger}$ $(\hat{R}^{(3)} = \hat{R}^{(3)\dagger})$.

The solutions of these equations are (see [1, 7]) $(h \in R)$

$$\textbf{1.}\quad R^{(3)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & r & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \ r \in R ; \quad \textbf{2.}\quad R^{(3)} = \begin{bmatrix} 1 & 0 & -h & 0 \\ -h & 1 & 0 & h \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h & 1 \end{bmatrix} . \tag{11}$$

They characterize the two h-deformed Lorentz groups, which will be denoted $L_h^{(1)}$ and $L_h^{(2)}$ respectively. Using (11) in (8), the commutation relations among the entries of M and $M^\dagger$ read (see also [7])

1. $L_h^{(1)}$:

$$\begin{array}{lll} [a,a^*] = rc^*c \,, & [a,b^*] = rd^*c \,, & [a,d^*] = 0 \,, \\ [b,b^*] = r(d^*d - aa^*) \,, & [b,d^*] = -rc^*a \,, & [d,d^*] = -rc^*c \,, \\ [c,M^\dagger] = 0 \,. & & \end{array} \tag{12}$$

2. $L_h^{(2)}$:

$$\begin{array}{ll} [a,a^*] = -h(c^*a + a^*c) \,, & [a,b^*] = h(aa^* - d^*a - b^*c) \,, \\ [a,c^*] = hc^*c \,, & [a,d^*] = h(ac^* + d^*c) \,, \\ [b,b^*] = -h(ab^* + ba^* + b^*d + d^*b) \,, & [b,c^*] = h(c^*d + ac^*) \,, \\ [b,d^*] = h(d^*d - ad^* - bc^*) \,, & [c,c^*] = 0 \,, \\ [c,d^*] = hc^*c \,, & [d,d^*] = -h(cd^* + dc^*) \,. \end{array} \tag{13}$$

3. h-Deformed Minkowski spaces. To introduce the deformed Minkowski *algebra* $\mathcal{M}_h^{(j)}$ associated with a deformed Lorentz group $L_h^{(j)}$ (j=1,2) it is natural to extend $K' = AKA^\dagger$ above by stating that in the deformed case the corresponding K generates a comodule algebra for the coaction ϕ defined by

$$\phi : K \longmapsto K' = MKM^\dagger \ , \ K'_{is} = M_{ij}M^\dagger_{ls}K_{jl} \ , \ K = K^\dagger \ , \ \Lambda = M \otimes M^* \ , \tag{14}$$

where, as usual, it is assumed that the elements of K, which now do not commute among themselves, commute with those of M and $M^\dagger$. As in Sec. 1 for h-two-vectors (rather, h-two-*spinors*) we now demand that the commuting properties of the entries of K are preserved by (14). Covariance arguments to characterize the algebra generated by the elements of K have been extensively used, and the resulting equations are associated with the name of reflection equations [14, 15] or, in a more general setting, braided algebras [16, 17]. In the present $SL_h(2)$ case, the commutation properties of the entries of the hermitian matrix K generating a deformed Minkowski algebra $\mathcal{M}_h$ are given by a reflection equation of the form

$$R_h K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R_h^\dagger \quad , \tag{15}$$

where the $R^{(3)} = R^{(2)\dagger}$ matrices are those given in (11). Indeed, writing equation (15) for $K' = MKM^\dagger$, it follows that the invariance of the commutation properties of K under the associated deformed Lorentz transformation (14) is achieved if relations (8) are satisfied.

The deformed Minkowski length and metric, invariant under a Lorentz transformation (14) of $L_h^{(j)}$, are defined through the quantum determinant of K given by [1]

$$(det_h K) P_{h-} P_{h-}^\dagger = -P_{h-} K_1 \hat{R}^{(3)} K_1 P_{h-}^\dagger \quad , \tag{16}$$

where $P_{h-}P_{h-}^\dagger$ is a projector since $(P_{h-}P_{h-}^\dagger)^2 = \left(\frac{2+h^2}{2}\right)^2 P_{h-}P_{h-}^\dagger$. The above h-determinant is invariant, central and, since $\hat{R}^{(3)}$ and K are hermitian, real; thus, it defines the *deformed Minkowski length* l_h for the h-deformed spacetimes $\mathcal{M}_h^{(j)}$.

