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NONCOMMUTATIVE 3-SPHERE AS AN EXAMPLE OF NONCOMMUTATIVE CONTACT ALGEBRAS

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Introduction. The notion of deformation quantization was introduced by F.Bayen, M.Flato *et al.* in [1]. The basic idea is to formally deform the pointwise commutative multiplication in the space of smooth functions $C^\infty(M)$ on a symplectic manifold M to a noncommutative associative multiplication, whose first order commutator is proportional to the Poisson bracket.

It is of interest to compute this quantization for naturally occuring cases. In this paper, we discuss deformations of contact algebras and give a definition of deformations of algebras slightly different from the deformation quantization of Poisson algebras. Since the standard 3-sphere is a basic example of a contact manifold, we study the properties of the noncommutative 3-sphere obtained by this reduction.

We remark that the parameter of the deformation of a contact algebra is not in the center, while the deformation quantization of Poisson algebras is given by algebras of formal power series of functions on a manifold; in particular, the deformation parameter is a central element.

Details and related results will appear in [6] and [7].

1. Reduction of the Wick product. Let us first recall some well-known facts about the Wick product on $\mathbf{C}^2$. Consider an associative algebra W over $\mathbf{C}$ generated by $\{\hbar, \zeta_1,$

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$\bar{\zeta}_1$,ζ_2, $\bar{\zeta}_2\}$ with the commutation relations:

$$[\zeta_i, \bar{\zeta}_j] = -2\hbar\delta_{ij}, \quad [\zeta_i, \zeta_j] = [\bar{\zeta}_i, \bar{\zeta}_j] = 0, \quad \hbar \in \text{center}. \tag{1}$$

The algebra W is called the *Wick algebra*. W has a canonical involutive anti-automorphism $a \to \bar{a}$. We denote its product by $*$.

The Fock space representation is the representation of W with the vacuum $|0\rangle$ on the vector space $V^{(\infty)} = W/\mathcal{L}$, where $\mathcal{L}$ is a left ideal generated by $\bar{\zeta}_i$ for $i = 1, 2$, i.e. $\bar{\zeta}_i|0\rangle = 0$. Here, we extend the algebra W by adjoining $\sqrt{2\hbar}$, $\sqrt{2\hbar}^{-1}$. These adjoined elements remain in the center. We see that $V^{(\infty)} = \sum \oplus V^{(m)}$, where

$$V^{(m)} = span\frac{1}{\sqrt{2\hbar}^m}\{\frac{1}{\sqrt{m!}}\zeta_1^m, \cdots, \frac{1}{\sqrt{k!l!}}\zeta_1^k * \zeta_2^l, \cdots, \frac{1}{\sqrt{m!}}\zeta_2^m\} \quad (m = k + l). \tag{2}$$

in the extended algebra. The left action of $w \in W$ on $V^{(\infty)}$ gives a representation of W. This action can be expressed as a matrix of infinite rank and it gives the representation of the Wick algebra. For $w \in W$, $\hat{w}$ denotes the matrix representation of w on $V^{(\infty)}$. We also have $\hat{\bar{\zeta}}_i = {}^t\hat{\zeta}_i$. Notice that almost all elements in W are represented as unbounded operators in general.

In what follows, w will be substituted for the matrix representation $\hat{w}$, whenever it creates no confusion. We will still denote by $a * b$ the product of the matrix a and b. Note that there are elements which are not well defined as elements of W but have rigorous meanings as matrices of infinite rank. We now consider a matrix given by

$$r = {}^*\!\sqrt{\bar{\zeta}_1 * \zeta_1 + \zeta_2 * \bar{\zeta}_2}\,, \tag{3}$$

where ${}_*\!\sqrt{}$ denotes the square root of the matrix. It is easily seen that r is given as a diagonal matrix. We see that $\bar{r} = r$ and r is invertible. We call r the *radial element*. We now set

$$\mu = -2\hbar r^{-2}, \quad \Xi_i = r^{-1} * \zeta_i, \quad \Xi_i^* = \bar{\zeta}_i * r^{-1} \quad (i = 1, 2). \tag{4}$$

Lemma 1. *The following relations hold*:

$$[\mu^{-1}, \Xi_i] = -\Xi_i, \quad [\mu^{-1}, \quad \Xi_i^*] = \Xi_i^*, \quad [\Xi_1, \Xi_2] = 0. \tag{5}$$

$$\Xi_i * \Xi_j^* - (1 - \mu)\Xi_j^* * \Xi_i = \mu\delta_{ij} \quad \textit{for } i, j = 1, 2. \tag{6}$$

$$\Xi_1^* * \Xi_1 + \Xi_2^* * \Xi_2 = 1. \tag{7}$$

Definition 1. We denote by $\mathcal{A}$ the algebra generated by $\{\mu, \Xi_1, \Xi_2, \Xi_1^*, \Xi_2^*\}$ with relations (5-7).