Similarly, it is possible to write the $L_h^{(j)}$-invariant scalar product of *contravariant* (transforming as the matrix K, eq. (14)) and *covariant* ($Y \mapsto Y' = (M^\dagger)^{-1} Y M^{-1}$) matrices (Minkowski four-vectors) as the quantum trace ([5, 18]) of a matrix product [12, 1]. We define the h-deformed trace of a matrix B by

$$tr_h(B) := tr(D_h B) \quad , \quad D_h := tr_{(2)}(\mathcal{P}((R_h^{t_1})^{-1})^{t_1}) = \begin{pmatrix} 1 & -2h \\ 0 & 1 \end{pmatrix} \; , \tag{17}$$

where $tr_{(2)}$ means trace in the second space. This deformed trace is invariant under the quantum group coaction $B \mapsto MBM^{-1}$ since the expression of D_h above guarantees that $D_h^t = M^t D_h^t (M^{-1})^t$ is fulfilled. To check this explicitly, we start by transposing the first eq. in (8) in the first space, obtaining

$$M_1^{t_1} R_h^{t_1} M_2 = M_2 R_h^{t_1} M_1^{t_1} \quad . \tag{18}$$

Inverting this expression and multiplying by $M_1^{t_1}$ from left and right, we get

$$(M_1^{t_1})_{ia} (M_2^{-1})_{jb} (R_h^{t_1})^{-1}_{ab,kl} = (R_h^{t_1})^{-1}_{ij,st} (M_2^{-1})_{tl} (M_1^{t_1})_{sk} \quad . \tag{19}$$

Setting $l = k$ and summing over k this gives

$$M_{ia}^t (M^{-1})_{jb} (R_h^{t_1})^{-1}_{ab,kk} = (R_h^{t_1})^{-1}_{ij,ss} \quad , \quad (R_h^{t_1})^{-1}_{ij,ss} = M_{ia}^t (R_h^{t_1})^{-1}_{ab,kk} (M^{-1})^t_{bj} \quad . \tag{20}$$

This allows us to define D_h as $D_{h\,ij} = (R_h^{t_1})^{-1}_{ji,ss}$ so that (17) is obtained.

Let us now find the expression of the metric tensor. Consider

$$K_{ij}^\epsilon := \hat{R}^\epsilon_{h\,ij,kl} K_{kl} \quad , \quad \hat{R}_h^\epsilon \equiv (1 \otimes (\epsilon_h^{-1})^t) \hat{R}^{(3)} (1 \otimes (\epsilon_h^{-1})^\dagger) \quad . \tag{21}$$

Then, if K is contravariant [(14)], K^ϵ is covariant *i.e.*, $K^\epsilon \mapsto (M^\dagger)^{-1} K^\epsilon M^{-1}$. This may be checked by using the property of $\hat{R}_h^\epsilon$,

$$\hat{R}_h^\epsilon (M \otimes (M^\dagger)^t) = ((M^\dagger)^{-1} \otimes (M^{-1})^t) \hat{R}_h^\epsilon \;\text{ or }\; \hat{R}_h^\epsilon M_1 M_2^* = (M_1^\dagger)^{-1} (M_2^{-1})^t \hat{R}_h^\epsilon \; , \tag{22}$$

which follows from the preservation of the h-symplectic metric ϵ_h. Now, using the expression of ϵ_h in (7), $(P_{h-})_{ij,kl} = -\frac{1}{2} \epsilon_{h\,ij} \epsilon^{-1}_{h\,kl}$ and $D_h = -\epsilon_h (\epsilon_h^{-1})^t$, it follows that the *h-deformed Minkowski length l_h* and *h-metric g_h* are given by

$$l_h = det_h K = \tfrac{1}{2+h^2} tr_h K K^\epsilon \equiv g_{h\, ij,kl} K_{ij} K_{kl} \quad , \quad g_{h\, ij,kl} = \tfrac{1}{2+h^2} D_{h\, si} \hat{R}^\epsilon_{h\, js,kl} \; . \tag{23}$$

The h-metric is preserved under h-Lorentz transformations $\Lambda = M \otimes M^*$,

$$\Lambda^t g_h \Lambda = g_h \quad , \quad g_h = \tfrac{1}{2+h^2} (D^t_h \otimes 1) \mathcal{P} \hat{R}^\epsilon_h \quad . \tag{24}$$

This is checked using eq. (22) and that $D^t_h = M^t D^t_h (M^{-1})^t$:

$$\begin{aligned} \Lambda^t g_h \Lambda &= (M \otimes M^*)^t g_h (M \otimes M^*) = M^t_1 M^\dagger_2 g_h M_1 M^*_2 \\ &= \tfrac{1}{2+h^2} M^t_1 M^\dagger_2 D^t_{h\,1} \mathcal{P} \hat{R}^\epsilon_h M_1 M^*_2 = \tfrac{1}{2+h^2} \mathcal{P} M^t_2 M^\dagger_1 D^t_{h\,2} \hat{R}^\epsilon_h M_1 M^*_2 \\ &= \tfrac{1}{2+h^2} \mathcal{P} M^t_2 M^\dagger_1 D^t_{h\,2} (M^\dagger_1)^{-1} (M^{-1}_2)^t \hat{R}^\epsilon_h = \tfrac{1}{2+h^2} \mathcal{P} M^t_2 D^t_{h\,2} (M^{-1}_2)^t \hat{R}^\epsilon_h \\ &= \tfrac{1}{2+h^2} \mathcal{P} D^t_{h\,2} \hat{R}^\epsilon_h = \tfrac{1}{2+h^2} D^t_{h\,1} \mathcal{P} \hat{R}^\epsilon_h = g_h \quad . \end{aligned} \tag{25}$$

The deformed h-Minkowski algebras $\mathcal{M}^{(j)}_h$

Using (11) in eqs. (15), (16) and (23), the h-Minkowski algebras associated with $SL_h(2)$ as well as the deformed Minkowski length and metric read

1. $\mathcal{M}^{(1)}_h$: Here, $R^{(3)}$ is the first matrix in (11). Then, eq. (15) gives (h real)

$$\begin{array}{ll} [\alpha, \beta] = -h\beta^2 - r\beta\delta + h\delta\alpha - h\beta\gamma + h^2\delta\gamma \; , & [\alpha, \delta] = h(\delta\gamma - \beta\delta) \; , \\ [\alpha, \gamma] = h\gamma^2 + r\delta\gamma - h\alpha\delta + h\beta\gamma - h^2\beta\delta \; , & [\beta, \delta] = h\delta^2 \; , \\ [\beta, \gamma] = h\delta(\gamma + \beta) + r\delta^2 \; , & [\gamma, \delta] = -h\delta^2 \; ; \end{array} \tag{26}$$

$$det_h K = \frac{2}{h^2+2} (\alpha\delta - \beta\gamma + h\beta\delta) \quad ; \tag{27}$$

$$\hat{R}^\epsilon_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & h \\ 0 & 0 & -1 & h \\ 1 & -h & -h & h^2 - r \end{bmatrix} , \; K^\epsilon = \begin{bmatrix} \delta & -\beta + h\delta \\ -\gamma + h\delta & \alpha - h(\beta + \gamma) + (h^2 - r)\delta \end{bmatrix} \; ; \tag{28}$$

$$g_h = \frac{1}{2+h^2} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & h \\ 0 & -1 & 0 & -h \\ 1 & -h & h & -r - h^2 \end{bmatrix} \quad . \tag{29}$$