2. Noncommutative 3-sphere. We extend the algebra $\mathcal{A}$ in Definition 1 to a more suitable setting in a smooth category. Namely, we give another approach to obtain a complete topological algebra containing $\mathcal{A}$ densely via the reduction of the deformation quantization of $\mathbf{C}^2 - \{0\}$. Let ζ_1 and ζ_2 be complex coordinates on $\mathbf{C}^2$ and $\mathbf{C}[\zeta, \bar{\zeta}, \hbar]$ the space of all polynomials on $\mathbf{C}^2$ with coefficients in the polynomials of $\hbar$. The Wick algebra W is linearly isomorphic to $\mathbf{C}[\zeta, \bar{\zeta}, \hbar]$ and its associative product $*$ is given by the Moyal product formula:

$$a * b = a \exp\hbar\{\overleftarrow{\partial_\zeta} \cdot \overrightarrow{\partial_{\bar{\zeta}}} - \overleftarrow{\partial_{\bar{\zeta}}} \cdot \overrightarrow{\partial_\zeta}\}b, \tag{8}$$

where

$$a\left(\overleftarrow{\partial_\zeta}\cdot\overrightarrow{\partial_{\bar\zeta}}-\overleftarrow{\partial_{\bar\zeta}}\cdot\overrightarrow{\partial_\zeta}\right)b=\sum\nolimits_i(\partial_{\zeta_i}a\partial_{\bar\zeta_i}b-\partial_{\bar\zeta_i}a\partial_{\zeta_i}b).$$

The formula (8) extends naturally to the associative product on $C^\infty(\mathbf{C}^2)[[\hbar]]$. The associative algebra $(C^\infty(\mathbf{C}^2)[[\hbar]],*)$ is called the *deformation quantization* of $C^\infty(\mathbf{C}^2)$. Here, $C^\infty(\mathbf{C}^2)[[\hbar]]$ is the set of formal power series with values in $C^\infty(\mathbf{C}^2)$ with the formal parameter $\hbar$. We endow $C^\infty(\mathbf{C}^2)$ and $C^\infty(\mathbf{C}^2)[[\hbar]]$ the C^∞ topology and the direct product topology, respectively. Then the Wick algebra W is a dense subalgebra of $(C^\infty(\mathbf{C}^2)[[\hbar]],*)$.

It is easy to see that

$$r^2=\bar\zeta_1*\zeta_1+\zeta_2*\bar\zeta_2=\bar\zeta_1\cdot\zeta_1+\zeta_2\cdot\bar\zeta_2. \tag{9}$$

As each $\hbar^k$-term in the Moyal product formula (8) is expressed as a bidifferential operator, the star-product $*$ has locality. Hence, for any open subset U of $\mathbf{C}^2$, we can define the star-product $*$ of the deformation quantization $C^\infty(U)[[\hbar]]$ by the same formula (8). Note that any maximal 2-sided ideals (classical points) of $C^\infty(U)[[\hbar]]$ correspond to points of U. In the following, we will work mainly on $\mathbf{C}^2_*=\mathbf{C}^2-\{0\}$. We consider a function r as the square root of r^2 with respect to the ordinary commutative product $\cdot$ on the space $\mathbf{C}^2_*$, which is regarded as the radial element defined in §1.

A one parameter group of automorphisms

$$R(e^t):C^\infty(\mathbf{C}^2_*)[[\hbar]]\to C^\infty(\mathbf{C}^2_*)[[\hbar]] \tag{10}$$

is defined as follows:

$$R(e^t)\zeta_i=e^t\zeta_i,\quad R(e^t)\bar\zeta_i=e^t\bar\zeta_i,\quad R(e^t)\hbar=e^{2t}\hbar.$$

Set a closed subalgebra $\mathcal{A}^\infty$ of $C^\infty(\mathbf{C}^2_*)[[\hbar]]$ by

$$\mathcal{A}^\infty=\{f\in C^\infty(\mathbf{C}^2_*)[[\hbar]];R(e^t)f=f\}. \tag{11}$$

Under the relative topology from $C^\infty(\mathbf{C}^2_*)[[\hbar]]$, $\mathcal{A}^\infty$ is a complete topological associative algebra. We put

$$\mu=-2\hbar r^{-2},\quad \Xi_i=r^{-1}*\zeta_i,\quad \Xi_i^*=\bar\zeta_i*r^{-1}\quad(i=1,2).$$