2. $\mathcal{M}^{(2)}_h$: In this case, $R^{(3)}$ is the second matrix in (11). Then,

$$\begin{array}{ll} [\alpha, \beta] = 2h\alpha\delta + h^2\beta\delta \quad , & [\alpha, \delta] = 2h(\delta\gamma - \beta\delta) \quad , \\ [\alpha, \gamma] = -h^2\delta\gamma - 2h\delta\alpha \quad , & [\beta, \delta] = 2h\delta^2 \quad , \\ [\beta, \gamma] = 3h^2\delta^2 \quad , & [\gamma, \delta] = -2h\delta^2 \quad ; \end{array} \tag{30}$$

$$det_h K = \frac{2}{h^2+2} (\alpha\delta - \beta\gamma + 2h\beta\delta) \quad ; \tag{31}$$

$$\hat{R}^\epsilon_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 2h \\ 0 & 0 & -1 & 2h \\ 1 & 0 & 0 & h^2 \end{bmatrix} \quad , \quad K^\epsilon = \begin{bmatrix} \delta & -\beta + 2h\delta \\ -\gamma + 2h\delta & \alpha + h^2\delta \end{bmatrix} \quad ; \tag{32}$$

$$g_h = \frac{1}{2+h^2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2h \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 2h & -3h^2 \end{bmatrix} \quad . \tag{33}$$

We might define a 'time' generator for these h-deformed spacetimes as proportional to $tr_h K$ ($=2x^0$ in the undeformed case). However, the resulting algebra element has undesirable properties: $tr_h K$ is neither real nor central. The time generator is central only for the q-deformed Minkowski space of [10, 11] ($\mathcal{M}_q^{(1)}$ in the notation of [12, 1]).

4. Differential calculus on $\mathcal{M}_h^{(j)}$. To describe the differential calculus on h-Minkowski spaces, we need to express the different commutation relations among the fundamental objects: deformed coordinates, derivatives and one-forms. Following the approach of [19, 12, 20] to the differential calculus on Minkowski algebras associated with the standard deformation $SL_q(2)$, we introduce the reflection equations expressing the commutation relations defining the algebras of h-derivatives and h-one-forms (h-differential calculi have been considered in [4] and in [21] for quantum N-dimensional homogeneous spaces [1]). The triangularity property of R_h provides these algebras with some advantages with respect to the q-deformed ones; namely, the invariance requirement leads in both cases to only one reflection equation. This is due to the fact that, in general, the equivalent 'FRT' equations

$$R_{12}M_1M_2 = M_2M_1R_{12} \quad , \quad R_{21}^{-1}M_1M_2 = M_2M_1R_{21}^{-1} \quad , \tag{34}$$

allow us to take in $R^{(1)}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R^{(4)}$ (cf. (15)) $R^{(1)} = R_{12}$ or R_{21}^{-1}, $R^{(4)} = R_{12}^{\dagger}$ or $(R_{21}^{-1})^{\dagger}$ (see [1]). When the triangularity condition holds, however, $R_{12} = R_{21}^{-1}$ and there is only one possibility. This argument also applies to algebras other than the algebra of coordinates. Moreover, we shall show that it is possible to introduce 'coordinates' and 'derivatives' which are respectively and *simultaneously* hermitian and antihermitian.

The algebras of h-deformed derivatives $\mathcal{D}_h^{(j)}$

As in [12], we introduce the derivatives by means of an object Y transforming covariantly *i.e.*,

$$Y \longmapsto Y' = (M^{\dagger})^{-1}YM^{-1} \quad , \quad Y = \begin{bmatrix} \partial_\alpha & \partial_\gamma \\ \partial_\beta & \partial_\delta \end{bmatrix} \quad ; \tag{35}$$

The commutation properties of the derivatives are described by

$$R_h^{\dagger}Y_2R^{(2)\,-1}Y_1 = Y_1R^{(3)\,-1}Y_2R_h \quad , \tag{36}$$

where $R^{(3)} = \mathcal{P}R^{(2)}\mathcal{P}$ is given in (11), and are preserved under the h-Lorentz coaction. Since the covariance requirement is the main ingredient in our approach let us check explicitly that (36) is invariant under (35). Multiplying (36) by $(M_1^{\dagger})^{-1}(M_2^{\dagger})^{-1}$ from the

[1]We are grateful to the referee for drawing our attention to [21], where similar results were independently obtained.

left and by $M_2^{-1}M_1^{-1}$ from the right we get, using the first and the last equations in (8),

$$R_h^\dagger (M_2^\dagger)^{-1} Y_2 (M_1^\dagger)^{-1} R^{(2)\,-1} M_2^{-1} Y_1 M_1^{-1} = (M_1^\dagger)^{-1} Y_1 (M_2^\dagger)^{-1} R^{(3)\,-1} M_1^{-1} Y_2 M_2^{-1} R_h \ . \quad (37)$$

Finally, using the second and third third eqs. in (8), we obtain

$$R_h^\dagger \, (M_2^\dagger)^{-1} Y_2 M_2^{-1} \, R^{(2)\,-1} \, (M_1^\dagger)^{-1} Y_1 M_1^{-1} = (M_1^\dagger)^{-1} Y_1 M_1^{-1} \, R^{(3)\,-1} \, (M_2^\dagger)^{-1} Y_2 M_2^{-1} R_h \ , \quad (38)$$

which is eq. (36) for $Y' = (M^\dagger)^{-1} Y M^{-1}$.

The h-deformed d'Alembertian may be introduced by using the h-trace

$$\Box_h \equiv \frac{1}{2+h^2} tr_h (Y^\epsilon Y) \quad , \quad Y^\epsilon = (\hat{R}_h^\epsilon)^{-1} Y \quad . \quad (39)$$

As l_h, $\Box_h$ is Lorentz invariant, real and central in the algebra $\mathcal{D}_h^{(j)}$ of derivatives.

Using (11) in eqs. (36), the commutation relations for $\mathcal{D}_h^{(j)}$ read

1. $\mathcal{D}_h^{(1)}$:

$$\begin{array}{ll} [\partial_\alpha, \partial_\beta] = -h\partial_\alpha^2 \ , & [\partial_\beta, \partial_\delta] = h(\partial_\beta^2 + \partial_\beta\partial_\gamma - \partial_\alpha\partial_\delta) - r\partial_\beta\partial_\alpha - h^2\partial_\alpha\partial_\gamma \ , \\ [\partial_\alpha, \partial_\gamma] = h\partial_\alpha^2 \ , & [\partial_\gamma, \partial_\delta] = -h(\partial_\gamma^2 + \partial_\beta\partial_\gamma - \partial_\delta\partial_\alpha) + r\partial_\alpha\partial_\gamma + h^2\partial_\beta\partial_\alpha \ , \\ [\partial_\alpha, \partial_\delta] = h(\partial_\alpha\partial_\beta - \partial_\gamma\partial_\alpha) \ , & [\partial_\beta, \partial_\gamma] = h\partial_\alpha(\partial_\beta + \partial_\gamma) - r\partial_\alpha^2 \ ; \end{array} \quad (40)$$

2. $\mathcal{D}_h^{(2)}$:

$$\begin{array}{ll} [\partial_\alpha, \partial_\beta] = -2h\partial_\alpha^2 \quad , & [\partial_\alpha, \partial_\delta] = 2h(\partial_\beta\partial_\alpha - \partial_\alpha\partial_\gamma) \ , \\ [\partial_\alpha, \partial_\gamma] = 2h\partial_\alpha^2 \quad , & [\partial_\beta, \partial_\delta] = h^2\partial_\beta\partial_\alpha - 2h\partial_\delta\partial_\alpha \ , \\ [\partial_\beta, \partial_\gamma] = 5h^2\partial_\alpha^2 \quad , & [\partial_\gamma, \partial_\delta] = 2h\partial_\alpha\partial_\delta - h^2\partial_\alpha\partial_\gamma \ . \end{array} \quad (41)$$