Since the above elements have the same relations as in Lemma 1, the algebra $\mathcal{A}$ is densely embedded in $\mathcal{A}^\infty$. For the algebra $\mathcal{A}^\infty$, we have the following:

THEOREM A. *Set* $B=\mathcal{A}^\infty\cap C^\infty(\mathbf{C}^2_*)$

(A.1) $[\mu,\mathcal{A}^\infty]\subset\mu*\mathcal{A}^\infty*\mu$

(A.2) $[\mathcal{A}^\infty,\mathcal{A}^\infty]\subset\mu*\mathcal{A}^\infty$, *where* $[a,b]=a*b-b*a$ *is the commutator bracket.*

(A.3) $\mathcal{A}^\infty=B\oplus\mu*\mathcal{A}^\infty$ (*topological direct sum*).

(A.4) *The mappings* $\mu*:\mathcal{A}^\infty\to\mu*\mathcal{A}^\infty$, $*\mu:\mathcal{A}^\infty\to\mathcal{A}^\infty*\mu$ *defined by* $a\to\mu*a$, $a\to a*\mu$ *respectively are linear isomorphisms.*

(A.5) $a\to\bar a$ *is an involutive anti-automorphism such that* $\bar\mu=\mu$.

By property (A.3), we see for any positive integer N, $\mathcal{A}^{\infty}$ decomposes as follows:

$$\mathcal{A}^{\infty} = B \oplus \mu{*}B \oplus \cdots \oplus \mu^{N-1}{*}B \oplus \mu^{N}{*}\mathcal{A}^{\infty}. \tag{12}$$

$\mathcal{A}^{\infty}$ satisfies

$$\bigcap_{k} \mu^{k}{*}\mathcal{A}^{\infty} = \{0\}. \tag{A.6}$$

3. Noncommutative contact algebras. We call a complete topological associative algebra $\tilde{\mathcal{A}}$ a *regulated algebra*, (or more explicitly *μ-regulated algebra*) if there exists an element μ and a closed subspace $\tilde{B}$ satisfying (A.1)−(A.5). μ is called the *regulator* of $\tilde{\mathcal{A}}$. Note that (12) holds for any μ-regulated algebra. A μ-regulated algebra $\tilde{\mathcal{A}}$ is called *formal* if it satisfies (A.6). By (12), a formal μ-regulated algebra $\tilde{\mathcal{A}}$ may be denoted by $\tilde{\mathcal{A}} = \tilde{B}[[\mu]]$.

On any formal μ-regulated algebra $\tilde{\mathcal{A}}$, the axioms (A.1) and (A.4) permit us to introduce the formal symbol μ^{-1} such that $\mu^{-1} * \mu = \mu * \mu^{-1} = 1$. It gives a derivation $[\mu^{-1}, a]$ of $\tilde{\mathcal{A}}$ defined by

$$[\mu^{-1}, a] = -\mu^{-1} * [\mu, a] * \mu^{-1} \tag{13}$$

It is easy to see that

$$[\mu^{-1}{*}\tilde{\mathcal{A}}, \tilde{\mathcal{A}}] \subset \tilde{\mathcal{A}}, \quad [\mu^{-1}{*}\tilde{\mathcal{A}}, \mu^{-1}{*}\tilde{\mathcal{A}}] \subset \mu^{-1}{*}\tilde{\mathcal{A}}. \tag{14}$$

Let (m_0, m_1) be the maximal integers such that

$$[\mu^{-1}, \tilde{\mathcal{A}}] \subset \mu^{m_0}{*}\tilde{\mathcal{A}}, \quad [\tilde{\mathcal{A}}, \tilde{\mathcal{A}}] \subset \mu^{m_1}{*}\tilde{\mathcal{A}}.$$

If $[\mu^{-1}, \tilde{\mathcal{A}}] = \{0\}$, we put $m_0 = \infty$. We call (m_0, m_1) the *weight* of $\tilde{\mathcal{A}}$. A formal μ-regulated algebra $(\tilde{\mathcal{A}}, *)$ of weight $(\infty, 1)$ will be called a *quantized Poisson algebra*. In particular, the deformation quantization $C^{\infty}(\mathbf{C}^2)[[\hbar]]$ of $C^{\infty}(\mathbf{C}^2)$ is a formal $\hbar$-regulated algebra of weight $(\infty, 1)$, and $\mathcal{A}^{\infty}$ in Theorem A is a formal μ-regulated algebra of weight $(0, 1)$ respectively.

For any formal μ-regulated algebra $\tilde{\mathcal{A}} = \tilde{B}[[\mu]]$ of the weight $(0, 1)$, its associative product $*$ is determined by giving $a * b$ for $a, b \in \tilde{B}$: Set

$$a * b = \sum_{k\geq 0} \mu^{k} * \pi_k(a, b), \quad \pi_k(a, b) \in \tilde{B}. \tag{15}$$

We put

$$[\mu^{-1}, a] = \xi_0(a) + \cdots + \mu^{k} * \xi_k(a) + \cdots. \tag{16}$$

(2.11) is used for computing the following:

$$a * \mu = \mu * a + \mu^{2} * \xi_0(a) + \mu^{3} * (\xi_1(a) + \xi_0^{2}(a)) + \cdots \tag{17}$$

A commutative associative product $\cdot$ on $\tilde{\mathcal{A}}/\mu\tilde{\mathcal{A}}$ induces one on $\tilde{B}$ by the identification $\tilde{B}$ with $\tilde{\mathcal{A}}/\mu\tilde{\mathcal{A}}$.

It is easy to see that π_1 in (15) is a biderivation of $(\tilde{B}, \cdot)$ and ξ_0 in (16) is a derivation of $(\tilde{B}, \cdot)$. We remark here that one can change the filtration by a linear isomorphism $a \to a + \mu * L(a)$ of $\tilde{B}[[\mu]]$ defined by any continuous linear operator $L : \tilde{B} \to \tilde{B}$.

DEFINITION 1. A formal (μ-)regulated algebra $\tilde{\mathcal{A}}$ will be called a (μ-)regulated *smooth algebra* if there exists a filtration $\tilde{\mathcal{A}} = \tilde{B}[[\mu]]$ satisfying the following:

(i) $\tilde{B}$ in (A.3) is isomorphic to a subalgebra of the commutative algebra $C^\infty(M)$ of all C^∞ functions on a finite dimensional manifold M, and $\tilde{B} \supset C_0^\infty(M)$ the space of all support compact functions.

(ii) With $\tilde{B}$ considered as a subalgebra of $C^\infty(M)$, ξ_k in (16) is a linear operator of $\tilde{B}$ into $\tilde{B}$ expressed as a differential operator on M for any $k \geq 0$.

(iii) For any $k \geq 0$, π_k in (15) is a bilinear operator of $\tilde{B} \times \tilde{B}$ into $\tilde{B}$ expressed as a bidifferential operator on M.

In any smooth algebra, ξ_0 in (16) is a C^∞ vector field on M, called the characteristic vector field, and π_1 in (17) is a C^∞ bivector field on M.

DEFINITION 2. Let π_1^- be the skew symmetric part of π_1. A smooth algebra of weight $(0,1)$ is called a *noncommutative contact algebra* if the rank of π_1^- in (15) is maximal at each point of M.

The notion of the classical contact algebras can be obtained by considering the first term (μ^0-term) and the second term (μ^1-term) of the product (15) and (16) in noncommutative contact algebras. The space of C^∞ functions on a contact manifold naturally forms a contact algebra. Moreover, any contact algebra extends to a noncommutative contact algebra, that is *any contact algebra is quantizable.* Various properties on noncommutative contact algebras are shown in [6] and [7].

The following is easy to see :

PROPOSITION 3. *The noncommutative contact algebra $\mathcal{A}^\infty = B[[\mu]]$ given in Theorem A is a μ-regulated smooth algebra with $B = C^\infty(S^3)$.*

Although, as seen in §1, $\mu, \Xi_1, \Xi_2, \Xi_1^*, \Xi_2^*$ are represented as matrices, we regard them as elements of $C^\infty(S^3)[[\mu]]$ without matrix representations.

4. Remarks. The noncommutative 3-sphere we present here has additional structure corresponding to the Hopf fibration of the 3-sphere over the Riemann sphere. Instead of the canonical circle action on the standard 3-sphere, $ad(\mu^{-1})$ plays the role of 'noncommutatizing' the Hopf fibration for the noncommutative 3-sphere. As a result, we can construct the noncommutative Riemannian sphere. More generally, on Kähler manifolds, we can define noncommutative Kähler manifolds. Although this is essentially the same notion as in Karabegov [4], we work strictly in the noncommutative algebra setting.

The noncommutative algebra $\mathcal{A}^\infty$ also furnishes a representation of the noncommutative Riemann sphere. This agrees with the work on geometric quantization for Kähler manifolds by Berezin [2] and Cahen-Gutt-Rawnsley [3]. In addition to the representation of the noncommutative Riemannian sphere, elements in the algebra $\mathcal{A}^\infty$ are representable; in particular, μ^{-1} is representable. More generally, these considerations also provide a representation for Kähler manifolds M of integral class whenever the associated line bundle of the S^1-bundle over M has a nontrivial holomorphic section. This should be compared with the recent work by Guillemin [5].

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