Commutation relations for coordinates and derivatives

The commutation relations among the entries of K and Y may be expressed by an inhomogeneous reflection equation (see [19, 12])

$$Y_2 R_h K_1 R^{(2)} = R^{(3)} K_1 R_h^\dagger Y_2 + \eta R^{(3)} \mathcal{P} \quad , \quad (42)$$

which extends to the h-deformed case the undeformed relation $\partial_\mu x^\nu = \delta_\mu^\nu + x^\nu \partial_\mu$. This equation is consistent with the commutation relations defining the algebras $\mathcal{M}_h^{(j)}$, $\mathcal{D}_h^{(j)}$, and is invariant under h-Lorentz transformations (as already mentioned, there is only one L_h-invariant reflection equation due to the triangularity property of R_h). The invariance is seen by multiplying eq. (42) by $(M_2^\dagger)^{-1} M_1$ from the left and by $M_1^\dagger M_2^{-1}$ from the right and using the commutation relations in (8).

It is a common feature of all q-deformed Minkowski spaces that the covariance transformation properties for 'coordinates' and 'derivatives' are consistent with their hermiticity. The mixed commutation relations (as expressed by an inhomogeneous reflection equation), however, do not allow in general for *simultaneously* hermitian coordinates and derivatives, a feature of non-commutative geometry already noted in [11, 22]. Let us then look at the hermiticity properties of K and Y for our h-deformed Minkowski spaces. Clearly, eqs. (15) and (36) allow us to take both K and Y hermitian. Keeping the physically reasonable assumption that K is hermitian, eq. (42) gives

$$R^{(2)\dagger} K_1 R_h^\dagger Y_2^\dagger = Y_2^\dagger R_h K_1 R^{(3)\dagger} + \eta^* \mathcal{P} R^{(3)\dagger} \quad . \quad (43)$$

Since $R^{(3)\dagger} = R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$, we get that $(-Y^\dagger)$ satisfies (for $\eta^* = \eta$) the same commutation relations as Y. Thus, eqs. (15), (35) and (42) are compatible with the hermiticity of K and the antihermiticity of Y. This linear conjugation structure, absent in the q-deformation, may facilitate the formulation of invariant field equations on $\mathcal{M}_h^{(j)}$.

Using the $R^{(3)}$ matrices in (11) in eq. (42) and setting $\eta = 1$, the mixed commutation relations are found to be (we give only a few cases)

1. $\mathcal{M}_h^{(1)} \times \mathcal{D}_h^{(1)}$:

$$\begin{array}{ll} [\partial_\alpha, \alpha] = 1 + h\beta\partial_\alpha - h\gamma\partial_\alpha - h^2\delta\partial_\alpha \quad , & [\partial_\alpha, \beta] = -h\delta\partial_\alpha \quad , \\ [\partial_\alpha, \gamma] = h\delta\partial_\alpha \quad , & [\partial_\alpha, \delta] = 0 \quad , \end{array} \tag{44}$$

$$\begin{aligned} &[\partial_\beta, \alpha] = (r + h^2)\gamma\partial_\alpha + h(r - h^2)\delta\partial_\alpha - h(\alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\beta) + h^2(\delta\partial_\beta + \beta\partial_\alpha) \ , \\ &[\partial_\beta, \beta] = 1 + (r - h^2)\delta\partial_\alpha + h(\beta\partial_\alpha - \delta\partial_\beta) \ , \\ &[\partial_\beta, \gamma] = -h(\gamma\partial_\alpha + \delta\partial_\beta) + h^2\delta\partial_\alpha \ , \\ &[\partial_\beta, \delta] = h\delta\partial_\alpha \quad . \end{aligned}$$

2. $\mathcal{M}_h^{(2)} \times \mathcal{D}_h^{(2)}$:

$$\begin{array}{ll} [\partial_\alpha, \alpha] = 1 + 2h(\beta\partial_\alpha - \gamma\partial_\alpha) - 4h^2\delta\partial_\alpha \ , & [\partial_\alpha, \beta] = -2h\delta\partial_\alpha \ , \\ [\partial_\alpha, \gamma] = 2h\delta\partial_\alpha \ , & [\partial_\alpha, \delta] = 0 \ , \\ & \\ [\partial_\beta, \alpha] = -2h\alpha\partial_\alpha - h^2\gamma\partial_\alpha - 2h^3\delta\partial_\alpha \ , & [\partial_\beta, \delta] = 2h\delta\partial_\alpha \ , \\ [\partial_\beta, \beta] = 1 - h^2\delta\partial_\alpha \ , & [\partial_\beta, \gamma] = 4h^2\delta\partial_\alpha \end{array} \tag{45}$$

The algebras of h-deformed one-forms $\Lambda_h^{(j)}$

To determine now the commutation relations for the h-de Rham complex we now introduce the exterior derivative d following [11] (see also [12, 19]). The algebra of the h-forms is generated by the entries of a matrix dK. Clearly, d commutes with the Lorentz coaction, so that

$$dK' = M\, dK\, M^\dagger \quad . \tag{46}$$

Applying d to eq. (15) we obtain

$$R_h dK_1 R^{(2)} K_2 + R_h K_1 R^{(2)} dK_2 = dK_2 R^{(3)} K_1 R_h^\dagger + K_2 R^{(3)} dK_1 R_h^\dagger \quad . \tag{47}$$

Its only solution is given by

$$R_h dK_1 R^{(2)} K_2 = K_2 R^{(3)} dK_1 R_h^\dagger \tag{48}$$

(which implies $R_h K_1 R^{(2)} dK_2 = dK_2 R^{(3)} K_1 R_h^\dagger$). From eq. (48), it follows that

$$R_h dK_1 R^{(2)} dK_2 = -dK_2 R^{(3)} dK_1 R_h^\dagger \quad . \tag{49}$$

Again, it is easy to check that these relations are invariant under hermitian conjugation. Notice that the reflection equations (15), (48) and (49) have the same R-matrix structure. In the h-deformed case, the reflection equation giving the commutation relations among the generators of two differential algebras is determined only by the transformation (covariant or contravariant) law of these generators. Thus, there are only three types of reflection equations, those of (15), (36) and (42), as a consequence of the triangularity of $SL_h(2)$. In contrast, in the q-deformation (based on $SL_q(2)$), the number of reflection equation types is larger.

The exterior derivative is given for the two h-Minkowski algebras by

$$d = tr_h(dK\,Y) = d\alpha\partial_\alpha + d\beta\partial_\beta + d\gamma\partial_\gamma + d\delta\partial_\delta - 2h(d\gamma\partial_\alpha + d\delta\partial_\beta)\ . \tag{50}$$

To conclude, let us mention that the additive braided group [16, 17] structure of the above algebras may be easily found. It suffices to impose *e.g.* that eq. (15) is also satisfied by the sum $K+K'$ of two *copies* K and K' of the given h-Minkowski algebra. This leads to an equation of the same type as (15) (see the comment above)

$$R_h K_1' R^{(2)} K_2 = K_2 R^{(3)} K_1' R_h^\dagger \quad . \tag{51}$$

Eq. (51) is clearly preserved by (14). The above discussion could be extended easily to obtain a braided differential calculus. The unified braided structure of all h- (and q-, we note in passing) deformed Minkowski spaces was given in [1] (see [23] for the particular q-deformed case of [11]). Given the generality of our presentation, it is a trivial exercise to introduce a *unified* additive[2] braided differential calculus valid for all the h- (or q-) Minkowski algebras. For the case of the q-Minkowski space in [11] we refer to [24].

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